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MODULATIONAL INSTABILITY IN MAGNETIC FLUIDS*

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Abstract. A weakly nonlinear evolution of two dimensional wave packets on the surface of a magnetic fluid in the presence of a tangential magnetic field is presented taking into account the surface tension. It is shown that the magnetic field has a stabilizing influence on the modulational instability.

1. Introduction. The propagation of plane waves in ferro-fluids in the presence of a tangential magnetic field has been investigated theoretically as well as experimentally by Zelazo and Melcher [1]. These authors have demonstrated that the magnetic field has a stabilizing influence on the stability of the fluid surface. In their experiment, a plane wave of specific wavelength, consistent with the boundary conditions, was imposed on the interface, and the subsequent frequency shift for various strengths of the magnetic field was measured. Both theoretical and experimental results show an upward shift of frequency of the imposed wavelength as a function of the magnetic fluid. This, however, is in constrast when the magnetic field is normal to the interface where beyond a certain critical magnetic field strength, Cowley and Rosensweig [2] report the existence of an instability leading to the appearance of the regular hexagonal cells.

The aim of the present paper is to study the nonlinear propagation of wave packets on the surface of a magentic fluid. In hydrodynamics, this classical problem has received considerable attention (see Lighthill [3], Benjamin and Feir [4], Whitham [5]). The analysis carried out by Benjamin and Feir [4] reveals that a uniform wavetrain of weakly nonlinear dispersive waves is unstable against the side band perturbation. Such an instability is confined to the long wave modulations, and possesses a much higher cutoff waveneumber. These findings generated considerable efforts by various authors [5, 6, 7] towards obtaining the system of equations governing the amplitude evolution of the unstable wavetrain. Hasimoto and Ono [7] derived the nonlinear Schrödinger equation for the evolution of the finite amplitude gravity wave packets on the fluid surface with the use of the derivative expansion method, and succeeded in recovering the results reported earlier [4].

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In this presentation, we plan to develop the nonlinear Schrödinger equation describing the evolution of the finite amplitude wave packet on the surface of a ferro-fluid in the presence of a uniform tangential magnetic field. The basic equations with the accompanying boundary conditions are given in Sec. 2. The first order theory and the linear dispersion relation are obtained in Sec. 3. In Sec. 4, we have derived the second order solutions and the nonlinear Schrödinger equation governing the amplitude modulation. It is well known that the solution of this equation can be represented in terms of Jacobian elliptic functions. The solitary wave, phase jump, and the progressive wave of constant amplitude are just the special cases. The wavetrain solution of constant amplitude are unstable against modulation if the product of the group velocity rate and the nonlinear interaction parameter is negative. We have shown in this paper that the wavenumber at which the modulational instability sets in is highly sensitive to the magnetic field strength. Furthermore, we have established that the magnetic field has a stabilizing influence on the modulation instability.

2. Basic equations. We consider two dimensional wave motion of an inviscid, incompressible, magnetic fluid with density ρ and magentic permeability μ_1. The fluid is occupying the half space $z < 0$, and the medium $z > 0$ is of magnetic permeability μ_2 but of negligible density. There is a magnetic field $\underline{H}(H, 0, 0)$ along the fluid interface. The motion under gravity $\underline{g}(0, 0, -g)$ is assumed to be irrotational. The basic equations which govern the velocity potential $\phi(\underline{v} = \underline{\nabla}\phi)$ and the magnetic potential $\psi(\underline{H} = -\underline{\nabla}\psi)$ are:

$$\nabla^2\phi = 0, \tag{1}$$

$$\nabla^2\psi^{(j)} = 0 \qquad (j = 1, 2). \tag{2}$$

The boundary conditions at the free interface $z = \eta(x, t)$ are (See Malik and Singh [8]) given by

$$\frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial z} = -\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x}, \tag{3}$$

$$\mu H_n^{(1)} = H_n^{(2)}, \tag{4}$$

$$H_T^{(1)} = H_T^{(2)}, \tag{5}$$

$$\begin{aligned} &\frac{\partial\phi}{\partial t} + g\eta + \frac{1}{2}(\nabla\phi)^2 - \frac{T}{\rho}\frac{\partial^2\eta}{\partial x^2}\left(1 + \left(\frac{\partial\eta}{\partial x}\right)^2\right)^{-3/2} \\ &= \frac{(\mu - 1)}{8\pi\rho}\left[H_T^{(1)} + \mu H_n^{(2)}\right] \end{aligned} \tag{6}$$

where $\mu = \mu_1/\mu_2$. Here T, η, H_n and H_T repesent the surface tension, the elevation of the free surface, the normal and tangential components of the magnetic field, respectively.

