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ON NON-UNIQUENESS IN THE TRACTION BOUNDARY-VALUE PROBLEM FOR A COMPRESSIBLE ELASTIC SOLID*

By

R. W. OGDEN

Brunel University

Abstract. For a compressible isotropic elastic solid local and global non-uniqueness of the homogeneous deformation resulting from prescribed dead-load boundary tractions is examined. In particular, for the plane-strain problem with equibiaxial in-plane tension, equations governing the paths of deformation branching from the bifurcation point on a deformation path corresponding to in-plane pure dilatation are derived. Explicit calculations are given for a specific strain-energy function and the stability of the branches is discussed. Some general results are then given for an arbitrary form of strain-energy function.

Introduction and basic equations. In [1] the branching and global non-uniqueness of the solution to the plane-strain problem of a homogeneous *incompressible* isotropic elastic solid subject to in-plane equibiaxial dead-load tractions was examined. In particular, bifurcation from a homogeneously-deformed configuration with equal in-plane principal stretches was found to occur at a certain critical value of the stress, dependent on the form of strain-energy function. An equation governing the global development of the deformation branches emanating from the bifurcation point was obtained and examined in detail with regard to stability and in respect of different choices of strain-energy function. Related problems for incompressible materials have been considered in [2–4].

In the present paper we give a parallel analysis for compressible materials for which, apparently, no results have appeared previously. We begin by summarizing the basic equations and notation.

A point of the material body $\mathcal{B}$ is identified by its position vector $\mathbf{X}$ relative to some fixed origin in a given reference configuration. Let the deformation χ be defined by

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}), \qquad \mathbf{X} \in \mathcal{B}, \tag{1}$$

where $\mathbf{x}$ is the position vector of the point $\mathbf{X}$ in the current configuration. The deformation gradient

$$\mathbf{A} = \operatorname{Grad} \boldsymbol{\chi} \tag{2}$$

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is subject to

$$\det \mathbf{A} > 0, \tag{3}$$

and we shall make use of the polar decomposition

$$\mathbf{A} = \mathbf{RU}, \tag{4}$$

where **R** is proper orthogonal and **U** is positive definite and symmetric. The notation

$$J = \det \mathbf{A} = \det \mathbf{U} \tag{5}$$

is also used.

The elastic strain-energy function W, per unit reference volume, satisfies the objectivity condition

$$W(\mathbf{A}) = W(\mathbf{U}). \tag{6}$$

In what follows we use both the nominal stress **S**, given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{A}}, \tag{7}$$

and the symmetric Biot stress $\mathbf{T}^{(1)}$, given by

$$\mathbf{T}^{(1)} = \frac{\partial W}{\partial \mathbf{U}}. \tag{8}$$

For an isotropic material W depends on **U** only through its principal values $\lambda_1, \lambda_2, \lambda_3$, the principal stretches, and is subject to the pairwise symmetries.

$$W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_1, \lambda_3, \lambda_2) = W(\lambda_3, \lambda_1, \lambda_2). \tag{9}$$

The principal components of $\mathbf{T}^{(1)}$ are then given by

$$t_i^{(1)} = \frac{\partial W}{\partial \lambda_i}, \qquad i = 1, 2, 3, \tag{10}$$

and we also have the important connection [5, 6],

$$\mathbf{S} = \mathbf{T}^{(1)}\mathbf{R}^T, \tag{11}$$

where $\mathbf{R}^T$ is the transpose of **R**.

Since we are concerned with a homogeneous material homogeneously deformed under the action of dead-load tractions, prescription of these tractions is equivalent to prescription of **S**. Equation (11) can be regarded as a polar decomposition of **S** in terms of symmetric $\mathbf{T}^{(1)}$ and proper orthogonal **R**. Once $\mathbf{T}^{(1)}$ and **R** are determined the deformation gradient **A** is given by (4) when **U** is found by inversion of (8). However, unlike **U** in (4), $\mathbf{T}^{(1)}$ in (11) need not be positive definite and so the polar decomposition (11) is not unique. The extent of its non-uniqueness has been discussed in [5, 6] and here we summarize briefly the relevant results.

If the principal values of $\mathbf{SS}^T$ are distinct then the orientation of the principal axes of $\mathbf{T}^{(1)}$, and hence of **U** (since $\mathbf{T}^{(1)}$ is coaxial with **U** for an isotropic material), is uniquely determined by **S** and there are just four possible combinations of signs of the principal values $t_1^{(1)}, t_2^{(1)}, t_3^{(1)}$. Only one of these satisfies the inequalities

$$t_i^{(1)} + t_j^{(1)} > 0, \qquad i \neq j, \tag{12}$$

which are necessary for the considered configuration to be stable (necessary and sufficient conditions for stability are given in Sec. 2). Subject to the above restrictions the deformation resulting from $\mathbf{S}$ is unique if and only if (10) is uniquely invertible.

