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A NEW MODEL FOR THIN PLATES WITH RAPIDLY VARYING THICKNESS. III: COMPARISON OF DIFFERENT SCALINGS*

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1. Introduction. This work continues a series of papers [1, 2] on the bending of thin, symmetric plates with rapidly varying thickness. Motivated by recent developments in structural optimization [3, 4, 5] we have studied plates with thickness of order ε varying on a length scale of order ε^a. There are three different regimes, depending on whether $a < 1$ (the case of relatively slow thickness variation), $a = 1$ (when the variation is on the same scale as the mean thickness), or $a > 1$ (the case of relatively fast thickness variation). Each determines an effective rigidity tensor $M_{\alpha\beta\gamma\delta}$ relating bending moment to midplane curvature; in the limit as $\varepsilon \to 0$, the vertical displacement of the midplane solves an equation of the form

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(M_{\alpha\beta\gamma\delta} \frac{\partial^2}{\partial x_\gamma \partial x_\delta} w \right) = F.$$

It is natural to ask which scaling produces the most rigid structure for a given thickness profile. The present paper addresses that issue. We shall consider the periodic case, in which the plate thickness is $\varepsilon h(x_1/\varepsilon^a, x_2/\varepsilon^a)$, where $h(\boldsymbol{\eta})$ is a periodic function of $\boldsymbol{\eta} = (\eta_1, \eta_2)$ and ε is a small parameter. Our models are easily extended to plates whose microstructure varies slowly from point to point, i.e., to those with thickness $\varepsilon h(x_1, x_2; x_1/\varepsilon^a, x/\varepsilon^a)$ [1]. In that case $M_{\alpha\beta\gamma\delta}$ varies with $\mathbf{x} = (x_1, x_2)$. The "slow variation" is irrelevant for comparisons of the type performed here, hence all our conclusions apply equally well to that case.

Our analysis begins in Sec. 2 with a brief review of the three models. As a new element we formulate a set of dual variational principles for the effective rigidities $M_{\alpha\beta\gamma\delta}$. All comparisons will be established by using suitable test fields in these variational principles.

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Section 3 is devoted to the study of plates with one family of stiffeners. This means that the thickness variation h is a function of one variable, e.g. $h = h(\eta_1)$. If the three-dimensional elastic law is isotropic we show that

$$M^{a>1}_{\alpha\beta\gamma\delta}t_{\alpha\beta}t_{\gamma\delta} \leqslant M^{a=1}_{\alpha\beta\gamma\delta}t_{\alpha\beta}t_{\gamma\delta} \leqslant M^{a<1}_{\alpha\beta\gamma\delta}t_{\alpha\beta}t_{\gamma\delta}. \tag{1.1}$$

In other words $a < 1$ (relatively slow variation) gives the most rigid structure, and $a > 1$ (relatively fast variation) the least rigid. The left inequality in (1.1) is a general fact for plates with one family of stiffeners, valid for an arbitrary three-dimensional elastic law. The right inequality, however, is more or less specific to the isotropic case: we shall give an example of an anisotropic material and a class of thickness variations such that $M^{a=1}_{1111} > M^{a<1}_{1111}$.

For a general thickness function $h(\eta_1, \eta_2)$ our results are less complete. Section 4 establishes the validity of (1.1) for a rather limited class of elastic laws, including the isotropic one with Poisson's ratio zero. We expect also the left inequality of (1.1) to fail for some choices of the thickness variation $h(\boldsymbol{\eta})$ and the three-dimensional elastic material.

Our conclusions have obvious implications for structural optimization. It is known that plates with rapidly varying thickness may be stronger, in some design contexts, than any conventional, slowly varying structure [3, 4, 5]. However, there are at least three different types of rapid variation, corresponding to our models $a < 1$, $a = 1$, and $a > 1$ respectively. For maximal strength one should choose the scaling with the largest effective rigidity quadratic form. We shall explore this issue further and report on numerical experiments in forthcoming papers [6, 7].

Remark. We take this opportunity to correct a misprint in [1]. The second equation in (7.3) has an incorrect factor of ν, and should read

$$M^{a=1}_{2222} = \frac{2}{3}\frac{E}{1-\nu^2}m - \left(\frac{E\nu}{1-\nu^2}\right)^2 \mathscr{E}^*.$$

(The results in Tables 1 and 2a–d of [1] were computed using the correct formula.)

2. Variational principles. For each of the cases $a < 1$, $a = 1$, and $a > 1$, the effective rigidity tensor $M_{\alpha\beta\gamma\delta}$ was defined in [1, 2] in terms of the energies of auxiliary functions obtained by solving certain elliptic boundary value problems. For making comparisons it is convenient to use variational characterizations of the associated quadratic form $M_{\alpha\beta\gamma\delta}t_{\alpha\beta}t_{\gamma\delta}$, where $t = (t_{\alpha\beta})$ is any symmetric tensor. For each case $a < 1$, $a = 1$, and $a > 1$, we shall give two variational principles, involving "displacement energy" and "complementary energy" respectively. The functionals to be extremized are the same for $a < 1$ and $a > 1$, but the one involving displacement energy differs slightly when $a = 1$. Moreover, the class of admissible test fields differs in each of the three cases. These distinctions display rather clearly the differences between the models.

We adopt the notation of [1, 2]; in particular, $\eta = (\eta_1, \eta_2)$ and $\underline{\eta} = (\eta_1, \eta_2, \eta_3)$. Throughout, Q will be the rescaled period cell determined by the periodic function $h(\boldsymbol{\eta})$,

$$Q = \left\{\underline{\eta}: |\eta_\alpha| < L_\alpha/2, |\eta_3| < h(\boldsymbol{\eta})\right\},$$

where L_1 and L_2 are the periods of h. Its upper and lower boundaries are

$$\partial_{\pm} Q = \left\{ \underline{\eta}: |\eta_\alpha| < L_\alpha/2, \eta_3 = \pm h(\boldsymbol{\eta}) \right\},$$

with outward unit normal vector $\underline{\nu}$. We shall always assume that Q is a Lipschitz domain. The summation convention will be used, with Latin indices ranging from 1 to 3 and Greek ones from 1 to 2. The linear strain and stress associated with a displacement $\underline{\psi}$ are

$$E_{ij}(\underline{\psi}) = \frac{1}{2}\left(\frac{\partial \psi_j}{\partial x_i} + \frac{\partial \psi_i}{\partial x_j} \right), \qquad \Sigma_{ij}(\underline{\psi}) = B_{ijkl} E_{kl}(\underline{\psi}).$$