To investigate the modulation of a weakly nonlinear quasi-monochromatic wave with narrow band width spectrum, we employ the method of multiple scales by introducing the variables

$$x_n = \varepsilon^n x \quad \text{and} \quad t_n = \varepsilon^n t \qquad (n = 0, 1, 2, 3),$$

and letting

$$\eta(x,t) = \sum_{n=1}^{3} \varepsilon^n \eta_n(x_0, x_1, x_2; t_0, t_1, t_2) + O(\varepsilon^4), \tag{7}$$

$$\phi(x,t) = \sum_{n=1}^{3} \varepsilon^n \phi_n(x_0, x_1, x_2; t_0, t_1, t_2) + O(\varepsilon^4), \tag{8}$$

$$\psi^{(j)}(x,t) = \sum_{n=1}^{3} \varepsilon^n \psi_n^{(j)}(x_0, x_1, x_2; t_0, t_1, t_2) + O(\varepsilon^4), \tag{9}$$

where the small parameter ε characterises the steepness ratio of the wave. The short scale x_0 and the fast scale t_0 denote, respectively, the wavelength and the frequency of the wave. Here, t_1 and t_2 represent the slow temporal scales of the phase and the amplitude, respectively, whereas the long scales x_1, x_2 stand for the spatial modulations of the phase and the amplitude (see Nayfeh [9]). The expansions (7) to (9) are assumed to be uniformly valid for $-\infty < x < \infty$ and $0 < t < \infty$. The quantities appearing in the field Eqs. (1) to (2) and the boundary conditions (4) to (6) can now be expressed in Maclaurin Series expansions around $z = 0$. Then, we use Eqs. (7) to (9), and equate the coefficients of equal powers in ε to obtain the linear and the successive nonlinear partial differential equations of various orders.

3. Linear theory. Since there is no steady flow in the unperturbed state, the linear progressive wave solutions of the Eqs. (1) to (2) subject to the boundary conditions (3) to (6) yield

$$\eta_1 = A(x_1, x_2; t_1, t_2)\exp(i\psi) + \bar{A}(x_1, x_2; t_1, t_2)\exp(-i\psi), \tag{10}$$

$$\phi_1 = -i\omega k^{-1}\left[A(x_1, x_2; t_1, t_2)\exp(i\psi) - \bar{A}(x_1, x_2, t_1, t_2)\exp(-i\psi)\right] \cdot \exp(kz), \tag{11}$$

$$\psi_1^{(1)} = iB\left[A(x_1, x_2; t_1, t_2)\exp(i\psi) - \bar{A}(x_1, x_2; t_1, t_2)\exp(-i\psi)\right] \cdot \exp(kz), \tag{12}$$

$$\psi_1^{(2)} = iB\left[A(x_1, x_2; t_1, t_2)\exp(i\psi) - \bar{A}(x_1, x_2; t_1, t_2)\exp(-i\psi)\right] \cdot \exp(-kz), \tag{13}$$

where

$$\psi = kx_0 - \omega t_0, \qquad B = H(1-\mu)/(1+\mu). \tag{14}$$

Here, $\bar{A}$ denotes the complex conjugate of the amplitude A, and k, ω stand for the wavenumber and the frequency of the centre of the wave packet, respectively. The progressive wave solutiosn (10)–(14) lead to the dispersion relation:

$$\omega^2 = gk + \frac{T}{\rho}k^3 + V^2k^2, \tag{15}$$

where

$$V^2 = (\mu - 1)^2 H^2/4\pi(\mu + 1)\rho. \tag{16}$$

From the dispersion relation (15), $\omega^2 > 0$, implying that the tangential magentic field has a stabilizing influence on the wave motion. These theoretical results were obtained and confirmed experimentally by Zelazo and Melcher [1].

