QUARTERLY OF APPLIED MATHEMATICS
VOLUME XLIII, NUMBER 2
JULY 1985, PAGES 179-185

A NOTE ON LANGFORD'S CYLINDER FUNCTIONS $c_n(z, z_0)$ AND $e_n(z, z_0)$*

By

JAMES M. HILL AND JEFFREY N. DEWYNNE

The University of Wollongong, Australia

Abstract. New general expressions are given for Langford's cylinder functions, which occur in solutions of the Cauchy problem for the heat equation in cylindrical co-ordinates. These formulae are deduced by means of generating functions. In addition a new technique is used to obtain Langford's formal series, new basic formulae connecting Langford's various cylinder functions are established and their relevance in a formal series solution of a moving boundary problem is noted.

1. Introduction. Langford [4] gives general solutions of the Cauchy problem for a one dimensional heat equation in planar, spherical or cylindrical co-ordinates. Given arbitrary, analytic temperature and heat flux functions prescribed on a fixed plane, spherical or cylindrical surface formal series solutions are given in terms of these functions. These solutions are general in the sense that particular choices of the prescribed temperature and heat flux functions yield other known solutions of the heat equation. For example, the classical Fourier and Bessel series solutions of the heat equation may be generated in this manner (see [4]). The results for both planar and spherical geometries are relatively simple, and we refer the reader to [4] for these expressions. However the corresponding expressions for the cylinder are, as usual, awkward and Langford [4] gives only the first few terms of the series explictly, indicating a complex process by which the remaining terms may be calculated. Langford's functions are fundamental to the solution of the heat equation with cylindrical geometry and the purpose of this note is to give new general formulae for these functions, which we obtain from generating functions. In addition we give an alternative derivation of Langford's solutions, and we show that Langford's functions also arise in the formal series solution of the classical Stefan problem in a cylinder. An entirely analogous derivation is applicable to planar and spherical geometries.

Langford [4] states that the solution $T(r, t)$ of the problem

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}, \tag{1.1}$$

* Received January 4, 1984.

$$T(r_0, t) = A(t), \qquad \lim_{r \to r_0}\left(r\frac{\partial T}{\partial r}(r, t)\right) = B(t), \tag{1.2}$$

where r_0 is an arbitrary, non-zero, fixed radius, and $A(t)$ and $B(t)$ are analytic functions of time, is given by

$$T(r, t) = \sum_{n=0}^{\infty}\left\{A^{(n)}(t)c_n\left(\frac{r^2}{4}, \frac{r_0^2}{4}\right) + \frac{1}{2}B^{(n)}(t)e_n\left(\frac{r^2}{4}, \frac{r_0^2}{4}\right)\right\}. \tag{1.3}$$

Here $A^{(n)}(t)$ and $B^{(n)}(t)$ denote the nth derivatives of the functions $A(t)$ and $B(t)$ respectively. With the notation $z = r^2/4$ and $z_0 = r_0^2/4$ the first four of the functions $c_n(z, z_0)$ and $e_n(z, z_0)$ are given by

$$\begin{aligned} c_0(z, z_0) &= 1, \qquad c_1(z, z_0) = (z - z_0) - z_0 \log\frac{z}{z_0}, \\ c_2(z, z_0) &= \frac{(z - z_0)}{4}(z + 5z_0) - \frac{z_0}{2}(2z + z_0)\log\frac{z}{z_0}, \\ c_3(z, z_0) &= \frac{(z - z_0)}{36}(z^2 + 19zz_0 + 10z_0^2) - \frac{z_0}{12}(3z^2 + 6zz_0 + z_0^2)\log\frac{z}{z_0}, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} e_0(z, z_0) &= \log\frac{z}{z_0}, \qquad e_1(z, z_0) = 2(z_0 - z) + (z + z_0)\log\frac{z}{z_0}, \\ e_2(z, z_0) &= \frac{3}{4}(z_0 - z)(z + z_0) + \frac{1}{4}(z^2 + 4zz_0 + z_0^2)\log\frac{z}{z_0}, \\ e_3(z, z_0) &= \frac{(z_0 - z)}{108}(11z^2 + 38zz_0 + 11z_0^2) + \frac{(z + z_0)}{36}(z^2 + 8zz_0 + z_0^2)\log\frac{z}{z_0}. \end{aligned} \tag{1.5}$$