The tensor B_{ijkl} represents Hooke's law for the three-dimensional elastic material comprising the plate. It satisfies

$$B_{ijkl} = B_{jikl} = B_{ijlk} = B_{klij}; \tag{2.1}$$

in addition we require

$$B_{\alpha\beta\gamma 3} = 0 \quad \text{and} \quad B_{\alpha 333} = 0,$$

so that the horizontal planes are planes of elastic symmetry. We assume that the form $B_{ijkl} e_{ij} e_{kl}$ is positive definite on symmetric tensors (e_{ij}). The inverse of Hooke's law will be denoted by A_{ijkl}:

$$\Sigma_{ij} = B_{ijkl} E_{kl} \Leftrightarrow E_{ij} = A_{ijkl} \Sigma_{kl}.$$

As in [2], $\underline{\Gamma}^{\alpha\beta}$ denotes the displacement field

$$\underline{\Gamma}^{\alpha\beta} = \left(-\eta_3 \frac{\partial}{\partial \eta_1}\left(\frac{1}{2}\eta_\alpha \eta_\beta\right), -\eta_3 \frac{\partial}{\partial \eta_2}\left(\frac{1}{2}\eta_\alpha \eta_\beta\right), \right.$$
$$\left. \frac{1}{2}\eta_\alpha\eta_\beta + \frac{1}{2}\eta_3^2 \frac{B_{33\gamma\delta}}{B_{3333}} \frac{\partial^2}{\partial \eta_\gamma \partial \eta_\delta}\left(\frac{1}{2}\eta_\alpha\eta_\beta\right) \right).$$

Its key role is due to the fact that it produces the linear stress

$$\Sigma_{\gamma\delta}(\underline{\Gamma}^{\alpha\beta}) = -\eta_3 \tilde{B}_{\alpha\beta\gamma\delta}, \qquad \Sigma_{3j}(\underline{\Gamma}^{\alpha\beta}) = 0, \tag{2.2}$$

with

$$\tilde{B}_{\alpha\beta\gamma\delta} = B_{\alpha\beta\gamma\delta} - \left(B_{\alpha\beta 33} B_{\gamma\delta 33} / B_{3333} \right).$$

Finally, for any symmetric tensor $t = (t_{\alpha\beta})$ we define $\underline{\Gamma}^t = \underline{\Gamma}^{\alpha\beta} t_{\alpha\beta}$.

2*A*. *The* $a = 1$ *model*. Our variational principles for $M_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta}$ express it first as the minimum of an elastic energy functional over a suitable class of displacements $\underline{\psi} = (\psi_1, \psi_2, \psi_3)$, and then as the maximum of the corresponding complementary energy over a suitable class of stress fields $\Sigma = (\Sigma_{ij})$.

The space of admissible displacements for $a = 1$ consists of all $\underline{\psi}$ which have finite strain energy and are periodic in $\boldsymbol{\eta}$ with period $\mathbf{L}$:

$$\underline{\psi} \in V^{a=1} \quad \text{if} \int_Q \Sigma_{ij}(\underline{\psi}) E_{ij}(\underline{\psi})\, d\underline{\eta} < \infty \text{ and}$$
$$\underline{\psi}(\eta_1 + L_1, \eta_2, \eta_3) = \underline{\psi}(\eta_1, \eta_2 + L_2, \eta_3) = \underline{\psi}(\eta_1, \eta_2, \eta_3).$$

The space $W^{a=1}$ of admissible stress fields contains periodic, symmetric tensors Σ which have finite energy and are equilibrated:

$$\Sigma \in W^{a=1} \quad \text{if} \int_Q |\Sigma|^2 \, d\underline{\eta} < \infty, \qquad \operatorname{div} \Sigma = 0,$$
$$\Sigma \cdot \underline{\nu} = 0 \quad \text{on } \partial_{\pm} Q, \Sigma_{ij} = \Sigma_{ji}, \text{ and}$$
$$\Sigma(\eta_1 + L_1, \eta_2, \eta_3) = \Sigma(\eta_1, \eta_2 + L_2, \eta_3) = \Sigma(\eta_1, \eta_2, \eta_3).$$

The dual variational principles for $M^{a=1}_{\alpha\beta\gamma\delta}$ are these:

$$(\mathscr{P}_{a=1}) \quad M_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} = \min_{\underline{\psi} \in V^{a=1}} \frac{1}{L_1 L_2} \int_Q \Sigma_{ij}(\underline{\psi} + \underline{\Gamma}^t) E_{ij}(\underline{\psi} + \underline{\Gamma}^t) \, d\underline{\eta}$$

$$(\mathscr{D}_{a=1}) \quad = \max_{\Sigma \in W^{a=1}} \frac{1}{L_1 L_2} \int_Q \left(2\Sigma_{ij} E_{ij}(\underline{\Gamma}^t) - A_{ijkl} \Sigma_{ij} \Sigma_{kl}\right) d\underline{\eta}.$$

To justify $(\mathscr{P}_{a=1})$, we recall the definition of $M^{a=1}_{\alpha\beta\gamma\delta}$ given in [2]:

$$M_{\alpha\beta\gamma\delta} = \frac{1}{L_1 L_2} \int_Q \Sigma_{ij}\left(\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta}\right) E_{ij}\left(\underline{\phi}^{\gamma\delta} + \underline{\Gamma}^{\gamma\delta}\right) d\underline{\eta}, \tag{2.3}$$

where the auxiliary functions $\underline{\phi}^{\alpha\beta} \in V^{a=1}$ are chosen to satisfy

$$\int_Q \Sigma_{ij}\left(\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta}\right) E_{ij}(\underline{\psi}) \, d\underline{\eta} = 0 \quad \forall \underline{\psi} \in V^{a=1}. \tag{2.4}$$

Contracting (2.4) with $t_{\alpha\beta}$ one sees that the right side of $(\mathscr{P}_{a=1})$ is stationary at $\underline{\psi} = \underline{\phi}^{\alpha\beta} t_{\alpha\beta}$. By convexity this must be a minimum, and (2.3) leads to $(\mathscr{P}_{a=1})$.

The dual principle $(\mathscr{D}_{a=1})$ is just the principle of maximum complementary energy adapted to the present context. To prove it we consider the pointwise inequality

$$A_{ijkl} \Sigma_{ij} \Sigma_{kl} \geqslant 2\Sigma_{ij} E_{ij} - B_{ijkl} E_{ij} E_{kl}, \tag{2.5}$$

valid for any symmetric tensors Σ and E. If $\Sigma \in W^{a=1}$, we take $E = E((\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta}) t_{\alpha\beta})$, and note that

$$\int_Q \Sigma_{ij} E_{ij} \, d\underline{\eta} = \int_Q \Sigma_{ij} E_{ij}(\underline{\Gamma}^t) \, d\underline{\eta} \tag{2.6}$$

by Green's formula. Therefore integration of (2.5) over Q in combination with (2.4) gives

$$\frac{1}{L_1 L_2} \int_Q \left(2\Sigma_{ij} E_{ij}(\underline{\Gamma}^t) - A_{ijkl} \Sigma_{ij} \Sigma_{kl}\right) d\underline{\eta} \leqslant M^{a=1}_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta}. \tag{2.7}$$

To complete the proof we observe that equality is achieved in (2.5) and hence in (2.7) by $\Sigma = \Sigma((\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta}) t_{\alpha\beta})$, which is admissible by virtue of (2.4).

