A COORDINATE TRANSFORMATION FOR THE POROUS MEDIA EQUATION THAT RENDERS THE FREE-BOUNDARY STATIONARY

By

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1. Introduction. In this paper we study degenerate diffusion problems in which free boundaries occur, our major objective being the development of numerical procedures which effectively track these boundaries. For convenience, we introduce our ideas in terms of the one-dimensional porous media problem; generalizations are given at the end of Sec. 2 and in Sec. 4.

Consider the problem of determining $u(t, x)$, $t \geqslant 0$, $x \in R$, such that

$$u_t = (u^2)_{xx}, \qquad u(0, x) = u_0(x), \tag{$\mathcal{P}$}$$

with initial data u_0 supported on a finite interval:

$$\begin{aligned} u_0(x) &> 0, \qquad -a < x < a, \\ u_0(x) &= 0 \quad \text{otherwise}. \end{aligned} \tag{1.1}$$

This is (a special case of) the porous media problem (cf., e.g., [1]). As is known ([2–6]): there exists a unique weak solution u; at each t the support of u is a finite interval

$$\zeta_-(t) < x < \zeta_+(t);$$

inside its support u is smooth, but across $\zeta_\pm(t)$, u_x is generally discontinuous;

$$\dot{\zeta}_\pm(t) = -2u_x(t, \zeta_\pm(t)). \tag{1.2}$$

There exist numerical procedures for Problem ($\mathcal{P}$) (cf., e.g., [7–9]), all of which seem to encounter some difficulty in tracking the free boundaries $x = \zeta_\pm(t)$. The main feature of our method is the determination of a family of curves

$$x = X(t, p), \qquad X(0, p) = p$$

along which the free boundary propagates:

$$\zeta_\pm(t) = X(t, \pm a). \tag{1.3}$$

Writing $(\mathcal{P})_1$ in the form of a mass balance law

$$u_t + (uv)_x = 0 \tag{1.4}$$

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with $u(t, x)$ a "density" and

$$v = -2u_x \tag{1.5}$$

a "velocity" (cf., e.g., [10]), we see, using (1.2), that the free boundary propagates with the velocity $v(t, x)$ of the medium; hence property (1.3) will follow provided we take $X(t, p)$ to be the solution of

$$X_t(t, p) = v(t, X(t, p)), \qquad X(0, p) = p. \tag{1.6}$$

Within this context $x = X(t, p)$ represents the motion of the medium with material points labeled by their positions p at $t = 0$.

Using these ideas, we are able to reduce $(\mathcal{P})$ to the following initial-value problem for X:

$$X_p^3 X_t = 2u_0 X_{pp} - 2u_0' X_p, \qquad X(0, p) = p. \tag{$\mathcal{P}^*$}$$

(Here and in what follows $X_p^m = (X_p)^m$.) The free boundary is then given by (1.3), while $U(t, p) = u(t, X(t, p))$ satisfies

$$U = X_p^{-1} u_0, \tag{1.7}$$

a relation which expresses balance of mass in material (Lagrangian) coordinates. Our procedure for solving $(\mathcal{P})$ consists in solving $(\mathcal{P}^*)$ on the *fixed* interval $-a \leqslant p \leqslant a$ of support of u_0.

In Section 2 we establish a uniqueness theorem for $(\mathcal{P}^*)$, and we show that given a sufficiently regular solution X of this problem, (1.7) generates, at least locally in time, a weak solution u of our original problem $(\mathcal{P})$.

The determination of $u(t, x)$ from $U(t, p)$ requires that $X(t, p)$, as a function of p, be invertible at each t, a condition related to the nonvanishing of X_p. We show, in Sec. 2, that $X_p \geqslant 1$ for all time whenever $u_0'' < 0$ on its support, and that $X_p(t, a)$, say, tends to zero in a finite time T whenever $u_0'(a) = 0$, $u_0''(a) > 0$. We show further that under the latter two conditions $X(t, a) \equiv a$ for $0 \leqslant t < T$, so that the free boundary is vertical until $t = T$ (cf. [6, 11]). We also establish a growth estimate for the $L^2(-a, a)$ norm of $X_p(t, \cdot)$.

In Sec. 3 we describe a simple difference scheme for Problem $(\mathcal{P}^*)$ and give some calculations which demonstrate the utility of our procedure; in particular, we show that even with a fairly crude mesh the free boundary is tracked quite accurately.