In the following sections we give a new derivation of (1.3) and obtain general expressions for $c_n(z, z_0)$ and $e_n(z, z_0)$. In the final section we show the moving boundary problem

$$\begin{gathered} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}, \qquad R(t) < r < 1, \\ T(1, t) + \beta\frac{\partial T}{\partial r}(1, t) = 1, \qquad T(R(t), t) = 0, \\ \frac{\partial T}{\partial r}(R(t), t) = -\alpha\frac{dR}{dt}(t), \qquad R(0) = 1, \end{gathered} \tag{1.6}$$

has formal series solution

$$T(r, t) = \alpha\sum_{n=0}^{\infty}\frac{\partial^n}{\partial t^n}c_n\left(\frac{r^2}{4}, \frac{R(t)^2}{4}\right), \tag{1.7}$$

where $R(t)$ denotes the unknown moving boundary, and α is a strictly positive constant, β a non-negative constant. This formal result follows from an integral formulation of the problem given in a recent paper Dewynne and Hill [1] and the following basic relations for $c_n(z, z_0)$ and $e_n(z, z_0)$,

$$c_n(z, z_0) = \int_{z_0}^{z}\log\frac{z}{\omega}c_{n-1}(\omega, z_0)\,d\omega, \tag{1.8}$$

$$e_n(z, z_0) = \int_{z_0}^{z} \log\frac{z}{\omega} e_{n-1}(\omega, z_0)\, d\omega, \tag{1.9}$$

$$c_n(z, z_0) = \int_{z_0}^{z} e_{n-1}(z, \omega)\, d\omega, \tag{1.10}$$

which we establish in Sec. 2. In a practical context the formal series solution (1.7) of the moving boundary problem (1.6) has limited utility. However our results serve more to identify the essential mathematical structure, associated with a cylindrical problem, which is always far more complicated than the corresponding planar or spherical problem.

2. An alternative derivation of Langford's solutions. By multiplying (1.1) throughout by r, integrating twice with respect to r and changing the order of integration, we may deduce that

$$T(r, t) = A(t) + B(t)\log\frac{r}{r_0} + \int_{r_0}^{r} \xi \log\frac{r}{\xi} \frac{\partial T}{\partial t}(\xi, t)\, d\xi, \tag{2.1}$$

where (1.2) has been used for the prescribed temperature and flux at $r = r_0$. Substituting this expression for $T(r, t)$ into the right hand side of equation (2.1) we obtain

$$T(r, t) = A(t) + B(t)\log\frac{r}{r_0} + A^{(1)}(t)\int_{r_0}^{r} \xi\log\frac{r}{\xi} d\xi + B^{(1)}(t)\int_{r_0}^{r} \xi\log\frac{r}{\xi}\log\frac{\xi}{r_0} d\xi$$

$$+ \int_{r_0}^{r}\int_{r_0}^{\xi} \xi\eta \log\frac{r}{\xi}\log\frac{\xi}{\eta}\frac{\partial^2 T}{\partial t^2}(\eta, t)\, d\eta\, d\xi. \tag{2.2}$$

Repeating this process indefinitely, and assuming that the remainder tends to zero, leads to

$$T(r, t) = \sum_{n=0}^{\infty}\left\{A^{(n)}(t)c_n^*(r, r_0) + \frac{1}{2}B^{(n)}(t)e_n^*(r, r_0)\right\}, \tag{2.3}$$

where $c_n^*(r, r_0)$ and $e_n^*(r, r_0)$ are given by

$$c_n^*(r, r_0) = \int_{r_0}^{r} \xi\log\frac{r}{\xi} c_{n-1}^*(\xi, r_0)\, d\xi, \qquad c_0^*(r, r_0) = 1,$$

$$e_n^*(r, r_0) = \int_{r_0}^{r} \xi\log\frac{r}{\xi} e_{n-1}^*(\xi, r_0)\, d\xi, \qquad e_0^*(r, r_0) = 2\log\frac{r}{r_0}. \tag{2.4}$$