2*B*. *The $a > 1$ model*. For the $a > 1$ model, we define a modified stress $\hat{\Sigma}$ by ignoring the transverse shear:

$$\hat{\Sigma}_{13}(\underline{\psi}) = \hat{\Sigma}_{23}(\underline{\psi}) = 0, \qquad \hat{\Sigma}_{ij}(\underline{\psi}) = \Sigma_{ij}(\underline{\psi}) \quad \text{otherwise.}$$

The space $V^{a>1}$ of admissible displacements for the $a > 1$ plate consists of $\underline{\psi}$ for which ψ_3 depends only on η_3, and which have finite energy ignoring the shear:

$$\underline{\psi} \in V^{a>1} \quad \text{if} \int_Q \hat{\Sigma}_{ij}(\underline{\psi}) E_{ij}(\underline{\psi})\, d\underline{\eta} < \infty, \psi_3 = \psi_3(\eta_3),$$
$$\underline{\psi}(\eta_1 + L_1, \eta_2, \eta_3) = \underline{\psi}(\eta_1, \eta_2 + L_2, \eta_3) = \underline{\psi}(\eta_1, \eta_2, \eta_3).$$

This is not a subset of $V^{a=1}$, because $E_{\alpha 3}(\underline{\psi})$ may not be square integrable.

The space $W^{a>1}$ of admissible stresses contains fields with zero shear, equilibrated on each slice as far as the plane stresses are concerned, and for which Σ_{33} averages to zero on each slice.

$$\Sigma \in W^{a>1} \quad \text{if } \Sigma_{\alpha 3} = 0, \qquad \int_Q |\Sigma|^2\, d\underline{\eta} < \infty, \qquad \frac{\partial}{\partial \eta_\beta} \Sigma_{\alpha\beta} = 0,$$
$$\Sigma_{\alpha\beta}\nu_\beta = 0 \quad \text{on } \partial_{\pm} Q, \qquad \int_{Q \cap \{\eta_3 = t\}} \Sigma_{33}\, d\boldsymbol{\eta} = 0, \Sigma_{ij} = \Sigma_{ji} \text{ and}$$
$$\Sigma(\eta_1 + L_2, \eta_2, \eta_3) = \Sigma(\eta_1, \eta_2 + L_2, \eta_3) = \Sigma(\eta_1, \eta_2, \eta_3).$$

It is not contained in $W^{a=1}$, because $\Sigma_{33}\nu_3$ may be nonzero on $\partial_{\pm} Q$ and $\partial\Sigma_{33}/\partial\eta_3$ may not vanish.

The variational principles are similar to those for $a = 1$, except that we use $\hat{\Sigma}$ instead of Σ in $(\mathscr{P}_{a>1})$:

$$(\mathscr{P}_{a>1}) \quad M^{a>1}_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} = \min_{\underline{\psi} \in V^{a>1}} \frac{1}{L_1 L_2} \int_Q \hat{\Sigma}_{ij}(\underline{\psi} + \underline{\Gamma}^t) E_{ij}(\underline{\psi} + \underline{\Gamma}^t)\, d\underline{\eta}$$
$$(\mathscr{D}_{a>1}) \quad = \max_{\Sigma \in W^{a>1}} \frac{1}{L_1 L_2} \int_Q \left(2\Sigma_{ij} E_{ij}(\underline{\Gamma}^t) - A_{ijkl}\Sigma_{ij}\Sigma_{kl}\right) d\underline{\eta}.$$

The justification of $(\mathscr{P}_{a>1})$ follows the pattern of the preceding case. The definition of $M^{a>1}_{\alpha\beta\gamma\delta}$ given in [2] is

$$M^{a>1}_{\alpha\beta\gamma\delta} = \frac{1}{L_1 L_2} \int_Q \hat{\Sigma}_{ij}(\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta}) E_{ij}(\underline{\phi}^{\gamma\delta} + \underline{\Gamma}^{\gamma\delta})\, d\underline{\eta}, \tag{2.8}$$

in terms of auxiliary functions $\underline{\phi}^{\alpha\beta} \in V^{a>1}$ for which

$$\int_Q \hat{\Sigma}_{ij}(\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta}) E_{ij}(\underline{\psi})\, d\underline{\eta} = 0 \quad \forall \underline{\psi} \in V^{a>1}. \tag{2.9}$$

Contraction of (2.9) with $t_{\alpha\beta}$ shows that $\underline{\psi} = \underline{\phi}^{\alpha\beta} t_{\alpha\beta}$ is stationary for $(\mathscr{P}_{a>1})$; by convexity it must be a minimizer. Evaluation of the integral by means of (2.8) leads to $(\mathscr{P}_{a>1})$.

The proof of $(\mathscr{D}_{a>1})$ is parallel to that of $(\mathscr{D}_{a=1})$. We again integrate (2.5) with $E = E((\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta})t_{\alpha\beta})$, this time using the auxiliary functions defined by (2.9). Since $\hat{\Sigma}((\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta})t_{\alpha\beta}) \in W^{a>1}$ (cf. [1]) and since (2.6) remains valid for $\Sigma \in W^{a>1}$, the argument used before leads directly to $(\mathscr{D}_{a>1})$.

2C. *The $a<1$ model.* The space $V^{a<1}$ of admissible displacements is formed using the ansatz of standard Kirchhoff plate theory:

$$\underline{\psi} \in V^{a<1} \quad \text{if} \int_Q \hat{\Sigma}(\underline{\psi}) E(\underline{\psi})\, d\underline{\eta} < \infty \quad \text{and}$$

$$\underline{\psi} = \left(-\eta_3 \frac{\partial \chi}{\partial \eta_1}, -\eta_3 \frac{\partial \chi}{\partial \eta_2}, \chi + \frac{1}{2}\eta_3^2 \frac{B_{\alpha\beta 33}}{B_{3333}} \frac{\partial^2 \chi}{\partial \eta_\alpha \partial \eta_\beta} \right) \text{ for some}$$

function $\chi \in H^2_{\text{per}}([-L_1/2, L_1/2] \times [-L_2/2, L_2/2])$.

The space $W^{a<1}$ of admissible stresses is also formed using the Kirchhoff ansatz:

$$\Sigma \in W^{a<1} \quad \text{if} \int_Q |\Sigma|^2\, d\underline{\eta} < \infty, \qquad \Sigma_{i3} = \Sigma_{3i} = 0, \quad \text{and} \quad \Sigma_{\alpha\beta} = -\eta_3 m_{\alpha\beta}(\eta)$$

for some $m_{\alpha\beta} = m_{\beta\alpha}$ with $\partial^2\left(h^3 m_{\alpha\beta}\right)/\partial\eta_\alpha\partial\eta_\beta = 0$

and $m_{\alpha\beta}(\eta_1 + L_1, \eta_2) = m_{\alpha\beta}(\eta_1, \eta_2 + L_2) = m_{\alpha\beta}(\eta_1, \eta_2)$.