4. Amplitude modulation of traveling waves. Since our aim is to study the amplitude modulation when $\omega^2 > 0$, we now proceed to the second order problem in $O(\varepsilon^2)$. With the use of the first order solutions given by the equations (10) to (14), we obtain the equations for the second order problem:

$$\frac{\partial^2\phi_2}{\partial z^2} + \frac{\partial^2\phi_2}{\partial x_0^2} = -2\omega\left(\frac{\partial A}{\partial x_1}\right)\exp(i\theta + kz) + \text{c.c.}, \tag{17}$$

$$\frac{\partial^2\psi_2^{(1)}}{\partial z^2} + \frac{\partial^2\psi_2^{(1)}}{\partial x_0^2} = 2kB\left(\frac{\partial A}{\partial x_1}\right)\exp(i\theta + kz) + \text{c.c.}, \tag{18}$$

$$\frac{\partial^2\psi_2^{(2)}}{\partial z^2} + \frac{\partial^2\psi_2^{(2)}}{\partial x_0^2} = 2kB\left(\frac{\partial A}{\partial x_1}\right)\exp(i\theta - kz) + \text{c.c.}, \tag{19}$$

and the boundary conditions at $z = 0$:

$$\frac{\partial\eta_2}{\partial t_0} - \frac{\partial\phi_2}{\partial z} = -\frac{\partial A}{\partial t_1}\exp(i\theta) - 2i\omega kA^2\exp(2i\theta) + \text{c.c.}, \tag{20}$$

$$\left(\mu\frac{\partial\psi_2^{(1)}}{\partial z} - \frac{\partial\psi_2^{(2)}}{\partial z}\right) - H(1-\mu)\frac{\partial\eta_2}{\partial x_0} = H(1-\mu)\frac{\partial A}{\partial x_1}\exp(i\theta)$$
$$-2ik^2B(\mu - 1)A^2\exp(2i\theta) + \text{c.c.}, \tag{21}$$

$$\frac{\partial\psi_2^{(1)}}{\partial x_0} - \frac{\partial\psi_2^{(2)}}{\partial x_0} = 4Bk^2A^2\exp(2i\theta) + \text{c.c.}, \tag{22}$$

$$\rho\frac{\partial\phi_2}{\partial t_0} + \rho g\eta_2 - T\frac{\partial^2\eta_2}{\partial x_0^2} + \frac{H(\mu-1)}{4\pi}\frac{\partial\psi_2^{(1)}}{\partial x_0}$$
$$= i\left[2kT\frac{\partial A}{\partial x_1} + \rho\frac{\omega}{k}\frac{\partial A}{\partial t_1} - \frac{HB}{4\pi}(\mu - 1)\frac{\partial A}{\partial x_1}\right]\exp(i\theta)$$
$$+\left[\omega^2\rho - k^2V^2\frac{(\mu+3)}{(1+\mu)}\right]A^2\exp(2i\theta) + \text{c.c.}. \tag{23}$$

The non secularity conditions for the existence of the uniformly valid solution are

$$2\omega k^{-1}\left(\frac{\partial A}{\partial t_1}\right) + \left(2k\frac{T}{\rho} + \frac{\omega^2}{k^2} + V^2\right)\frac{\partial A}{\partial x_1} = 0, \tag{24}$$

and its complex conjugate relation. The group velocity of the wave is given by

$$V_g = \frac{d\omega}{dk} = \frac{k}{2\omega}\left(2k\frac{T}{\rho} + \frac{\omega^2}{k^2} + V^2\right). \tag{25}$$

The equation (24) shows that in the second order theory, the amplitude A is constant in a frame of reference moving with the group velocity V_g of the waves.

The Eqs. (17) to (25) furnish the second order solutions:

$$\eta_2 = \Lambda A^2 \exp(2i\theta) + \text{c.c.}, \tag{26}$$

$$\phi_2 = \frac{1}{k}\left[\frac{\partial A}{\partial t_1} + \frac{\omega}{k}(1 - zk)\frac{\partial A}{\partial x_1}\right]\exp(i\theta + kz) - i\frac{\omega}{k}(\Lambda - k)A^2\exp(2i\theta + 2kz) + \text{c.c.}, \tag{27}$$

$$\psi_2^{(1)} = \frac{i}{k}B(1 - zk)\frac{\partial A}{\partial x_1}\exp(i\theta + kz) + iB(\Lambda - k)A^2\exp(2i\theta + 2kz) + \text{c.c.}, \tag{28}$$

$$\psi_2^{(2)} = \frac{i}{k}B(1 + zk)\frac{\partial A}{\partial x_1}\exp(i\theta - kz) + iB(\Lambda + k)A^2\exp(2i\theta - 2kz) + \text{c.c.}, \tag{29}$$