Assuming that both $r > 0$ and $r_0 > 0$, it is easily shown inductively that

$$|c_n^*(r, r_0)| \leqslant \frac{M_1^n}{n!}|r - r_0|^n, \qquad |e_n^*(r, r_0)| \leqslant M_2\frac{M_1^n}{n!}|r - r_0|^n, \tag{2.5}$$

where

$$M_1 = \sup\left\{\left|\xi\log\frac{r}{\xi}\right| : \xi \in [r_0, r]\right\}, \qquad M_2 = \sup\left\{\left|\log\frac{\xi}{r_0}\right| : \xi \in [r_0, r]\right\}. \tag{2.6}$$

Observing that $M_1 \to 0$ as $r \to r_0$, it follows that we may find some $r_1 \neq r_0$ such that $M_1|r_1 - r_0|$ equals the smaller of the radii of convergence of the Taylor series expansions of $A(t)$ and $B(t)$ about t. Then clearly for $r \in [r_0, r_1)$ the series (2.3) converges absolutely, and term-wise operations may be used to show that it does indeed represent a solution of (1.1)–(1.2).

If we let

$$c_n(z, z_0) = c_n^*(r, r_0), \qquad e_n(z, z_0) = e_n^*(r, r_0), \tag{2.7}$$

where as before $z = r^2/4$, $z_0 = r_0^2/4$, then from (2.3) and (2.4) with $\omega = \xi^2/4$ we have

$$\begin{gathered} T(r, t) = \sum_{n=0}^{\infty} \left\{ A^{(n)}(t) c_n(z, z_0) + \frac{1}{2} B^n(t) e_n(z, z_0) \right\}, \\ c_n(z, z_0) = \int_{z_0}^{z} \log \frac{z}{\omega} c_{n-1}(\omega, z_0)\, d\omega, \qquad c_0(z, z_0) = 1, \\ e_n(z, z_0) = \int_{z_0}^{z} \log \frac{z}{\omega} e_{n-1}(\omega, z_0)\, d\omega, \qquad e_0(z, z_0) = \log \frac{z}{z_0}. \end{gathered} \tag{2.8}$$

Thus (1.8) and (1.9) are established. To prove (1.10) we proceed by induction. The case $n = 1$ is trivially true, so assuming the result for some $n > 1$ we have

$$\begin{aligned} \int_{z_0}^{z} e_n(z, \omega)\, d\omega &= \int_{z_0}^{z} \int_{\omega}^{z} \log \frac{z}{\rho} e_{n-1}(\rho, \omega)\, d\rho\, d\omega \\ &= \int_{z_0}^{z} \int_{z_0}^{\rho} \log \frac{z}{\rho} e_{n-1}(\rho, \omega)\, d\omega\, d\rho \\ &= \int_{z_0}^{z} \log \frac{z}{\rho} c_n(\rho, z_0)\, d\rho = c_{n+1}(z, z_0), \end{aligned} \tag{2.9}$$

establishing (1.10).

3. New expressions for Langford's functions. We now proceed to deduce explicit expressions for $c_n(z, z_0)$ and $e_n(z, z_0)$. From (2.8) or Langford [4], it may be verified that both sequences (c_n) and (e_n) satisfy the recursion relations,

$$\left.\begin{aligned} \frac{\partial}{\partial z}\left(z \frac{\partial f_n}{\partial z}(z, z_0) \right) &= f_{n-1}(z, z_0), \\ f_n(z_0, z_0) = \frac{\partial f_n}{\partial z}(z_0, z_0) &= 0, \end{aligned}\right. \qquad (n \geqslant 1) \tag{3.1}$$

with f_0 being taken as 1 or $\log(z/z_0)$ respectively. Since the first function in the sequence, f_0 is an homogeneous function of degree 0, it follows inductively that the nth function in the sequence, f_n, is homogeneous of degree n. Thus we define a new sequence (g_n) such that

$$f_n(z, z_0) = z_0^n g_n\left(\frac{z}{z_0} \right) \qquad (n \geqslant 0), \tag{3.2}$$

and from (3.1) it is apparent that the sequence (g_n) is determined recursively by

$$\left.\begin{aligned} &\xi g_n''(\xi) + g_n'(\xi) = g_{n-1}(\xi), \\ &g_n(1) = g_n'(1) = 0, \end{aligned}\right. \qquad (n \geqslant 1) \tag{3.3}$$