The dual variational principles involve the same functionals as for $a > 1$:

$$(\mathscr{P}_{a<1}) \quad M^{a<1}_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} = \min_{\underline{\psi} \in V^{a<1}} \frac{1}{L_1 L_2} \int_Q \hat{\Sigma}_{ij}\left(\underline{\psi} + \underline{\Gamma}'\right) E_{ij}\left(\underline{\psi} + \underline{\Gamma}'\right) d\underline{\eta}$$

$$(\mathscr{D}_{a<1}) \qquad = \max_{\Sigma \in W^{a<1}} \frac{1}{L_1 L_2} \int_Q \left(2\Sigma_{ij} E_{ij}(\underline{\Gamma}') - A_{ijkl}\Sigma_{ij}\Sigma_{kl}\right) d\underline{\eta}.$$

To justify $(\mathscr{P}_{a<1})$, we reformulate slightly the definition of $M^{a<1}_{\alpha\beta\gamma\delta}$ given in [1]:

$$M^{a<1}_{\alpha\beta\gamma\delta} = \frac{1}{L_1 L_2} \int_Q \hat{\Sigma}_{ij}\left(\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta}\right) E_{ij}\left(\underline{\phi}^{\gamma\delta} + \underline{\Gamma}^{\gamma\delta}\right) d\underline{\eta}, \tag{2.10}$$

where the auxiliary functions $\underline{\phi}^{\alpha\beta} \in V^{a<1}$ are chosen so that

$$\int_Q \hat{\Sigma}_{ij}\left(\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta}\right) E_{ij}(\underline{\psi})\, d\underline{\eta} = 0 \quad \forall \underline{\psi} \in V^{a<1}. \tag{2.11}$$

(Formulas (3.3) and (3.2) of [1] are obtained from (2.10) and (2.11) by doing the integrations in η_3 explicitly.) Arguing as in the other two cases, one verifies that $\underline{\psi} = \underline{\phi}^{\alpha\beta} t_{\alpha\beta}$ is the minimizer for $(\mathscr{P}_{a<1})$, and that $\hat{\Sigma}((\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta}) t_{\alpha\beta})$ is extremal for $(\mathscr{D}_{a<1})$.

3. Plates with one family of stiffeners. This section investigates the case when h is a function of a single variable; without loss of generality we shall take $h = h(\eta_1)$. It was conjectured in [1] that

$$M^{a>1}_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} \leqslant M^{a=1}_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} \leqslant M^{a<1}_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} \tag{3.1}$$

for any isotropic material. The validity of this assertion is a consequence of Propositions 3.1 and 3.3. The left inequality is rather easy, and it actually holds for any choice of the elastic law. The right side is more difficult, and our proof applies only to the isotropic case. Proposition 3.4 gives a simple example of an anisotropic material and a thickness variation $h(\eta_1)$ for which the right of (3.1) is false.

If M and N are rigidity tensors, we shall say that $M \leqslant N$ whenever $M_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} \leqslant N_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta}$ for all symmetric second order tensors t.

PROPOSITION 3.1. *If $h = h(\eta_1)$ then $M^{a>1} \leqslant M^{a=1}$ for every choice of the elastic law B_{ijkl}.*

Proof. When $h = h(\eta_1)$ the auxiliary functions $\underline{\phi}^{\alpha\beta}$ for the $a > 1$ model can be written explicitly (see [1] for the isotropic case). One finds that $\Sigma_{33}(\underline{\phi}^{\alpha\beta}) = 0$, and therefore that the extremal $\Sigma((\underline{\phi}^{\alpha\beta} + \underline{\Gamma}^{\alpha\beta}) t_{\alpha\beta})$ for $(\mathscr{D}_{a>1})$ is in $W^{a=1}$. Use of this stress in the variational principle $(\mathscr{D}_{a=1})$ yields the desired conclusion. ●

Comparing $M^{a=1}$ and $M^{a<1}$ is more subtle. As a first, relatively easy step we compare $M_{1212}^{a=1}$ and $M_{1212}^{a<1}$.

PROPOSITION 3.2. *If $h = h(\eta_1)$ and the elastic law satisfies $\tilde{B}_{1112} = 0$ then $M_{1212}^{a=1} \leqslant M_{1212}^{a<1}$.*

Proof. Under this hypothesis the auxiliary function $\underline{\phi}^{12}$ for $a < 1$ is equal to $(0, 0, 0)$. Since $\Sigma_{\alpha 3}(\underline{\phi}^{12} + \underline{\Gamma}^{12}) = 0$, use of $\underline{\phi}^{12}$ as a test function in $(\mathscr{P}_{a=1})$ establishes the proposition. ●

We recall that the elastic law B_{ijkl} is called isotropic if

$$\begin{aligned} B_{iiii} &= \lambda + 2\mu, \\ B_{iijj} &= \lambda, \qquad i \neq j, \\ B_{ijij} &= \mu, \qquad i \neq j \end{aligned}$$

(no summation convention), with all other B_{ijkl} equal to zero (except those determined by the symmetries (2.1)). The parameters λ and μ are the Lamé coefficients, given in terms of Poisson's ratio $0 < \nu < \frac{1}{2}$ and Young's modulus $E > 0$ by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}.$$

For an isotropic elastic law and $h = h(\eta_1)$ it is not difficult to compute that

$$\begin{aligned} &\left(M_{\alpha\beta\gamma\delta}^{a<1} - M_{\alpha\beta\gamma\delta}^{a=1}\right) t_{\alpha\beta} t_{\gamma\delta} \\ &\quad = \left(M_{1111}^{a<1} - M_{1111}^{a=1}\right)(t_{11} + \nu t_{22})^2 + 4\left(M_{1212}^{a<1} - M_{1212}^{a=1}\right) t_{12}^2. \end{aligned} \tag{3.2}$$

This relation arises because the auxiliary functions $\underline{\phi}^{11}$ and $\underline{\phi}^{22}$ are linearly dependent, *cf.* [1]. Since $M_{1212}^{a=1} \leqslant M_{1212}^{a<1}$ by Proposition 3.2, we see that $M^{a=1} \leqslant M^{a<1}$ for an isotropic law if and only if $M_{1111}^{a=1} \leqslant M_{1111}^{a<1}$. This last relation will be proved by projecting the minimizer of $(\mathscr{D}_{a=1})$ into $W^{a<1}$, then using the result as a test field for $(\mathscr{D}_{a<1})$. The relevant projection is defined by the following lemma.