If, on the other hand, two principal values of $\mathbf{SS}^T$ are equal then the orientation of the principal axes of $\mathbf{T}^{(1)}$, and hence of $\mathbf{U}$, is arbitrary in the principal plane defined by the equal principal values. Subject to this arbitrariness the deformation is found by determining $\lambda_1, \lambda_2, \lambda_3$ from (10), given that (12) holds.

In Secs. 3 and 4 we shall consider the inversion of (10) in detail, but first we examine the local counterpart of the inversion of (7).

2. Incremental deformations and stability. We now consider a small increment, $\dot{\chi}$ say, in the deformation χ and introduce the notation

$$\dot{\mathbf{A}} = \operatorname{Grad} \dot{\chi}. \tag{13}$$

The incremental form of the constitutive equation (7) may be written

$$\dot{\mathbf{S}} = \mathcal{A}\dot{\mathbf{A}} \tag{14}$$

correct to the first order in $\dot{\mathbf{A}}$, where $\dot{\mathbf{S}}$ is the nominal stress increment and

$$\mathcal{A} = \frac{\partial \mathbf{S}}{\partial \mathbf{A}} = \frac{\partial^2 W}{\partial \mathbf{A}^2} \tag{15}$$

is the (fourth-order) tensor of elastic moduli associated with $(\mathbf{S}, \mathbf{A})$.

The incremental constitutive law (14) is invertible provided $\mathcal{A}$ is non-singular. This is guaranteed, in particular, if the *exclusion condition*

$$\operatorname{tr}\{(\mathcal{A}\dot{\mathbf{A}})\dot{\mathbf{A}}\} > 0 \tag{16}$$

holds for all $\dot{\mathbf{A}} \neq \mathbf{0}$. If (16) holds the deformation χ is *stable* in the classical sense.

We are concerned with a path of loading which leads through a sequence of stable configurations and reaches a configuration where (16) just fails, i.e. where

$$\operatorname{tr}\{(\mathcal{A}\dot{\mathbf{A}})\dot{\mathbf{A}}\} \geqslant 0 \tag{17}$$

for all $\dot{\mathbf{A}}$, with equality holding for some $\dot{\mathbf{A}} \neq \mathbf{0}$.

In such a configuration extremization of (17) shows that

$$\dot{\mathbf{S}} = \mathcal{A}\dot{\mathbf{A}} = 0 \tag{18}$$

for the $\dot{\mathbf{A}}$ which gives equality in (17). Thus, since both $\mathbf{A}$ and $\mathbf{A} + \dot{\mathbf{A}}$ correspond to nominal stress $\mathbf{S}$ in the considered configuration bifurcation occurs there. Of particular interest, therefore, are configurations χ in which $\mathcal{A}$ is singular. We refer to such configurations as *bifurcation points*. In what follows we shall be concerned with the determination of such points and the global development of the branches of χ emanating from them.

To be more specific we note first that the components of $\mathscr{Q}$ referred to the principal axes of $\mathbf{U}$ are given by

$$\left.\begin{aligned} \mathscr{Q}_{iijj} &= \frac{\partial^2 W}{\partial\lambda_i \partial\lambda_j}, \\ \mathscr{Q}_{ijij} &= \frac{\lambda_i t_i^{(1)} - \lambda_j t_j^{(1)}}{\lambda_i^2 - \lambda_j^2}, \quad i \neq j, \\ \mathscr{Q}_{ijji} &= \frac{\lambda_j t_i^{(1)} - \lambda_i t_j^{(1)}}{\lambda_i^2 - \lambda_j^2}, \quad i \neq j \end{aligned}\right\} \tag{19}$$

(for a detailed derivation see [6]). The exclusion condition (16) is then easily shown to be equivalent to

$$\left(\partial t_i^{(1)}/\partial\lambda_j\right) \quad \text{is positive definite,} \tag{20}$$

$$t_i^{(1)} + t_j^{(1)} > 0, \qquad i \neq j, \tag{21}$$

$$\left(t_i^{(1)} - t_j^{(1)}\right)/\left(\lambda_i - \lambda_j\right) > 0, \qquad i \neq j \tag{22}$$

jointly, and $\mathscr{Q}$ is singular where any one of these just fails.