While our paper is devoted to the one-dimensional porous media problem, our method seems to have considerable generality: in Section 4 we derive the analog of $(\mathcal{P}^*)$ for the porous media problem in R^n; in a future paper we will discuss applications to more general equations and to Stefan problems.

2. The initial-value problem for $X(t, p)$. We first proceed formally. Let u be a solution of $(\mathcal{P})$ with initial data u_0 subject to (1.1), let $X(t, p)$ be the solution of the initial-value problem (1.6), and define

$$U(t, p) = u(t, X(t, p)), \tag{2.1}$$

or more generally, for any function $f(t, x)$,

$$f^*(t, p) = f(t, X(t, p)). \tag{2.2}$$

Then by (1.4), (1.6), and (2.1),

$$U_t = (u_t + u_x v)^* = -(uv_x)^*, \qquad X_{pt} = (v_x)^* X_p,$$

so that

$$(UX_p)_t = U_t X_p + UX_{pt} = 0. \tag{2.3}$$

But by $(1.6)_2$,

$$U(0, p) = u(0, p) = u_0(p), \qquad X_p(0, p) = 1;$$

hence

$$U = X_p^{-1} u_0. \tag{2.4}$$

Further, by (2.1),

$$U_p = (u_x)^* X_p,$$

and (1.5), (1.6) yield

$$X_t = -2U_p X_p^{-1}. \tag{2.5}$$

Equations (2.4) and (2.5) form the basis of our method. It is these equations that we will solve numerically in Sec. 3. Note that we can use (2.4) to eliminate U from (2.5); this leads to the initial-value problem ($\mathcal{P}^*$) for X.

Our procedure for solving ($\mathcal{P}$) is based on solving ($\mathcal{P}^*$) for $t \geqslant 0$ and p in the *fixed* interval $[-a, a]$, the support f u_0. (We will give a uniqueness theorem to show that boundary conditions at $p = \pm a$ are not needed.) Let X be a solution of ($\mathcal{P}^*$). Since $X_p = 1$ at $t = 0$, there exists a $T > 0$ such that

$$X_p > 0 \quad \text{on } [0, T) \times [-a, a].$$

Let

$$\zeta_\pm(t) = X(t, \pm a),$$

$$\Omega_T = \{(t, x) : \zeta_-(t) < x < \zeta_+(t), 0 \leqslant t < T\}.$$

Then for each $t \in [0, T)$ the mapping $p \mapsto X(t, p)$ is a bijection of $[-a, a]$ onto the interval $[\zeta_-(t), \zeta_+(t)]$; letting $P(t, x)$ be such that

$$X(t, P(t, x)) = x$$

(i.e., $P(t, \cdot)$ is the inverse of $X(t, \cdot)$), we define

$$u(t, x) = \begin{cases} U(t, P(t, x)), & (t, x) \in \Omega_T \\ 0, & \text{otherwise} \end{cases} \tag{2.6}$$

with U given by (2.4). We then expect the resulting function u to be the weak solution of ($\mathcal{P}$). We now show that this expectation is indeed justified. To avoid repeated hypotheses we assume, for the remainder of the section, that

$$u_0 > 0 \text{ on } (-a, a), \ u_0 = 0 \text{ otherwise}, \ u_0 \in C(R) \cap C^2[-a, a]. \tag{2.7}$$

Further, we will use the term **solution of** ($\mathcal{P}^*$) **on** $[0, T]$ for a solution X on $[0, T) \times [-a, a]$ with X, X_t, X_p, X_{pp}, X_{tp}, and X_{ppp} continuous on $[0, T) \times [-a, a]$; if, in addition, $X_p > 0$ on $[0, T) \times [-a, a]$, then X is **regular**.

THEOREM 1 (CONSISTENCY). Let X be a regular solution of $(\mathcal{P}^*)$ on $[0, T)$, and let u be defined by (2.6). Then

(i) $u > 0$ on Ω_T, $u = 0$ otherwise;
(ii) $u(x,0) = u_0(x)$ for $x \in R$;
(iii) $u_t = (u^2)_{xx}$ in $\mathring{\Omega}_T$;
(iv) $\dot{\zeta}_\pm(t) = -2u_x(t, \zeta_\pm(t))$ and $\pm\dot{\zeta}_\pm(t) \geq 0$ for $t \in [0, T)$.

Proof. Assertion (i) follows from (2.4), (2.6), (2.7), and the inequality $X_p > 0$; (ii) is a consequence of (2.6), (2.7), and the indentities $P(0, x) = x$, $X_p(0, p) = 1$.