LEMMA 3.1. *If $h \in C^1$ and $\Sigma^{a=1}$ is an element of $W^{a=1}$, then the field defined by*

$$\begin{aligned} \Sigma_{\alpha\beta}^{a<1} &= \frac{3}{2}\eta_3 h^{-3}(\boldsymbol{\eta}) \int_{-h}^{h} \Sigma_{\alpha\beta}^{a=1}(\boldsymbol{\eta}, \xi)\xi \, d\xi \\ \Sigma_{i3}^{a<1} &= \Sigma_{3i}^{a<1} = 0 \end{aligned} \tag{3.3}$$

is contained in $W^{a<1}$.

Proof. Since $\Sigma_{\alpha\beta}^{a<1} = -\eta_3 m_{\alpha\beta}(\boldsymbol{\eta})$ with

$$m_{\alpha\beta} = \frac{-3}{2} h^{-3}(\boldsymbol{\eta}) \int_{-h}^{h} \Sigma_{\alpha\beta}^{a=1}(\boldsymbol{\eta}, \xi)\xi \, d\xi,$$

it clearly suffices to prove that

$$\frac{\partial}{\partial \eta_\alpha} \frac{\partial}{\partial \eta_\beta}\left(\int_{-h}^{h} \Sigma_{\alpha\beta}^{a=1}(\boldsymbol{\eta}, \xi)\xi \, d\xi\right) = 0. \tag{3.4}$$

We may assume that $\Sigma_{\alpha\beta}^{a=1}$ is odd in η_3, since the integral vanishes for its even part. A simple calculation gives that

$$\frac{\partial}{\partial \eta_\beta}\left(\int_{-h}^{h} \Sigma_{\alpha\beta}^{a=1}(\boldsymbol{\eta}, \xi)\xi \, d\xi\right)$$
$$= \int_{-h}^{h} \frac{\partial}{\partial \eta_\beta}\Sigma_{\alpha\beta}^{a=1}(\boldsymbol{\eta}, \xi)\xi \, d\xi + 2\Sigma_{\alpha\beta}^{a=1}(\boldsymbol{\eta}, h)h\frac{\partial h}{\partial \eta_\beta}.$$

Since $\operatorname{div} \Sigma^{a=1} = 0$ and $\Sigma^{a=1} \cdot \underline{\nu} = 0$ on $\partial_+ Q$ this last expression equals

$$-\int_{-h}^{h} \frac{\partial}{\partial \xi}\Sigma_{\alpha 3}^{a=1}(\boldsymbol{\eta}, \xi)\xi \, d\xi + 2\Sigma_{\alpha\beta}^{a=1}(\boldsymbol{\eta}, h)h\frac{\partial h}{\partial \eta_\beta}$$
$$= \int_{-h}^{h} \Sigma_{\alpha 3}^{a=1}(\boldsymbol{\eta}, \xi) \, d\xi - 2\Sigma_{\alpha 3}^{a=1}(\boldsymbol{\eta}, h)h + 2\Sigma_{\alpha\beta}^{a=1}(\boldsymbol{\eta}, h)h\frac{\partial h}{\partial \eta_\beta}$$
$$= \int_{-h}^{h} \Sigma_{\alpha 3}^{a=1}(\boldsymbol{\eta}, \xi) \, d\xi.$$

Thus to verify (3.4) we must show that

$$\frac{\partial}{\partial \eta_\alpha}\left(\int_{-h}^{h} \Sigma_{\alpha 3}^{a=1}(\boldsymbol{\eta}, \xi) \, d\xi\right) = 0.$$

Calculating as before we obtain

$$\frac{\partial}{\partial \eta_\alpha}\left(\int_{-h}^{h} \Sigma_{\alpha 3}^{a=1}(\boldsymbol{\eta}, \xi) \, d\xi\right) = \int_{-h}^{h} \frac{\partial}{\partial \eta_\alpha}\Sigma_{\alpha 3}^{a=1}(\boldsymbol{\eta}, \xi) \, d\xi + 2\Sigma_{\alpha 3}^{a=1}(\boldsymbol{\eta}, h)\frac{\partial h}{\partial \eta_\alpha}$$
$$= -2\Sigma_{33}^{a=1}(\boldsymbol{\eta}, h) + 2\Sigma_{\alpha 3}^{a=1}(\boldsymbol{\eta}, h)\frac{\partial h}{\partial \eta_\alpha} = 0,$$

precisely the desired conclusion. ●

PROPOSITION 3.3. *If $h = h(\eta_1) \in C^1$ and the elastic law is isotropic then $M^{a=1} \leqslant M^{a<1}$.*

Proof. Choosing $t_{\alpha\beta}$ as $t_{11} = 1$, $t_{12} = t_{22} = 0$, let $\Sigma^{a=1}$ be the maximizer for $(\mathscr{D}_{a=1})$, and consider its projection $\Sigma^{a<1}$ defined by (3.3). It is not hard to see that

$$\int_Q \Sigma_{ij}^{a<1} E_{ij}(\underline{\Gamma}^{11}) \, d\underline{\eta} = \int_Q \Sigma_{ij}^{a=1} E_{ij}(\underline{\Gamma}^{11}) \, d\underline{\eta}; \tag{3.5}$$

the verification uses (3.3) and the relation

$$\int_Q \Sigma_{33}^{a=1}\eta_3 \, d\underline{\eta} = 0,$$

which follows from (2.4) with $\underline{\psi} = (0, 0, \frac{1}{2}\eta_3^2)$. We shall show that

$$\int_Q A_{ijkl}\Sigma_{ij}^{a<1}\Sigma_{kl}^{a<1} \, d\underline{\eta} \leqslant \int_Q A_{ijkl}\Sigma_{ij}^{a=1}\Sigma_{kl}^{a=1} \, d\underline{\eta}. \tag{3.6}$$

Using this, (3.5), $(\mathscr{D}_{a=1})$, and $(\mathscr{D}_{a<1})$,

$$M_{1111}^{a=1} = \frac{1}{L_1 L_2}\int_Q \left(2\Sigma_{ij}^{a=1}E_{ij}(\underline{\Gamma}^{11}) - A_{ijkl}\Sigma_{ij}^{a=1}\Sigma_{kl}^{a=1}\right) d\underline{\eta}$$

$$\leqslant \frac{1}{L_1 L_2}\int_Q \left(2\Sigma_{ij}^{a<1}E_{ij}(\underline{\Gamma}^{11}) - A_{ijkl}\Sigma_{ij}^{a<1}\Sigma_{kl}^{a<1}\right) d\underline{\eta} \leqslant M_{1111}^{a<1}.$$

As explained earlier, this inequality suffices to establish that $M^{a=1} \leqslant M^{a<1}$.

