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SECOND HARMONIC RESONANCE IN ELECTROHYDRODYNAMICS*

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1. Introduction. The stability of the nonlinear waves on the surface of a perfectly conducting fluid in the presence of an applied electric field was investigated by Kant et al. [1] (hereafter referred to as Paper 1). In this presentation, we consider the effects of an electric field on second harmonic nonlinear resonant interactions. The phenomenon of second harmonic resonance occurs when the frequencies and wave numbers of two interacting waves satisfy the conditions $\omega_2 = 2\omega_1$ and $k_2 = 2k_1$. In such a case, the fundamental and the first harmonic travel with the same phase speed. This type of resonance has been studied for water waves by Wilton [2], Simmons [3], McGoldrick [4], and Nayfeh [5]. We obtain in this note the dynamical equations involving the fundamental and first harmonics. The analysis of these equations shows that for certain values of electric fields the fluid surface is unstable.

2. Derivation of dynamical equations and stability. The basic equations and the perturbation scheme are developed in Paper 1. It is shown there that the frequency ω and the wave number k satisfy the dispersion relation

$$\omega^2 = k\left(k^2 + 1 - \alpha k \coth(kb)\right)\tanh(ka), \tag{1}$$

where

$$\alpha = \Phi_c^2/4\pi b^2. \tag{2}$$

We examine the condition under which the two waves can interact resonantly. For the occurrence of such a resonant interaction, it is necessary that both (k, ω) and $(nk, n\omega)$ for some integer $n \geqslant 2$ must satisfy dispersion relation (1). The first resonant wave number k_1, corresponding to $n = 2$, is given by the equation

$$\begin{aligned}&\left(4k_1^2 + 1 - 2\alpha k_1 \operatorname{Coth} 2k_1 b\right)\tanh(2k_1 a)\\ &\quad = 2\left(k_1^2 + 1 - \alpha k_1 \operatorname{Coth}(k_1 b)\right)\tanh(k_1 a).\end{aligned} \tag{3}$$

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The resonant wave number k_1 is a function of α, a, and b, respectively. In the short wavelength approximation (i.e., $kb \gg 1$) Eq. (3) reduces to

$$\left(4k_1^2 + 1 - 2\alpha k_1\right)\tanh(2k_1 a) = 2\left(k_1^2 + 1 - \alpha k_1\right)\tanh(k_1 a). \tag{4}$$

It is interesting to note that as a approaches infinity, k_1^2 tends to 0.5 for all values of the parameter α. On solving (4), when $\alpha = 3\sqrt{2}/2$, we get $k_1^2 = 0.5$ for all a. The waves are weakly dispersive for the small values of a and $k_1 = \alpha/3$. The second harmonic resonance is not possible for the small values of kb and large values of a. The variation of second harmonic resonant wave number k_1 with respect to the electric field parameter α for various values of a is shown in Fig. 1. In this note, we assume that ω^2 is positive definite, so that (1) represents a progressive wave train.

With a view to describe the resonant interaction at or near k_1, we write

$$\eta_1 = \sum_{n=1}^{2}\left(A_n(x_1, t_1)\exp(i\theta_n) + \bar{A}_n(x_1, t_1)\exp(-i\theta_n)\right), \tag{5}$$

$$\Omega_1 = -i\sum_{n=1}^{2}\frac{\omega_n}{k_n}\frac{\cosh k_n(a+z)}{\sinh k_n a}\left(A_n\exp(i\theta_n) - \bar{A}_n\exp(-i\theta_n)\right), \tag{6}$$

$$\phi_1 = \frac{1}{b}\sum_{n=1}^{2}\frac{\sinh k_n(z-b)}{\sinh k_n b}\left(A_n\exp(i\theta_n) + \bar{A}_n\exp(-i\theta_n)\right), \tag{7}$$

where

$$\theta_n = k_n x_0 - \omega_n t_0, \tag{8}$$

$$\omega_n^2 = k_n\left(k_n^2 + 1 - \alpha k_n \coth k_n b\right)\tanh k_n a, \tag{9}$$