Next, $(\mathcal{P}^*)_1$ and (2.4) imply (2.5) and (2.3), while (2.1) is a consequence of (2.6). By (2.1) and (2.5),

$$X_t = -2(u_x)^*, \qquad X_{tp} = -2(u_{xx})^* X_p, \tag{2.8}$$

where we have used the notation (2.2). Also,

$$U_t = (u_t)^* + (u_x)^* X_t = \left(u_t - 2u_x^2\right)^*,$$

and hence (2.3) and $(2.8)_2$ imply

$$0 = X_p\left(u_t - 2u_x^2 - 2uu_{xx}\right)^* = X_p\left[u_t - (u^2)_{xx}\right]^*;$$

since $X_p > 0$, we have (iii).

Eq. $(2.8)_1$ leads to the first of (iv). Finally, since $u_0(\pm a) = 0$, $(\mathcal{P}^*)_1$ implies

$$\dot{\zeta}_\pm(t) = X_t(t, \pm a) = -2u_0'(\pm a)X_p(t, \pm a)^{-2}; \tag{2.9}$$

but by (2.7), $\pm u_0'(\pm a) < 0$, and the remainder of (iv) follows. This completes the proof.

THEOREM 2 (UNIQUENESS). For any $T > 0$ there exists at most one regular solution of $(\mathcal{P}^*)$ on $[0, T)$.

Proof. Let X and Y be regular solutions on $[0, T)$. Choose $t_0 \in (0, T)$; it suffices to show that $X = Y$ on $[0, t_0] \times [-a, a]$. By $(\mathcal{P}^*)_1$,

$$X_t = -\left(u_0 X_p^{-2}\right)_p - u_0' X_p^{-2}, \qquad Y_t = -\left(u_0 Y_p^{-2}\right)_p - u_0' Y_p^{-2}.$$

If we subtract these equations, multiply by $X - Y$, and integrate from $p = -a$ to $p = a$, we obtain, after an integration by parts using $u_0(\pm a) = 0$,

$$\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} (X - Y)^2\, dp = \int_{-a}^{a} u_0\left(X_p^{-2} - Y_p^{-2}\right)(X_p - Y_p)\, dp - \int_{-a}^{a} u_0'\left(X_p^{-2} - Y_p^{-2}\right)(X - Y)\, dp. \tag{2.10}$$

Since

$$\left(X_p^{-2} - Y_p^{-2}\right)(X_p - Y_p) = -(X_p + Y_p)(X_p - Y_p)^2 X_p^{-2} Y_p^{-2} \leq 0,$$

the first term on the right side of (2.10) is ≤ 0. Next, letting

$$\psi = u_0'(X_p + Y_p)X_p^{-2}Y_p^{-2},$$

we find that

$$-\int_{-a}^{a} u_0'\left(X_p^{-2} - Y_p^{-2}\right)(X - Y)\,dp = \frac{1}{2}\int_{-a}^{a}\left[(X - Y)^2\right]_p \psi\,dp$$

$$= -\frac{1}{2}\int_{-a}^{a}(X - Y)^2\psi_p\,dp + \frac{1}{2}\left[(X - Y)^2\psi\right]_{-a}^{a}.$$

Since $\pm u_0'(\pm a) \leqslant 0$, we have $\pm\psi(\pm a, t) \leqslant 0$; hence $[(X - Y)^2\psi]_{-a}^{a} \leqslant 0$. Further, as the solutions X and Y are both regular, X_p and Y_p are bounded away from zero on $[0, t_0] \times [-a, a]$; hence there is a constant K such that $|\psi_p| \leqslant K$ on $[0, t_0] \times [-a, a]$.

Let

$$\xi(t) = \int_{-a}^{a} [X(t, p) - Y(t, p)]^2\,dp$$

and note that $\xi(0) = 0$, since X and Y satisfy the same initial condition. Thus if we integrate (2.10) from 0 to t and use the above remarks, we arrive at the Gronwall inequality $\xi(t) \leqslant K\int_0^t \xi(\tau)\,d\tau$ for $0 \leqslant t \leqslant t_0$, which clearly implies $X = Y$ on $[0, t_0] \times [-a, a]$.

Remark. We have not been able to establish (directly) existence for Problem ($\mathcal{P}^*$). We note that existence can be inferred from existence for ($\mathcal{P}$) in conjunction with (1.6).