where primes denote differentiation with respect to the argument $\xi = z/z_0$. Thus if we define a generating function by

$$G(\xi, s) = \sum_{n=0}^{\infty} g_n(\xi) s^n, \tag{3.4}$$

since in both the cases $g_0(\xi) = 1$ and $g_0(\xi) = \log \xi$ (corresponding to $c_n(z, z_0)$ and $e_n(z, z_0)$ respectively), the relation

$$\xi g_0''(\xi) + g_0'(\xi) = 0, \tag{3.5}$$

is found to hold, we find from $(3.3)_1$ and (3.4) that

$$\xi \frac{\partial^2 G}{\partial \xi^2} + \frac{\partial G}{\partial \xi} - sG = 0.$$

If we let $C(\xi, s)$ be the generating function corresponding to the sequence of $c_n(z, z_0)$, we may deduce the following initial value problem,

$$\begin{gathered} \xi \frac{\partial^2 C}{\partial \xi^2} + \frac{\partial C}{\partial \xi} - sC = 0, \\ C(1,s) = 1, \qquad \frac{\partial C}{\partial \xi}(1, s) = 0, \end{gathered} \tag{3.6}$$

and it is not difficult to establish that the appropriate solution is

$$C(\xi, s) = 2\sqrt{s}\left\{\mathbf{I}_0(2\sqrt{s\xi})\mathbf{K}_1(2\sqrt{s}) + \mathbf{K}_0(2\sqrt{s\xi})\mathbf{I}_1(2\sqrt{s})\right\}, \tag{3.7}$$

where $\mathbf{I}_0$, $\mathbf{I}_1$, $\mathbf{K}_0$ and $\mathbf{K}_1$ denote the usual modified Bessel functions of the first and second kind. Similarly, letting $E(\xi, s)$ denote the generating function for the functions $e_n(z, z_0)$, we have

$$\begin{gathered} \xi \frac{\partial^2 E}{\partial \xi^2} + \frac{\partial E}{\partial \xi} - sE = 0, \\ E(1, s) = 0, \qquad \frac{\partial E}{\partial \xi}(1, s) = 1, \end{gathered} \tag{3.8}$$

which has the solution

$$E(\xi, s) = 2\left\{\mathbf{I}_0(2\sqrt{s\xi})\mathbf{K}_0(2\sqrt{s}) - \mathbf{K}_0(2\sqrt{s\xi})\mathbf{I}_0(2\sqrt{s})\right\}. \tag{3.9}$$

Expanding the Bessel functions as power series,

$$\begin{aligned} \mathbf{I}_0(x) &= \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\left(\frac{x}{2}\right)^{2n}, \qquad \mathbf{I}_1(x) = \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1}, \\ \mathbf{K}_0(x) &= -\left(\gamma + \log\frac{x}{2}\right)\mathbf{I}_0(x) \\ &\quad + \sum_{n=0}^{\infty}\left\{1 + \frac{1}{2} + \cdots + \frac{1}{n}\right\}\frac{1}{(n!)^2}\left(\frac{x}{2}\right)^{2n}, \\ \mathbf{K}_1(x) &= \left(\gamma + \log\frac{x}{2}\right)\mathbf{I}_1(x) + \frac{1}{x} \\ &\quad - \sum_{n=0}^{\infty}\left[\left\{1 + \frac{1}{2} + \cdots + \frac{1}{n}\right\} + \frac{1}{2(n+1)}\right]\frac{1}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1}, \end{aligned} \tag{3.10}$$

where $\gamma = 0.5772\ldots$ denotes Euler's constant, and as usual $\{1 + \frac{1}{2} + \cdots + \frac{1}{n}\}$ is taken to be zero when $n = 0$, we may expand (3.7) and (3.9) to deduce the following expressions for $c_n(z, z_0)$ and $e_n(z, z_0)$

$$\frac{c_n(z, z_0)}{z_0^n} = \frac{2}{[(n-1)!]^2} \sum_{j=0}^{n-1} \binom{n-1}{j}^2 \left\{1 + \frac{1}{2} + \cdots + \frac{1}{n-j-1}\right\} \cdot \left\{\frac{\xi^{n-j-1}}{(j+1)} - \frac{\xi^j}{(n-j)}\right\} + \frac{\xi^n}{(n!)^2} - \frac{1}{(n!)^2} \sum_{j=0}^{n-1} \binom{n}{j}^2 \xi^j (1 + (n-j)\log \xi), \tag{3.11}$$