The proof of (3.6) relies heavily on the form of h and the elastic law. Since h depends only on η_1, both $\Sigma^{a=1}$ and $\Sigma^{a<1}$ depend only on η_1 and η_3. Also, since $t_{12} = 0$, $\Sigma_{12}^{a=1} = \Sigma_{32}^{a=1} = 0$, and indeed $\Sigma^{a=1} = \Sigma(\underline{\phi}^{11} + \underline{\Gamma}^{11})$ where $\underline{\phi}^{11} = (\phi_1^{11}, 0, \phi_3^{11})$ solves a problem of plane strain elasticity on the domain

$$Q^* = \left\{(\eta_1, \eta_3) \colon |\eta_1| < L_1/2, |\eta_3| < h(\eta_1)\right\},$$

cf. [1]. Because $\underline{\phi}^{11}$ is independent of η_2 it follows that

$$\Sigma_{22}^{a=1} = \nu\left(\Sigma_{11}^{a=1} + \Sigma_{33}^{a=1}\right). \tag{3.7}$$

For fixed η_1, $\Sigma_{\alpha\beta}^{a<1}$ is the L^2-projection of $\Sigma_{\alpha\beta}^{a=1}$ onto the line of multiples of η_3. Therefore

$$\Sigma_{22}^{a<1} = \nu\left(\Sigma_{11}^{a<1} + \tilde{\Sigma}_{33}\right), \tag{3.8}$$

were $\tilde{\Sigma}_{33}$ denotes the L^2-projection of $\Sigma_{33}^{a=1}$ onto multiples of η_3 (note that $\tilde{\Sigma}_{33}$ is in general different from $\Sigma_{33}^{a<1} = 0$). Using (3.7) and (3.8) we find that

$$\begin{aligned}
&A_{ijkl}\left(\Sigma_{ij}^{a=1}\Sigma_{kl}^{a=1} - \Sigma_{ij}^{a<1}\Sigma_{kl}^{a<1}\right)\\
&\quad= \frac{1+\nu}{E}\Big[\left(\Sigma_{11}^{a=1}\right)^2 + 2\left(\Sigma_{13}^{a=1}\right)^2 + \left(\Sigma_{33}^{a=1}\right)^2 - \nu\left(\Sigma_{11}^{a=1} + \Sigma_{33}^{a=1}\right)^2\\
&\qquad - \left(\Sigma_{11}^{a<1}\right)^2 - \nu^2\left(\Sigma_{11}^{a<1} + \tilde{\Sigma}_{33}\right)^2 + \frac{\nu}{1+\nu}\left((1+\nu)\Sigma_{11}^{a<1} + \nu\tilde{\Sigma}_{33}\right)^2\Big]\\
&\quad= \frac{1-\nu^2}{E}\left[\left(\Sigma_{11}^{a=1}\right)^2 - \left(\Sigma_{11}^{a<1}\right)^2 + \left(\Sigma_{33}^{a=1}\right)^2 - \frac{\nu^2}{1-\nu^2}\left(\tilde{\Sigma}_{33}\right)^2 + 2\left(\Sigma_{13}^{a=1}\right)^2\right]\\
&\qquad + \frac{2\nu(1+\nu)}{E}\left[\left(\Sigma_{13}^{a=1}\right)^2 - \Sigma_{11}^{a=1}\Sigma_{33}^{a=1}\right].
\end{aligned} \tag{3.9}$$

The constant $\nu^2/(1-\nu^2)$ is always less than $\frac{1}{3}$. Therefore the integral of the first term of (3.9) is nonnegative,

$$\int_Q \left[\left(\Sigma_{11}^{a=1}\right)^2 - \left(\Sigma_{11}^{a<1}\right)^2 + \left(\Sigma_{33}^{a=1}\right)^2 - \frac{\nu^2}{1-\nu^2}\left(\tilde{\Sigma}_{33}\right)^2 + 2\left(\Sigma_{13}^{a=1}\right)^2\right] d\underline{\eta} \geqslant 0,$$

using the fact that $\Sigma_{11}^{a<1}$ and $\tilde{\Sigma}_{33}$ are L^2-projections of $\Sigma_{11}^{a=1}$ and $\Sigma_{33}^{a=1}$ respectively.

The proof will be completed by showing that the second part has integral zero:

$$\int_Q \left[\left(\Sigma_{13}^{a=1}\right)^2 - \Sigma_{11}^{a=1}\Sigma_{33}^{a=1}\right] d\underline{\eta} = 0. \tag{3.10}$$

To this end we represent $\Sigma_{11}^{a=1}$, $\Sigma_{13}^{a=1}$ and $\Sigma_{33}^{a=1}$ by means of an Airy stress function $\phi(\eta_1, \eta_3)$:

$$\Sigma_{11}^{a=1} = \frac{\partial^2\phi}{\partial\eta_3^2}, \quad \Sigma_{13}^{a=1} = -\frac{\partial^2\phi}{\partial\eta_1\partial\eta_3}, \quad \Sigma_{33}^{a=1} = \frac{\partial^2\phi}{\partial\eta_1^2}. \tag{3.11}$$

Such a ϕ exists because the domain Q^* is simply connected. The field $\Sigma^{a=1}$ is contained in $W^{a=1}$ and therefore

$$0 = \Sigma_{11}^{a=1}\nu_1 + \Sigma_{13}^{a=1}\nu_3 = \frac{\partial^2\phi}{\partial\eta_3^2}\nu_1 - \frac{\partial^2\phi}{\partial\eta_1\partial\eta_3}\nu_3$$

$$0 = \Sigma_{13}^{a=1}\nu_1 + \Sigma_{33}^{a=1}\nu_3 = -\frac{\partial^2\phi}{\partial\eta_1\partial\eta_3}\nu_1 + \frac{\partial^2\phi}{\partial\eta_1^2}\nu_3$$

on $\partial_{\pm}Q^* = \{(\eta_1, \eta_3): |\eta_1| < L_1/2, \eta_3 = \pm h(\eta_1)\}$. It follows that

$$\frac{\partial\phi}{\partial\eta_1} \text{ and } \frac{\partial\phi}{\partial\eta_3} \text{ are constant on } \partial_+Q^* \left(\text{and } \partial_-Q^*\right). \tag{3.12}$$

From (3.11), (3.12) and the fact that $\Sigma^{a=1}$ is periodic in η_1 we conclude that ϕ has the form

$$\phi(\eta_1, \eta_3) = \tilde{\phi}(\eta_1, \eta_3) + l(\eta_1),$$

where $\tilde{\phi}$ is periodic in η_1 and l is a linear function in η_1. This immediately leads to

$$\frac{\partial\phi}{\partial\eta_1} \text{ and } \frac{\partial^2\phi}{\partial\eta_3^2} \text{ are periodic in } \eta_1. \tag{3.13}$$

Using (3.11), (3.13) and integration by parts we get that

$$\frac{1}{L_2}\int_Q \left[\left(\Sigma_{13}^{a=1}\right)^2 - \Sigma_{11}^{a=1}\Sigma_{33}^{a=1}\right] d\underline{\eta} = \int_{\partial_{\pm}Q^*} \frac{\partial\phi}{\partial\eta_1}\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial\eta_3}\right) ds, \tag{3.14}$$

where $\partial/\partial t$ denotes the derivative tangent to $\partial_{\pm}Q^*$ (clockwise) and ds is surface measure. The right hand side of (3.14) vanishes on account of (3.12), and this establishes (3.10). ●

The rest of this section is devoted to showing that the opposite inequality $M_{1111}^{a=1} > M_{1111}^{a<1}$ is possible for some choices of $h = h(\eta_1)$ and some anisotropic elastic laws. Our method is again to use the complementary energy variational principles. This time, however, we shall use an extremal stress for $(\mathscr{D}_{a<1})$ to construct a test field for use in $(\mathscr{D}_{a=1})$.