Thus far all of our results have been local, as the initial condition $X_p(0, p) = 1$ only insures $X_p > 0$ for sufficiently small time. There are situations in which we can give a global result.[1]

THEOREM 3. Suppose that $u_0'' < 0$ on $[-a, a]$. Let X be a solution of ($\mathcal{P}^*$) on $[0, T)$. Then X is regular; in fact, $X_p \geqslant 1$ on $[-a, a] \times [0, T)$.

Proof. We let $Z = X_p$ and differentiate $(\mathcal{P}^*)_1$ with respect to p to obtain

$$Z^3 Z_t = 2u_0 Z_{pp} - 6u_0 Z^{-1} Z_p^2 + 6u_0' Z_p - 2u_0'' Z. \tag{2.11}$$

At $t = 0$ we have $Z = 1$, $Z_p = Z_{pp} = 0$; hence $Z_t > 0$ at $t = 0$ and $Z > 1$ for small t. Suppose Z were ever equal to 1. Then there would be a first time $t_1 \in [0, T]$ at which this occurs and a point p_1 with $Z(t_1, p_1) = 1$. If $p_1 \in (-a, a)$, we would have $Z_p = 0$, $Z_{pp} \geqslant 0$, and $Z_t \leqslant 0$ at (t_1, p_1), which contradicts (2.11), since $u_0''(p_1) < 0$. If $p_1 = -a$, then $Z_t \leqslant 0$ and $Z_p \geqslant 0$ at (t_1, p_1); thus, since $u_0(-a) = 0$, $u_0'(-a) > 0$, and $u_0''(-a) < 0$, we again contradict (2.11). A similar argument applies at $x = a$.

For initial data u_0 which is not concave one cannot expect to have $X_p > 0$ for all time. Knerr [6], generalizing results of Aronson [11], has shown that if $u_0'(b) = 0$ for $b = a$, say, then the free boundary emanating from a is vertical for an interval $0 \leqslant t < T < \infty$. At $t = T$ there is a loss of smoothness and $\zeta^+(t)$ begins to increase. We now show that this phenomenon is related to the vanishing of X_p.

THEOREM 4 (BREAKDOWN). Suppose that $u_0'(b) = 0$ and $u_0''(b) > 0$ for $b = -a$ or $b = a$, and put

$$T = \left[6u_0''(b)\right]^{-1}. \tag{2.12}$$

[1] Cf. Graveleau and Jamet [8], who show that if u_0 is concave, then the solution u of ($\mathcal{P}$) will be concave for all time. With $u_{xx} < 0$ (2.8) clearly implies $X_p \geqslant 1$.

Let X be a solution of ($\mathcal{P}^*$) on $[0, T)$. Then

(i)

$$X(t, b) \equiv b \quad \text{for } 0 \leq t < T,$$

(ii)

$$X_p(t, b) \to 0 \quad \text{as } t \to T.$$

If, in addition, X is regular on $[0, T)$ and u is defined by (2.6), then

(iii)

$$u_{xx}(t, X(t, b)) \to \infty \quad \text{as } t \to T.$$

Proof. Let $z(t) = X_p(t, b)$. Since $z(0) = 1$, there is a t_0 such that $z(t) > 0$ for $0 \leq t < t_0 \leq T$. Thus, by (2.11), for $0 \leq t < t_0$,

$$z^2 z' = -2u_0''(b) \tag{2.13}$$

and

$$z^3(t) = 1 - 6u_0''(b)t.$$

Therefore, by continuity, we may take $t_0 = T$ and (ii) follows. Moreover, since $z > 0$ on $[0, T)$, ($\mathcal{P}^*$) yields (i). Finally, by $(2.8)_2$ and (2.13),

$$u_{xx}(t, X(t, b)) = u_0''(b)z(t)^{-3},$$

and (ii) implies (iii).

Under the hypotheses of Theorem 3 we have the following growth estimate for the L^2 norm of X_p.

THEOREM 5 (STABILITY). Suppose that $u_0'' < 0$ on $[-a, a]$, and put $c = 8[u_0'(-a) - u_0'(a)]$. Let X be a solution of ($\mathcal{P}^*$) on $[0, T)$. Then

$$\int_{-a}^{a} X_p^2(t, p)\, dp \leq 2a + ct \tag{2.14}$$

for $0 \leq t < T$.