$$\frac{e_n(z, z_0)}{z_0^n} = \frac{2}{(n!)^2} \sum_{j=0}^{n} \binom{n}{j}^2 \left\{1 + \frac{1}{2} + \cdots + \frac{1}{j}\right\} (\xi^{n-j} - \xi^j) + \frac{1}{(n!)^2} \sum_{j=0}^{n} \binom{n}{j}^2 \xi^{n-j} \log \xi,$$

where, as before, $\xi = z/z_0$. The reader may confirm that the first four functions given in (1.4)–(1.5) arise from the above formulae.

4. Formal moving boundary problem solution. Dewynne and Hill [1] show that the solution of (1.6) satisfies the integro-differential equation

$$T(r, t) = \frac{\partial}{\partial t} \int_{R(t)}^{r} \xi(\log r - \log \xi)[\alpha + T(\xi, t)]\, d\xi, \tag{4.1}$$

while the motion of the moving boundary is determined from

$$t = \int_{R(t)}^{1} \xi(\beta - \log \xi)[\alpha + T(\xi, t)]\, d\xi. \tag{4.2}$$

Proceeding as in Sec. 2, repeated application of (4.1) gives

$$T(r, t) = \alpha \frac{\partial}{\partial t} \int_{R(t)}^{r} \xi(\log r - \log \xi)\, d\xi + \frac{\partial^2}{\partial t^2} \int_{R(t)}^{r} \frac{\xi}{4}[(\xi^2 - r^2) + (\xi^2 + r^2)(\log r - \log \xi)][\alpha + T(\xi, t)]\, d\xi, \tag{4.3}$$

and on making the transformations

$$z = \frac{r^2}{4}, \quad Z(t) = \frac{R(t)^2}{4}, \quad \omega = \frac{\xi^2}{4}, \tag{4.4}$$

we find that (4.3) becomes

$$T(r, t) = \alpha \frac{\partial}{\partial t} \int_{Z(t)}^{z} e_0(z, \omega)\, d\omega + \alpha \frac{\partial^2}{\partial t^2} \int_{Z(t)}^{z} e_1(z, \omega)\, d\omega + \frac{\partial^2}{\partial t^2} \int_{Z(t)}^{z} e_1(z, \omega) T(2\sqrt{\omega}, t)\, d\omega. \tag{4.5}$$

Continuing this process, and again assuming that the remainder term containing the temperature function $T(r, t)$ tends to zero, we obtain the formal series solution

$$T(r,t) = \alpha \sum_{n=1}^{\infty} \frac{\partial^n}{\partial t^n} \int_{Z(t)}^{z} e_{n-1}(z, \omega)\, d\omega, \tag{4.6}$$

and (1.7) follows immediately on application of (1.10). We remark that (1.7) or (4.6) can be verified independently by direct substitution into either (1.6), or the integral formulation (4.1), assuming of course that the series is convergent. We note also from the formal series solution and either the surface condition of (1.6), or Eq. (4.2), we may deduce various formal eqautions for the motion of the boundary. These representations lead to unusual integral equations for $R(t)$, similar to those given by Grinberg and Chekmarera [2] for the planar geometry and Hill [3] for all three geometries. In particular the integral equation for the cylindrical geometry may be found in Hill [3]. Since these integral equations cannot be presently solved, we do not pursue the matter further.

REFERENCES

[1] J. N. Dewynne and J. M. Hill, *On an integral formulation for moving boundary problems*, Quart. Appl. Math. **41** (1984), 443–456

[2] G. A. Grinberg and O. M. Chekmarera, *Motion of the phase interface in the Stefan problem*, Sov. Phys. Tech. Phys. **15** (1971), 1579–1583

[3] J. M. Hill, *Some unusual unsolved integral equations for moving boundary diffusion problems*, Math. Scientist **9** (1984), 15–23

[4] D. Langford, *New analytic solutions of the one dimensional heat equation for temperature and heat flow rate both prescribed at the same fixed boundary (with applications to the phase change problem)*, Quart. Appl. Math. **24** (1967), 315–322