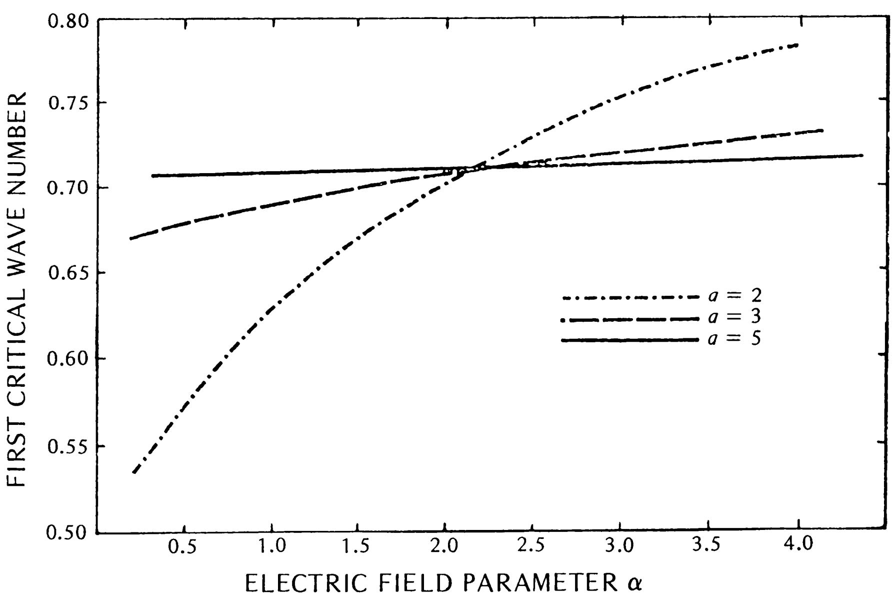

FIG. 1. The variation of second harmonic resonance wave number k_1 with α.

and

$$k_2 \simeq 2k_1, \qquad \omega_2 \simeq 2\omega_1. \tag{10}$$

The uniformly valid solutions of the second-order problem are:

$$\eta_2 = \sum_{n=1}^{2} \eta_{2n} \exp(i\theta_n) + \text{c.c.}, \tag{11}$$

$$\Omega_2 = -\sum_{n=1}^{2} \frac{\omega_n}{k_n}\left[(z+a)\frac{\sinh k_n(z+a)}{\sinh k_n a}\frac{\partial A_n}{\partial x_1} + i\frac{\cosh k_n(z+a)}{\sinh k_n a}\right]\Omega_{2n} A_n \exp i\theta_n + \text{c.c.}, \tag{12}$$

$$\phi_2 = -\frac{i}{b}\sum_{n=1}^{2}\left[(z-b)\frac{\cosh k_n(z-b)}{\sinh k_n b}\frac{\partial A_n}{\partial x_1} + i\frac{\sinh k_n(z-b)}{\sinh k_n b}\right]\phi_{2n} A_n \exp i\theta_n + \text{c.c.} \tag{13}$$

The dynamical equations for the coupled amplitudes are derived using Eqs. (5) to (13) into Eqs. (A11) to (A17) governing the second-order problem developed in Paper 1. A few algebraic manipulations allow us to obtain the following nonsecular conditions:

$$\frac{\partial A_1}{\partial t_1} + u_1\frac{\partial A_1}{\partial x_1} = iJ_1 A_2 \bar{A}_1 \exp(i\Gamma), \tag{14}$$

$$\frac{\partial A_2}{\partial t_1} + u_2\frac{\partial A_2}{\partial x_1} = iJ_2 A_1^2 \exp(-i\Gamma), \tag{15}$$

where

$$u_m = \frac{1}{2\omega_m \coth k_m a}\left[a\omega_m^2 \operatorname{cosech}^2 k_m a + 3k_m^2 + 1 - \alpha k_m\left(2\coth k_m b - k_m b \operatorname{cosech}^2 k_m b\right)\right] \quad \text{for } m = 1, 2, \tag{16}$$