Proof. Note first that, since $u_0(\pm a) = 0$, $(\mathcal{P}^*)_1$ implies

$$X_t(t, \pm a) = -2u_0'(\pm a)X_p^{-2}(t, \pm a). \tag{2.15}$$

We multiply $(\mathcal{P}^*)_1$ by $X_p^{-3}X_{pp}$ and integrate from $p = -a$ to $p = a$. After an integration by parts and the use of (2.15) we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_{-a}^{a} X_p^2\, dp - 2\left[u_0' X_p^{-1}\right]_{-a}^{a} = 2\int_{-a}^{a} u_0 X_p^{-3}X_{pp}^2\, dp + 2\int_{-a}^{a} u_0'\left(X_p^{-1}\right)_p dp.$$

If we integrate the last term on the right by parts, we find that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{-a}^{a} X_p^2\, dp &+ 2\int_{-a}^{a} u_0 X_p^{-3}X_{pp}^2\, dp \\ &= -4\left[u_0' X_p^{-1}\right]_{-a}^{a} + 2\int_{-a}^{a} u_0'' X_p^{-1}\, dp \\ &\leq 4\left[u_0'(-a)X_p^{-1}(t, -a) - u_0'(a)X_p^{-1}(t, a)\right] \leq \frac{c}{2}, \end{aligned}$$

since $u_0'' < 0$, $\pm u_0(\pm a) < 0$, and $X_p \geqslant 1$. The estimate (2.14) follows upon integration.

Observe that one obtains also the estimate

$$\int_0^t \int_{-a}^a u_0 X_p^{-3} X_{pp}^2 \, dp \, d\tau \leqslant 2a + ct;$$

in addition, since $X_p \geqslant 1$, (1.3) and (2.15) yield the inequality

$$|\dot{\zeta}_\pm(t)| \leqslant 2|u_0'(\pm a)|.$$

By (iv) of Theorem 1, $\pm X(t, \pm a) > 0$; hence for each t there is a p_1 such that $X(t, p_1) = 0$. The standard argument

$$|X(t, p)| = \left| \int_{p_1}^p X_p(t, \xi) \, d\xi \right| \leqslant \left[2a \int_{-a}^a X_p^2(t, \xi) \, d\xi \right]^{1/2}$$

and (2.14) therefore yield the estimate

$$|X(t, p)| \leqslant \sqrt{2a(2a + ct)}\,.$$

Thus the width of the support of $u(\cdot, t)$ is at most $O(t^{1/2})$.

Remark. Vasquez [12] has shown that this width is actually $O(t^{1/3})$ by proving that $u(t, x)$ is asymptotic to a certain "fundamental solution" $\hat{u}(t, x)$ of the form (3.5). Such fundamental solutions correspond to Dirac distributions at $t = 0$ and as such the corresponding coordinate transformation $\hat{X}(t, p)$ is not defined. If, however, one lets t_a denote the time for which the support of $\hat{u}(t_a, \cdot)$ has width $2a$, and considers $\hat{u}_a(t, x) = \hat{u}(t_a + t, x)$, then the corresponding coordinate transformation $\hat{X}_a$ is well defined; in fact,

$$\hat{X}_a(t, p) = \left(\frac{t_a + t}{t_a} \right)^{1/2} p.$$

Guided by the results of [12], we conjecture that $X(t, p)$ and $\hat{X}_a(t, p)$ are asymptotic as $t \to \infty$.

Problem $(\mathcal{P}^*)$ can be given a *weak formulation*, which we now deduce. We begin by writing $(\mathcal{P}^*)_1$ in the form

$$X_t = -\left(u_0 X_p^{-2}\right)_p - u_0' X_p^{-2}.$$

If we multiply this equation and $(\mathcal{P}^*)_2$ by an arbitrary C^1 function $w(p)$ and integrate from $p = -a$ to $p = +a$ we obtain, after an integration of parts,

$$\int_{-a}^a \left[\left(X_t + u_0' X_p^{-2} \right) w - u_0 X_p^{-2} w' \right] dp = 0,$$

$$\int_{-a}^a [X(0, p) - p] w(p) \, dp = 0. \tag{2.16}$$

Eq. (2.16) constitute the desired weak formulation of $(\mathcal{P}^*)$.

The weak form (2.16) admits an approximate formulation in terms of finite elements. In this connection it is important to note that the absence of boundary conditions allows one to operate in the space $H^1(-a, a)$. We expect these observations to be of great value in the extension to higher dimensions.