Consider any $\Sigma \in W^{a<1}$. By hypothesis it has the form

$$\Sigma = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{-\eta_3 m_{\alpha\beta}(\boldsymbol{\eta})} & 0 \\ & & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \tag{3.15}$$

with

$$\frac{\partial^2}{\partial\eta_\alpha\partial\eta_\beta}\left(h^3 m_{\alpha\beta}\right) = 0. \tag{3.16}$$

We define an associated test field $\tilde{\Sigma} \in W^{a=1}$ by

$$\tilde{\Sigma} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{-\eta_3 m_{\alpha\beta}(\boldsymbol{\eta})} & g_1 \\ & & g_2 \\ \hline g_1 & g_2 & G \end{array}\right) \tag{3.17}$$

with

$$g_\alpha = \frac{\partial}{\partial\eta_\beta}\left(\frac{1}{2}\left(\eta_3^2 - h^2\right)m_{\alpha\beta}\right), \qquad G = -\int_0^{\eta_3} \frac{\partial}{\partial\eta_\alpha} g_\alpha(\boldsymbol{\eta}, \xi)\, d\xi. \tag{3.18}$$

To be in $W^{a=1}$, $\tilde{\Sigma}$ must satisfy the six equilibrium conditions

$$\frac{\partial}{\partial \eta_j}\tilde{\Sigma}_{ij} = 0, \qquad \tilde{\Sigma}_{ij}\nu_j = 0 \quad \text{on } \partial_{\pm}Q.$$

The first five follow by direct computation from (3.17) and (3.18). The sixth is a consequence of (3.16), since that equation implies

$$0 = \frac{\partial^2}{\partial \eta_\alpha \partial \eta_\beta}\left(\int_0^h \frac{1}{2}\left(\eta_3^2 - h^2\right) m_{\alpha\beta}\, d\eta_3\right) = \frac{\partial}{\partial \eta_\alpha}\left(\int_0^h g_\alpha\, d\eta_3\right)$$

$$= g_\alpha \frac{\partial}{\partial \eta_\alpha} h|_{\eta_3 = h} + \int_0^h \frac{\partial g_\alpha}{\partial \eta_\alpha} d\eta_3.$$

This gives $\tilde{\Sigma}_{3j}\nu_j = 0$ on the upper boundary $\partial_+ Q$, and the same follows for $\partial_- Q$ by symmetry.

LEMMA 3.2. *Assume that* $h = h(\eta_1) \in C^1$. *Given* $t = (t_{\alpha\beta})$, *let* $\Sigma^{a<1}$ *be the maximizer for* $(\mathscr{D}_{a<1})$, *and let* $\Sigma^{a=1}$ *be the associated element of* $W^{a=1}$ *defined by* (3.17)–(3.18). *If*

$$\int_Q A_{ijkl}\Sigma_{ij}^{a=1}\Sigma_{kl}^{a=1}\, d\underline{\eta} < \int_Q A_{ijkl}\Sigma_{ij}^{a<1}\Sigma_{kl}^{a<1}\, d\underline{\eta} \tag{3.19}$$

then it follows that

$$M_{\alpha\beta\gamma\delta}^{a=1} t_{\alpha\beta} t_{\gamma\delta} > M_{\alpha\beta\gamma\delta}^{a<1} t_{\alpha\beta} t_{\gamma\delta}.$$

Proof. We recall that since $\Sigma^{a=1} \in W^{a=1}$,

$$\int_Q \Sigma_{33}^{a=1}\eta_3\, d\underline{\eta} = \int_Q \Sigma_{ij}^{a=1} E_{ij}\left(\left(0, 0, \frac{1}{2}\eta_3^2\right)\right) d\underline{\eta} = 0,$$

and therefore

$$\int_Q \Sigma_{ij}^{a=1} E_{ij}(\underline{\Gamma}^t)\, d\underline{\eta} = \int_Q \Sigma_{ij}^{a<1} E_{ij}(\underline{\Gamma}^t)\, d\underline{\eta}.$$

It follows from the variational principles and (3.19) that

$$M_{\alpha\beta\gamma\delta}^{a<1} t_{\alpha\beta} t_{\gamma\delta} = \frac{1}{L_1 L_2}\int_Q \left(2\Sigma_{ij}^{a<1} E_{ij}(\underline{\Gamma}^t) - A_{ijkl}\Sigma_{ij}^{a<1}\Sigma_{kl}^{a<1}\right) d\underline{\eta}$$

$$< \frac{1}{L_1 L_2}\int_Q \left(2\Sigma_{ij}^{a=1} E_{ij}(\underline{\Gamma}^t) - A_{ijkl}\Sigma_{ij}^{a=1}\Sigma_{kl}^{a=1}\right) d\underline{\eta} \leqslant M_{\alpha\beta\gamma\delta}^{a=1} t_{\alpha\beta} t_{\gamma\delta}. \quad \bullet$$

Now consider an elastic law of the form

$$\begin{aligned} &B_{iiii} = \lambda + 2\mu, \\ &B_{iijj} = \lambda, \qquad i \neq j, \\ &B_{1212} = \mu, \qquad B_{1313} = B_{2323} = \mu', \end{aligned} \tag{3.20}$$

(no summation convention) with λ, μ, $\mu' > 0$. The other components B_{ijkl} are zero, except of course those determined by the symmetries (2.1), and the complementary energy quadratic form is

$$A_{ijkl}\Sigma_{ij}\Sigma_{kl} = \frac{1}{2\mu}\left(\Sigma_{11}^2 + 2\Sigma_{12}^2 + \Sigma_{22}^2 + \Sigma_{33}^2\right) + \frac{1}{\mu'}\left(\Sigma_{13}^2 + \Sigma_{23}^2\right) - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\left(\Sigma_{11} + \Sigma_{22} + \Sigma_{33}\right)^2.$$

An isotropic law corresponds to $\mu = \mu'$, but for the present purposes we require instead that μ' be sufficiently large compared with μ.