$$J_1 = \frac{1}{2\omega_1 \coth(k_1 a)}\left[\omega_1 d_6 \coth(k_1 a) - \alpha b d_2 k_1^2 \coth(k_1 b) - k_1 d_4\right], \tag{17}$$

$$J_2 = \frac{1}{2\omega_2 \coth(k_2 a)}\left[\omega_2 d_5 \coth(k_2 a) - \alpha b d_1 k_2^2 \coth(k_2 b) - k_2 d_2\right], \tag{18}$$

$$d_1 = -\frac{k_1}{b}\coth(k_1 b), \tag{19}$$

$$d_2 = -\frac{1}{b}\left(k_2\coth(k_2 b) + k_1\coth(k_1 b)\right), \tag{20}$$

$$d_3 = \frac{\alpha k_1^2}{2}\left(3 - \coth^2(k_1 b)\right) + \frac{\omega_1^2}{2}\left(\coth^2(k_1 a) - 3\right), \tag{21}$$

$$d_4 = \alpha\left(k_1^2 + k_2^2 - k_1 k_2 \coth(k_1 b)\coth(k_2 b) - k_1 k_2\right) + \omega_1\omega_2\left(1 + \coth(k_1 a)\coth(k_2 a)\right) - \left(\omega_1^2 + \omega_2^2\right), \tag{22}$$

$$d_5 = -2\omega_1 k_1 \coth(k_1 a), \tag{23}$$

$$d_6 = -(k_2 - k_1)(\omega_1 \coth(k_1 a) + \omega_2(\coth k_2 a)), \tag{24}$$

and the detuning parameter

$$\Gamma = (k_2 - 2k_1)\varepsilon^{-1}x_1 - (\omega_2 - 2\omega_1)\varepsilon^{-1}t_1. \tag{25}$$

The interaction parameters J_1 and J_2 vanish for $\alpha = 3(2)^{-1/2}$ in the short wavelength approximations, implying thereby that there is no interaction between the fundamental and the first harmonic.

The general solutions of Eqs. (14)–(15) subject to arbitrary initial conditions are not yet available. We will, therefore, confine ourselves to the steady state solutions of these equations.

On letting $A_m = f_m(\S)\exp(i\mu m x_1), f_m(\S) = a_m \exp(i\psi_m)$ with $m = 1, 2$ and $\S = t_1 - \lambda x_1$, where μ, λ are constants, Eqs. (14) and (15) furnish the following conservation laws with $\Gamma = 0$:

$$\beta_1 a_1^2 + \beta_2 a_2^2 = E, \tag{26}$$

$$a_1^2(a_2 \cos\psi - \delta) = L, \tag{27}$$

where

$$\beta_m = J_m^{-1}(1 - \lambda u_m), \tag{28}$$

$$\delta = \mu\beta_1\left((J_1\beta_1)^{-1} - (J_2\beta_2)^{-1}\right), \tag{29}$$

and

$$\psi = 2\psi_1 - \psi_2. \tag{30}$$

The constant E is proportional to the energy density in the system and is independent of the relative phase illustrating the conservation of energy. Here L is a constant of integration. The motion is bounded provided $\nu = \beta_2\beta_1^{-1}$ is positive. The spatially homogeneous solutions of Eqs. (14) and (15) can be obtained by setting λ and μ equal to zero (see McGoldrick [4]). For the case $\mu = 0$, we recover the solutions by Simmons [3]. The bounded solutions of the dynamical equations can be written in terms of the Jacobian elliptic function and the elliptic integral of the third kind. The general motion in this case consists of both amplitude and phase-modulated waves. The pure phase-modulated waves, solitary waves, and the phase jump are special cases.

We now discuss the spatial variation of amplitudes and phases when ν is negative. It will be shown below that the fluid motions become unbounded in the long wavelength approximation.