Remarks. 1. The $p =$ constant trajectories $t \mapsto X(t, p)$ are analogs of characteristic curves, as the degeneracy in the porous media equation $(\mathcal{P})_1$ propagates along such trajectories. Indeed, this equation degenerates at $u = 0$, and

$$u(t, X(t, p)) = 0 \quad \text{if and only if } u_0(p) = 0,$$

at least when the solution is regular (cf. (2.4)).

2. Problem $(\mathcal{P})$ is a special case of the more general diffusion problem

$$u_t = q(u, u_x)_x, \qquad u(x,0) = u_0(x), \tag{2.17}$$

with (2.16) *degenerate* in the sense that

$$q_\xi(u, \xi)\begin{cases} > 0 & \text{for } u > 0 \\ = 0 & \text{for } u = 0. \end{cases}$$

As before, we seek curves $x = X(t, p)$ along which this degeneracy propagates. Formally, such curves are generated, via (1.6), by writing $(2.17)_1$ as a mass balance law (1.4), since this law has the Lagrangian form

$$u(t, X(t, p)) = X_p^{-1}(t, p)u_0(p)$$

(cf. (1.7)). If we do this, we find that the resulting initial-value problem for $X(t, p)$ is:

$$X_t = -\frac{X_p}{u_0}q\left(\frac{u_0}{X_p}, \frac{1}{X_p}\left(\frac{u_0}{X_p}\right)_p\right), \qquad X(0, p) = p. \tag{2.18}$$

As an example, the porous media equation is often considered in the form

$$u_t = (u^m)_{xx} \qquad (m \geqslant 2);$$

for this equation $(2.18)_1$ becomes

$$X_t = -\frac{mu_0^{m-2}}{X^{m-1}}\left(\frac{u_0}{X_p}\right)_p.$$

3. Numerical solutions. Here we describe a simple difference scheme for the initial-value problem $(\mathcal{P}^*)$. (We will actually work with (2.4), $(2.5)_1$ rather than $(\mathcal{P}^*)_1$.) For convenience, we assume that u_0 is symmetric: $u_0(x) = u_0(-x)$. Then by symmetry we can restrict ourselves to $0 \leqslant p \leqslant a$ provided we impose the additional conditions

$$X(0, t) \equiv 0, \qquad U_p(0, t) \equiv 0. \tag{3.1}$$

We want to solve the equations

$$X_t = -X_p^{-1}U_p, \qquad U = X_p^{-1}u_0. \tag{3.2}$$

We choose $\Delta t > 0$ and $\Delta p = aN^{-1}$ for some integer N, introduce the mesh $t_i = i\Delta t$, $i = 0, 1, 2, \ldots, p_n = n\Delta p$, $n = 0, 1, \ldots, N$, and write f_n^i for the value of a function f at the mesh point (t_i, p_n). We approximate (3.2) by the difference scheme:

$$\frac{X_n^{i+1} - X_n^i}{\Delta t} = \frac{L_n U^i}{M_n X_n^i}, \qquad U_n^{i+1} = \frac{u_n}{M_n X^{i+1}}. \tag{3.3}$$

In these formulae $u_n = u_0(p_n)$, while L_n and M_n represent the spatial difference operators

$$L_n f = \frac{1}{2\Delta p}\begin{cases} f_{n+1} - f_{n-1}, & 1 \leqslant n \leqslant N-1, \\ f_{N-2} - 4f_{N-1}, & n = N, \\ 0, & n = 0, \end{cases} \tag{3.4}$$

$$M_n f = \frac{1}{\Delta p}\begin{cases} f_{n+1} - f_{n-1}, & 1 \leqslant n \leqslant N-1, \\ 3f_N - 4f_{N-1} + f_{N-2}, & n = N, \\ 2f_1, & n = 0. \end{cases}$$

The formulae (3.4) are accurate to $O(\Delta p^2)$ for functions satisfying (3.1) with $U(t, a) \equiv 0$. Since the method (3.3) is partially explicit, we would expect it best to take $\Delta t = (\Delta p)^2$; with this choice we expect $O(\Delta p^2)$ accuracy.