Suppose now that λ_3 is fixed and attention is restricted to incremental deformations in the (1, 2)-plane. Then (20)–(22) reduce to

$$\begin{pmatrix} \dfrac{\partial^2 W}{\partial\lambda_1^2} & \dfrac{\partial^2 W}{\partial\lambda_1 \partial\lambda_2} \\ \dfrac{\partial^2 W}{\partial\lambda_1 \partial\lambda_2} & \dfrac{\partial^2 W}{\partial\lambda_2^2} \end{pmatrix} \quad \text{is positive definite,} \tag{23}$$

$$t_1^{(1)} + t_2^{(1)} \equiv \frac{\partial W}{\partial\lambda_1} + \frac{\partial W}{\partial\lambda_2} > 0, \tag{24}$$

$$\left(\frac{\partial W}{\partial\lambda_1} - \frac{\partial W}{\partial\lambda_2}\right) \Big/ \left(\lambda_1 - \lambda_2\right) > 0 \tag{25}$$

respectively.

. The bifurcation points corresponding to singularity of the matrix in (23) are associated with incremental modes of deformation which are coaxial with the underlying deformation. Those corresponding to failure of (24) and (25) involve shearing modes of incremental deformation [5, 6]. In the special case $\lambda_1 = \lambda_2$, $\partial^2 W/\partial\lambda_2^2 = \partial^2 W/\partial\lambda_1^2$ and the matrix (23) is singular when *either*

$$\frac{\partial^2 W}{\partial\lambda_1^2} + \frac{\partial^2 W}{\partial\lambda_1 \partial\lambda_2} = 0,$$

in which case the incremental deformation maintains equal in-plane principal stretches, *or*

$$\frac{\partial^2 W}{\partial\lambda_1^2} - \frac{\partial^2 W}{\partial\lambda_1 \partial\lambda_2} = 0, \tag{26}$$

which corresponds to a pure shear incremental mode. When $\lambda_1 = \lambda_2$, (25) also just fails where (26) holds.

3. Equibiaxial tension. Suppose that $t_1^{(1)} = t_2^{(1)} \equiv t^{(1)} > 0$ is prescribed and that λ_3 is fixed. Then (24) automatically holds. The equation

$$t_1^{(1)} = \frac{\partial W}{\partial \lambda_1} = \frac{\partial W}{\partial \lambda_2} = t_2^{(1)} \tag{27}$$

always has the solution $\lambda_1 = \lambda_2$. But on such a path of deformation bifurcation into a configuration with $\lambda_1 \neq \lambda_2$ can occur if the condition (26) is met as $t^{(1)}$ increases. To illustrate the type of result obtainable we now consider a specific form of strain-energy function prior to examining the general case in Section 4.

Let

$$W = \tfrac{1}{2}\mu\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\ln J\right) + \tfrac{1}{2}\kappa'(J-1)^2, \tag{28}$$

where

$$\kappa' = \kappa - \frac{2}{3}\mu \tag{29}$$

and $\mu > 0$, $\kappa > 0$ are the ground state shear and bulk moduli of elasticity. Then, from (27),

$$t^{(1)} = \mu\left(\lambda_1 - \lambda_1^{-1}\right) + \kappa'(J-1)\lambda_2\lambda_3 = \mu\left(\lambda_2 - \lambda_2^{-1}\right) + \kappa'(J-1)\lambda_1\lambda_3 \tag{30}$$

and also

$$\left(\frac{\partial W}{\partial \lambda_1} - \frac{\partial W}{\partial \lambda_2}\right) \Big/ (\lambda_1 - \lambda_2) = \mu + \mu\lambda_3 J^{-1} - \kappa'\lambda_3(J-1). \tag{31}$$

On setting $\lambda_1 = \lambda_2 = \lambda$ in (31) we deduce that the critical value of λ at which (26) holds is governed by

$$\kappa'\lambda_3^2\lambda^4 - (\mu + \kappa'\lambda_3)\lambda^2 - \mu = 0. \tag{32}$$

Assuming that $\kappa' > 0$ we see that (32) has only one positive real solution for λ, λ_c say, given by

$$\lambda_c^2 = \left\{\mu + \kappa'\lambda_3 + \left[(\mu + \kappa'\lambda_3)^2 + 4\mu\kappa'\lambda_3^2\right]^{1/2}\right\}/2\kappa'\lambda_3^2. \tag{33}$$

From (30) the associated critical value of $t^{(1)}$, $t_c^{(1)}$ say, is found to be simply

$$t_c^{(1)} = 2\mu\lambda_c. \tag{34}$$

Clearly,

$$\lambda_c^2\lambda_3 > 1. \tag{35}$$

Beyond the critical value a path of deformation with $\lambda_1 \neq \lambda_2$ is governed by the second equation in (30), which may be rearranged as

$$\kappa'\lambda_3^2\lambda_1^2\lambda_2^2 - (\mu + \kappa'\lambda_3)\lambda_1\lambda_2 - \mu = 0.$$