PROPOSITION 3.4. *Consider the thickness function*

$$h(\eta) = h_0(1 + \varepsilon\cos 2\pi\eta_1)$$

with h_0 constant. If

$$\frac{3}{\mu'} + \frac{8}{7}\frac{\mu + \lambda}{\mu(2\mu + 3\lambda)}\pi^2 h_0^2 < \frac{3\lambda}{2\mu(\mu + \lambda)} \tag{3.21}$$

in the elastic law (3.20) *then $M_{1111}^{a=1} > M_{1111}^{a<1}$ for ε sufficiently small.*

Proof. Of course we shall apply Lemma 3.2. When $h = h(\eta_1)$ the extremal stress for $(\mathscr{D}_{a<1})$ is easy to express in closed form. Taking $t_{11} = 1$, $t_{12} = t_{22} = 0$, and arguing as in [1], we see that it has the form (3.15) with

$$m_{\alpha\beta} = ch^{-3}\tilde{B}_{\alpha\beta 11},$$

where $c = \left(\int_0^1 h^{-3}\,d\eta_1\right)^{-1}$ is the harmonic mean of h^3 and

$$\tilde{B}_{1111} = \frac{4\mu(\mu + \lambda)}{2\mu + \lambda}, \quad \tilde{B}_{2211} = \frac{2\mu\lambda}{2\mu + \lambda}, \quad \tilde{B}_{1211} = 0.$$

Substitution in (3.18) gives $g_2 = 0$,

$$g_1 = \frac{c}{2}\tilde{B}_{1111}\frac{\partial}{\partial\eta_1}\left((\eta_3^2 - h^2)h^{-3}\right),$$

and

$$\begin{aligned} G &= -\frac{c}{2}\tilde{B}_{1111}\int_0^{\eta_3}\frac{\partial^2}{\partial\eta_1^2}\left[(\xi^2 - h^2)h^{-3}\right]d\xi \\ &= -\frac{c}{2}\tilde{B}_{1111}\left\{\eta_3 h^{-2}\left[h'' - 2h^{-1}(h')^2\right] - \eta_3^3 h^{-4}\left[h'' - 4h^{-1}(h')^2\right]\right\}, \end{aligned}$$

with $h' = \partial h/\partial\eta_1$. It follows with some computation that

$$\int_0^1\int_{-h}^{h} g_1^2\,d\eta_3\,d\eta_1 = \frac{4}{5}\tilde{B}_{1111}^2\pi^2 h_0^5\varepsilon^2 + O(\varepsilon^3),$$

$$\int_0^1\int_{-h}^{h} G^2\,d\eta_3\,d\eta_1 = \frac{32}{105}\tilde{B}_{1111}^2\pi^4 h_0^7\varepsilon^2 + O(\varepsilon^3),$$

$$\int_0^1\int_{-h}^{h} Gh^{-3}\eta_3\,d\eta_3\,d\eta_1 = -\frac{4}{5}\tilde{B}_{1111}\pi^2 h_0^2\varepsilon^2 + O(\varepsilon^3).$$

Now taking $Q = \{\underline{\eta}: |\eta_\alpha| < \frac{1}{2}, |\eta_3| < h(\eta_1)\}$, we obtain that

$$\int_Q A_{ijkl}\left(\Sigma_{ij}^{a=1}\Sigma_{kl}^{a=1} - \Sigma_{ij}^{a<1}\Sigma_{kl}^{a<1}\right) d\underline{\eta} = \mathrm{I} + \mathrm{II} + \mathrm{III},$$

with

$$\mathrm{I} = (\mu')^{-1}\int_Q g_1^2\, d\underline{\eta} = \frac{4}{5}\tilde{B}_{1111}^2(\mu')^{-1}\pi^2 h_0^5\varepsilon^2 + O(\varepsilon^3)$$

$$\mathrm{II} = \frac{1}{2\mu}\left(1 - \frac{\lambda}{2\mu + 3\lambda}\right)\int_Q G^2\, d\underline{\eta} = \frac{32}{105}\tilde{B}_{1111}^2\frac{\mu + \lambda}{\mu(2\mu + 3\lambda)}\pi^4 h_0^7\varepsilon^2 + O(\varepsilon^3)$$

$$\mathrm{III} = \frac{\lambda}{\mu(2\mu + 3\lambda)}c(\tilde{B}_{1111} + \tilde{B}_{2211})\int_Q Gh^{-3}\eta_3\, d\underline{\eta}$$

$$= -\frac{4}{5}\tilde{B}_{1111}(\tilde{B}_{1111} + \tilde{B}_{2211})\frac{\lambda}{\mu(2\mu + 3\lambda)}\pi^2 h_0^5\varepsilon^2 + O(\varepsilon^3).$$

One computes that

$$\mathrm{I} + \mathrm{II} + \mathrm{III} = \varepsilon^2\frac{\pi^2}{15}h_0^5\tilde{B}_{1111}^2\left[\frac{12}{\mu'} + \frac{32}{7}\frac{\mu + \lambda}{\mu(2\mu + 3\lambda)}\pi^2 h_0^2 - \frac{6\lambda}{\mu(\mu + \lambda)}\right] + O(\varepsilon^3),$$

and (3.21) is exactly the condition that this be negative for sufficiently small ε. When that occurs the desired conclusion follows from Lemma 3.2. ●

We note that (3.21) can be satisfied for any fixed μ and λ by taking μ' sufficiently large and h_0 sufficiently small. But it is never satisfied when $\mu' = \mu$, the case of an isotropic material, which is consistent with Proposition 3.3.

4. A result without geometric restrictions. When the thickness function h depends on both variables η_1 and η_2 we do not expect the inequality $M^{a>1} \leqslant M^{a=1} \leqslant M^{a<1}$ to hold in general, even for an isotropic elastic law. It is true, however, in case $B_{\alpha\beta 33} = 0$; this includes an isotropic law with Poisson's ratio set equal to 0.

PROPOSITION 4.1. *If $B_{\alpha\beta 33} = 0$ then $M^{a>1} \leqslant M^{a=1} \leqslant M^{a<1}$ for every choice of the thickness function h.*

Proof. With this hypothesis on the elastic law $\Sigma_{\alpha 3}(\underline{\psi} + \underline{\Gamma}') = 0$ for all $\underline{\psi} \in V^{a<1}$. Therefore use of the minimizer of $(\mathscr{P}_{a<1})$ as a test function in $(\mathscr{P}_{a=1})$ yields $M^{a=1} \leqslant M^{a<1}$.

To obtain the other inequality, we observe that the maximizer of $(\mathscr{D}_{a>1})$ has $\Sigma_{33} = 0$ when $B_{\alpha\beta 33} = 0$—for if not, then setting Σ_{33} equal to zero would preserve admissibility and increase the value of the functional. Therefore this tensor is admissible for $(\mathscr{D}_{a=1})$ and it follows that $M^{a>1} \leqslant M^{a=1}$. ●

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