The following explicit solution to the initial-value problem $(\mathcal{P})$—for initial data a Dirac distribution—is given by Pattle [13]:

$$\hat{u}(x, t) = \begin{cases} \lambda(t)^{-1}\left[1 - \dfrac{y^2}{\lambda(t)^2}\right], & |y| \leqslant \lambda(t), \\ 0, & |y| \geqslant \lambda(t), \end{cases} \tag{3.5}$$

$$y = x/x_0, \lambda(t) = (t/t_0)^{1/3}, \qquad x_0 = \Gamma(5/2)/\sqrt{\pi} = \tfrac{3}{4}, t_0 = x_0^2/12.$$

We attempted to approximate the solution $u(t, x) = \hat{u}(t+1, x)$. Thus we have $a = (12x_0)^{1/3} = 9^{1/3}$. The results are presented in Tables 1 and 2.

In Table 1 we give the approximate and theoretical values of $\zeta_+(t)$, together with the relative errors, for a sequence of times. These calculations are performed with ten subdivisions, so that $\Delta p = 0.208$. We observe that even with this fairly crude mesh the free boundary is tracked quite accurately. In Table 2 we present one of our calculations to determine the rate of convergence. The results confirm the expected rate of $(\Delta p)^2$.

Table 1

$\zeta^A(t)$ and $\zeta^T(t)$, the approximate and theoretical positions of the right-hand free boundary for $\Delta p = 0.208$, $\Delta t = (\Delta p)^2$.

t	$\zeta^A(t)$	$\zeta^T(t)$	relative error
5.327	3.63826	3.63274	0.00152
9.653	4.43406	4.42904	0.00113
13.980	5.01550	5.01093	0.00091
18.307	5.48638	5.48218	0.00079
22.634	5.88782	5.88391	0.00063
29.960	6.24088	6.23721	0.00059
31.287	6.55793	6.55445	0.00053
35.614	6.84696	6.84365	0.00048
39.941	7.11343	7.11027	0.00044
44.267	7.36131	7.35827	0.00041

Table 2

$\zeta^A(1.108)$ and $\zeta^T(1.108)$, the approximate and theoretical positions of the right-hand free boundary at $t = 1.108$ for $\Delta t = (\Delta p)^2$.

Δp	$\zeta^A(1.108)$	$\zeta^T(1.08)$	error
0.1040042	2.152773	2.152531	0.000242
0.0520021	2.152591	2.152531	0.000060
0.0260011	2.152546	2.152531	0.000015
0.0130005	2.152535	2.152531	0.000004

Problem ($\mathcal{P}$) with

$$u_0(x) = \begin{cases} \cos^2 \dfrac{\pi x}{2}, & x \leq 1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.6}$$

is discussed by Aronson [11], who shows that the corresponding free boundaries are vertical for an interval $0 \leq t < T$, a result consistent with the conclusions of Theorem 4, where for (3.6),

$$T = (3\pi^2)^{-1} \approx .03377.$$

We also performed a numerical experiment for the initial data (3.6); the results are shown in Table 3. It is seen that the free boundary is indeed roughly vertical for $t \leq T$.

Table 3

$\zeta^A(t)$, the approximate position of the right-hand free boundary for $u_0(x) = \cos^2(\pi X/2)$. Here $\Delta p = 0.025$, $\Delta t = (\Delta p)^2$.

t	$\zeta^A(t)$
0.00375	1.0000016
0.99750	1.0000039
0.01500	1.0000130
0.02000	1.0000293
0.02500	1.0000752
0.03000	1.0002330
0.03125	1.0003195
0.03250	1.0004445
0.03375	1.0006275
0.03500	1.0008996
0.03625	1.0013092
0.03750	1.0019303

4. Extension to R^n. The porous media problem in R^n consists in finding a scalar function $u(t, \mathbf{x})$, $t \geq 0$, $\mathbf{x} \in R^n$, such that

$$u_t = \Delta_{\mathbf{x}}(u^2), \qquad u(0, \mathbf{x}) = u_0(\mathbf{x}). \tag{$\mathcal{P}_n$}$$

(Here and in what follows Δ, div, and ∇, respectively, denote the Laplacian, divergence, and gradient operators in R^n.)