where

$$\Lambda = -\omega^2 + V^2k^2(\mu - 1)/(1 + \mu)/(2k^2T/\rho - g) \tag{30}$$

Furthermore, we assume that $k \neq (g\rho/2T)^{1/2}$. The case $k = (g\rho/2T)^{1/2}$ corresponds to the case of the second harmonic resonance. We now proceed to the third order problem in $O(\varepsilon^3)$. On using the first and the second order solutions in the equations for the third order theory, we deduce the following condition for the perturbation η_3 to be non secular:

$$i\left[\frac{2\omega}{k}\frac{\partial A}{\partial t_2} + \left(\frac{\omega^2}{k^2} + V^2 + \frac{2kT}{\rho}\right)\frac{\partial A}{\partial x_2}\right] - \frac{1}{k}\frac{\partial^2 A}{\partial t_1^2} + \left(T - \frac{\omega^3}{k^3} + \frac{V^2}{k^2}\right)\frac{\partial^2 A}{\partial x_1^2} - \frac{2\omega}{k^2}\frac{\partial^2 A}{\partial x_1 \partial t_1} = \left[2\Lambda\left\{\omega^2 - k^2V^2\left(\frac{\mu - 1}{\mu + 1}\right)\right\} + 2k(\omega^2 - k^2V^2) - \frac{3}{2}Tk^4\right]A^2\bar{A}. \tag{31}$$

On substituting (24) into (31), we get the nonlinear Schrödinger equation:

$$i\left(\frac{\partial A}{\partial t_2} + V_g\frac{\partial A}{\partial x_2}\right) + P\frac{\partial^2 A}{\partial x_1^2} = QA^2\bar{A}, \tag{32}$$

where

$$P = \tfrac{1}{2}dV_g/dk, \tag{33}$$

and

$$Q = \frac{k}{4\omega}\left[4\Lambda\left\{\omega^2 - k^2V^2\left(\frac{\mu - 1}{\mu + 1}\right)\right\} + 4k\{\omega^2 - k^2V^2\} - 3\frac{T}{\rho}k^4\right]. \tag{34}$$

It is appropriate now to introduce the transformations

$$\zeta = \varepsilon^{-1}(x_2 - V_g t_2) = x_1 - V_g t_1 = \varepsilon(x - V_g t) \quad \text{and} \quad \tau = t_2 = \varepsilon t_1 = \varepsilon^2 t.$$

The Eq. (31) is transformed to

$$i\frac{\partial A}{\partial \tau} + P\frac{\partial^2 A}{\partial \zeta^2} = QA^2\bar{A}. \tag{35}$$

It is apparent from Eq. (35) that the results of Hasimoto and Ono [6] and Whitham [5] are recovered on setting $\underline{H}$ equal to zero. It is also known that the solutions of the equation (35) are unstable against modulation if $PQ < 0$.

We shall now examine special cases of physical interest ensuing out of the nonlinear Schrödinger Eq. (35). In order to discuss the role of the magnetic fluid in the presence of a magnetic field, first we reproduce the results for the gravity and the capilliary waves in hydrodynamics.

(A) Letting $\underline{H} = T = 0$, we get $P = \omega/8k^2$, $Q = 2\omega k^2$. Thus, $PQ < 0$ for all values of the wavenumber k, implying thereby modulational instability of gravity waves [5].

(B) Letting $\underline{H} = 0$ and $T \to \infty$, we obtain $P = 3Tk/8\rho\omega$, $Q = k^5 T/2\rho\omega$. Hence, $PQ > 0$, implying thereby that the capilliary waves are stable. On the other hand if T is finite, then the waves are stable provided

$$\frac{2}{3}(3)^{1/2} < \frac{Tk^2}{\rho g} < \frac{1}{2}(2)^{1/2}. \tag{36}$$

Moreover, with k measured in units of $(T/\rho g)^{1/2}$, the inequality (36) indicates that the presence of surface tension leads to the stabilization of the gravity waves when the dimensionless wavenumber k_m lies between 0.393 and 0.707.