We let $A_m = \frac{1}{2}a_m \exp(i\psi_m)$ with $m = 1, 2$, where a_m and ψ_m are assumed to be slowly varying functions of the slower variable x_1. Equations (14)–(15) yield

$$u_1 \frac{\partial a_1}{\partial x_1} = -\frac{1}{2} J_1 a_1 a_2 \sin\alpha_1, \tag{31}$$

$$u_2 \frac{\partial a_2}{\partial x_1} = \frac{1}{2} J_2 a_1^2 \sin\alpha_1, \tag{32}$$

$$u_1 \frac{\partial \psi_1}{\partial x_1} = \frac{1}{2} J_1 a_2 \cos \alpha_1, \tag{33}$$

$$u_2 \frac{\partial \psi_2}{\partial x_1} = \frac{1}{2} J_2 a_2 \cos \alpha_1, \tag{34}$$

where

$$\alpha_1 = \psi_2 - 2\psi_1 + \Gamma. \tag{35}$$

From Eqs. (31) and (32), we get the following integral of motion:

$$a_1^2 + \nu a_2^2 = E, \qquad \nu = \frac{u_2 J_1}{u_1 J_2}. \tag{36}$$

Equation (4) for large values of a gives $k_1^2 = 0.5$ and $k_2^2 = 2.0$. The parameter ν becomes negative if α satisfies the inequality

$$\left(\tfrac{5}{4}\right)(2)^{1/2} < \alpha < \left(\tfrac{3}{2}\right)(2)^{1/2}. \tag{37}$$

When ν is negative, no conclusions regarding stability can be drawn from (36). In order to discuss stability, we combine Eqs. (33)–34) and obtain

$$\frac{d\alpha_1}{dx_1} = \sigma + \left(\frac{J_2 a_1^2}{2u_2 a_2} - \frac{J_1 a_2}{u_1} \right) \cos \alpha_1, \tag{38}$$

where

$$\sigma = (k_2 - 2k_1)\varepsilon^{-1}. \tag{39}$$

Equations (33), (36), and (38), on simplification yield

$$a_1^2 a_2 \cos \alpha_1 + \frac{\sigma a_2 u_2}{J_2} = L \quad \text{if } L \neq 0. \tag{40}$$

From (31), (38), and (40), we get

$$\left(\frac{d\chi}{dx_1} \right)^2 = \frac{J_2^2}{u_2^2} \left[\chi(1 - \nu\chi)^2 E - \left(\frac{L}{E} - \frac{\sigma u_2}{J_2} \chi \right)^2 \right] = G(\chi), \tag{41}$$

where

$$\chi = a_2^2 / E. \tag{42}$$

The stability of the fluid surface is dependent upon the roots of the algebraic equation $G(\chi) = 0$. If all the roots of the above equations are real, the surface is bounded and χ oscillates periodically between two positive roots when $G(x) > 0$ (see Nayfeh and Mook [6]). In this case the motion can be expressed in terms of the Jacobian elliptic function. If $G(\chi)$ has complex roots, the surface becomes unstable with increasing x_1.

To obtain a criterion of the instability, we take the initial conditions

$$a_1^2 = E, \quad a_2 = 0 \quad \text{at } x = x_1 = 0. \tag{43}$$

Equations (40) and (41) now give $L = 0$ and

$$(d\chi/dx_1)^2 = R\chi\left[(1 - \nu\chi)^2 - 2\Omega\chi\right], \tag{44}$$

where

$$R = EJ_2^2/u_2^2, \qquad \Omega = \sigma^2 u_2^2/2EJ_2^2. \tag{45}$$

The motion is stable only if

$$(I - \nu\chi)^2 - 2\Omega\chi = 0 \tag{46}$$

has positive real roots. This condition is satisfied for $\Omega > 2|\nu|$ if $\nu < 0$ and for all Ω if $\nu > 0$. The interface is unstable and the amplitude grows spatially if

$$\sigma^2 < 4|\nu|EJ_2^2/u_2^2. \tag{47}$$

Here σ is the detuning parameter. The fluid surface is always unstable for the case of perfect resonance (i.e., $\sigma = 0$). Thus there exist certain values of the electric fields for which the fluid motion becomes unbounded and hence results in an instability at the second harmonic resonance.

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