Guided by our one-dimensional analysis, we rewrite $(\mathcal{P}_n)_1$ as a balance law

$$u_t + \operatorname{div}_{\mathbf{x}}(u\mathbf{v}) = 0$$

with

$$\mathbf{v} = -2\nabla_{\mathbf{x}} u,$$

and we take $\mathbf{X}(t,\mathbf{p})$ to be the solution of the initial-value problem

$$\mathbf{X}_t(t,\mathbf{p}) = \mathbf{v}(t,\mathbf{X}(t,\mathbf{p})), \qquad \mathbf{X}(0,\mathbf{p}) = \mathbf{p}. \tag{4.1}$$

Proceding as before, we define

$$U(t,\mathbf{p}) = u(t,\mathbf{X}(t,\mathbf{p}))$$

and note that

$$U_t = (u_t + \mathbf{v}\cdot\nabla_{\mathbf{x}} u)^* = -(u\operatorname{div}_{\mathbf{x}}\mathbf{v})^*. \tag{4.2}$$

Let

$$\mathbf{Z} = \nabla_{\mathbf{p}}\mathbf{X},$$

and assume that $\det\mathbf{Z} > 0$. Then, using the identities (cf. e.g., [10], p. 77)

$$(\det\mathbf{Z})_t = (\det\mathbf{Z})\operatorname{tr}(\mathbf{Z}_t\mathbf{Z}^{-1}), \qquad \operatorname{tr}(\mathbf{Z}_t\mathbf{Z}^{-1}) = (\operatorname{div}_x\mathbf{v})^*$$

in conjunction with (4.2), we conclude that

$$(U\det\mathbf{Z})_t = 0$$

and hence that

$$U = (\det\mathbf{Z})^{-1}u_0.$$

On the other hand, by the chain-rule,

$$\nabla_{\mathbf{p}}U = \mathbf{Z}^T(\nabla_{\mathbf{x}}u)^*.$$

Thus, using (4.1), we arrive at the following initial-value problem for $\mathbf{X}$:

$$\nabla\mathbf{X}^T\mathbf{X}_t = -2\nabla U, \qquad U = (\det\nabla\mathbf{X})^{-1}u_0, \qquad (\mathcal{P}_n^*)$$
$$\mathbf{X}(0,\mathbf{p}) = \mathbf{p},$$

where $\nabla = \nabla_{\mathbf{p}}$.

A careful analysis of $(\mathcal{P}_n^*)$ is beyond the scope of this paper. Our ultimate hope is to show that when

$$u_0 > 0 \quad \text{on } A, \qquad u_0 = 0 \quad \text{otherwise}, \tag{4.3}$$

with $\bar{A}$ compact and connected, Problem $(\mathcal{P}_n)$ reduces to solving $(\mathcal{P}_n^*)$ on the *fixed* region A for all time.

In view of (4.3), $u_0(\mathbf{p}) = 0$ for $\mathbf{p}\in\partial A$. Thus, by $(\mathcal{P}_n^*)$, $\mathbf{X}_t(t,\mathbf{p}) \equiv 0$ at any $\mathbf{p}\in\partial A$ for which $\nabla u_0(\mathbf{p}) = \mathbf{0}$, at least as long as the solution remains regular.

Remarks. 1. The p = constant curves $t \mapsto \mathbf{X}(t,\mathbf{p})$ are analogs of bicharacteristic curves in the theory of partial differential equations (cf. Remark 1 at the end of Sec. 2).

2. For Problem $(\mathcal{P}_n)$ with $n \geqslant 2$, regularity of the free boundary $\mathcal{F}$ is essentially an open question (cf. [14]). The formulation $(\mathcal{P}_n^*)$ might be useful in attacking this question, as

$$\mathcal{F} = \{\mathbf{X}(t, \mathbf{p}) : \mathbf{p} \in \partial A, t \geqslant 0\}.$$

Moreover, since regularity is a local question, the problem of proving that $\mathbf{X}(t, \cdot)$ is a bijection is trivial: it follows, at least locally, from $\mathbf{X}(0, \mathbf{p}) = \mathbf{p}$, where we have chosen the time scale with $t = 0$ the time near which regularity is sought.

3. For the more general equation

$$u_t = \Delta_{\mathbf{x}}(u^m) \qquad (m \geqslant 2)$$

the first of $(\mathcal{P}_n^*)$ is replaced by

$$\nabla \mathbf{X}^T \mathbf{X}_t = -mU^{m-2} \nabla U.$$

Note added in proof. We have recently discovered the paper of J. G. Berryman (Evolution of a stable profile for a class of nonlinear diffusion equations, III; slow diffusion on the line, J. Math. Phys. **21**, 1326–1331 (1980)), where similar techniques are introduced. Berryman also uses Lagrangian coordinates (rather than the initial coordinate, Berryman takes $p = p(x, t)$ to be the total mass at t in the interval $(-\infty, x)$). Berryman's partial differential equation for X is simpler than ours, but his initial condition is more complicated.

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