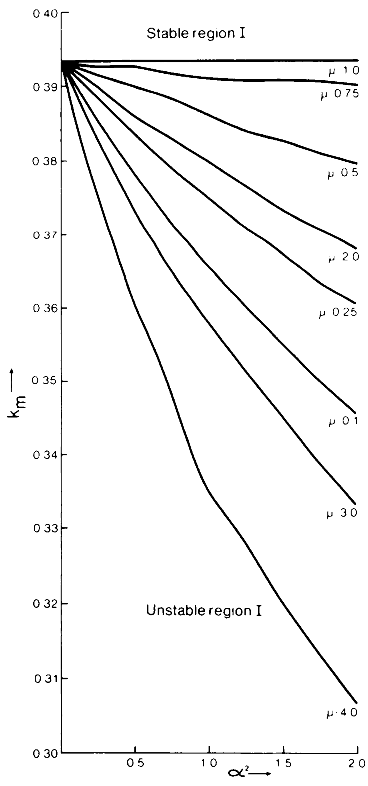

FIG. 1. The stability diagram showing the variation of k_m against magnetic field parameter α^2.

What we would like now is to consider a magnetic fluid in the presence of a magnetic field, and investigate wave propagation phenomena similar to the one described for hydrodynamics. Towards that goal, we normalize all physical quantities with respect to the characteristic length $l_c = (T/\rho g)^{1/2}$, characteristic time $t_c = (lc/g)^{1/2}$, and the characteristic magnetic field parameter $\alpha^2 = H^2 lc/4\pi T$. The group velocity rate P and the interaction parameter Q now become

$$P = \frac{1}{2}\left[\alpha^2 \frac{(\mu - 1)^2}{(\mu + 1)} + 3k - V_g^2\right], \tag{37}$$

$$Q = \frac{k}{4\omega}\left[4\Lambda\left\{\omega^2 - k^2\alpha^2\frac{(\mu - 1)^3}{(\mu + 1)^2} + 4k\left(\omega^2 - k^2\alpha^2\frac{(\mu - 1)^2}{(\mu + 1)}\right) - 3k^4\right], \tag{38}$$

where

$$\Lambda = \left[-\omega^2 + k^2\alpha^2\frac{(\mu - 1)^3}{(\mu + 1)^2}\right]\cdot[2k^2 - 1]^{-1}. \tag{39}$$

The modulational instability is characterised by the criterion $PQ < 0$, which yields the value of the wavenumber k_m at which the instability occurs. Such a criterion depends upon the wavenumber k, the ratio of the magnetic permeabilities μ, and the magnetic field parameter α^2. In Fig. 1 we have sketched the transition curves acros which P changes sign for different values of μ and α^2. Below the curves is region I where P is negative while Q is positive implying instability. It is interesting to observe from the graph that as the magnetic field increases, the region of instability shrinks quite significantly which makes us conclude that the modulational instability can be suppressed considerably with the application of strong magnetic fields. For a given α^2, as μ increases, the value of k_m initially increases, reaching a maximum when $\mu = 1$ and then starts decreasing. (Fig. 2)

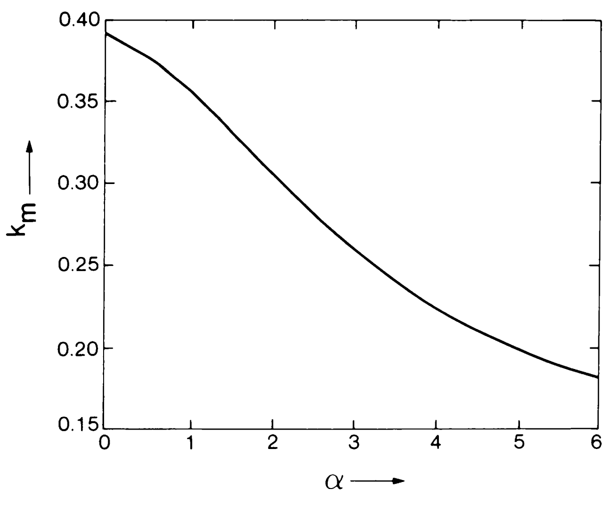

FIG. 2. The variation of k_m against magnetic field parameter α.

There is a second transition curve characterised by $k = (2)^{-1/2}$ which corresponds to the case of the second harmonic resonance. Here, the nonlinear interaction parameter Q changes sign across the transition curve. Above the curve, P is positive whereas Q is negative giving rise to the unstable region. This region is not shown in the figure. However, we wish to point out that the exact location of such a transition regions is only approximate in character since the analysis developed in this paper excludes the second harmonic resonance.

In conclusion, there are two unstable regions and one stable region. Furthermore, like water waves (see Zakharov and Shabat [10]), the modulational instability in a magnetic fluid shall cause an initial wave packet of arbitrary envelope to disintegrate into a series of solitons.

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