Comparison of this with (32) shows that

$$\lambda_1\lambda_2 = \lambda_c^2 \tag{36}$$

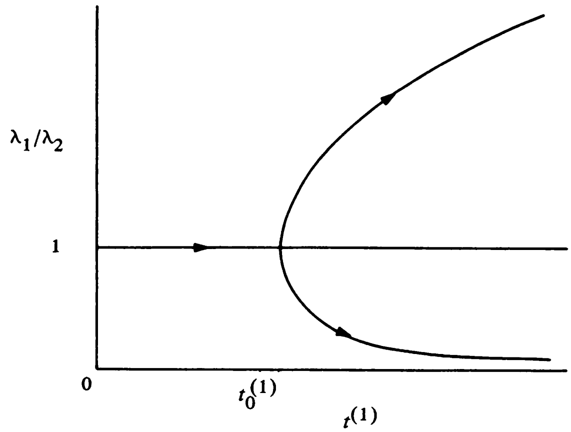

Fig. 1 Bifurcation diagram showing λ_1/λ_2 plotted against $t^{(1)}$ in respect of the strain-energy function (28).

and hence, from (30),

$$t^{(1)} = \mu(\lambda_1 + \lambda_1^{-1}\lambda_c). \tag{37}$$

With the help of (34), equation (37) is solved to give

$$\lambda_1 = \left\{t^{(1)} \pm \sqrt{t^{(1)2} - t_c^{(1)2}}\right\}\Big/2\mu. \tag{38}$$

Thus, for $t^{(1)} > t_c^{(1)}$ there are two deformation branches emanating from the critical value $\lambda_1 = \lambda_c$ on the deformation path $\lambda_1 = \lambda_2$. This is depicted in Fig. 1 where the ratio $\lambda_1/\lambda_2 \equiv \lambda_1^2/\lambda_c^2$ is plotted against $t^{(1)}$.

The basic deformation path $\lambda_1 = \lambda_2$ is clearly stable up to the bifurcation point $\lambda_1 = \lambda_c$ and unstable for $\lambda_1 > \lambda_c$. In order to assess the stability of the branches (with respect to in-plane incremental deformations) we observe first that (24) holds. After a little algebra, following substitution of (28) into the matrix (23) and use of (36), it is easy to show that (23) holds. However, since (27) holds with $\lambda_1 \neq \lambda_2$ the inequality (25) fails. Thus, the branches can be regarded as neutrally stable.

Whether or not branching occurs depends on the form of strain-energy function. To illustrate the point we note that for

$$W = \tfrac{1}{2}(\kappa' + \mu)(\lambda_1 + \lambda_2 - 2)^2 + \tfrac{1}{2}\mu(\lambda_1 - \lambda_2)^2$$

($\lambda_3 = 1$) there is no bifurcation point on the path $\lambda_1 = \lambda_2$. By contrast, for an incompressible isotropic elastic material, branching occurs for every sufficiently regular form of strain-energy function [1].

4. Some general results. At this point it is convenient to replace the variables λ_1, λ_2 by

$$J^* = \lambda_1\lambda_2, \qquad \lambda^* = \lambda_1 J^{*-1/2} \tag{39}$$

and to write

$$W^*(\lambda^*, J^*, \lambda_3) = W(\lambda^* J^{*1/2}, \lambda^{*-1} J^{*1/2}, \lambda_3). \tag{40}$$

The symmetry (9) leads to

$$W^*(\lambda^{*-1}, J^*, \lambda_3) = W^*(\lambda^*, J^*, \lambda_3). \tag{41}$$

Expressed in terms of λ^* and J^* the central equation in (27) becomes

$$(\lambda^* - \lambda^{*-1})J^* \frac{\partial W^*}{\partial J^*} = \frac{1}{2}(\lambda^{*2} + 1)\frac{\partial W^*}{\partial \lambda^*}, \tag{42}$$

and this is coupled with

$$t^{(1)} = J^{*-1/2}\frac{\partial W^*}{\partial \lambda^*} \Big/ (1 - \lambda^{*-2}). \tag{43}$$

On the fundamental path of deformation $\lambda^* = 1$ and the critical value J_c^* of J^* which defines the point of bifurcation is obtained from (42) by dividing by $\lambda^* - \lambda^{*-1}$ and taking the limit $\lambda^* \to 1$. This gives

$$J_c^* \frac{\partial W^*}{\partial J^*}(1, J_c^*, \lambda_3) = \frac{1}{2}\frac{\partial^2 W^*}{\partial \lambda^{*2}}(1, J_c^*, \lambda_3). \tag{44}$$

The corresponding critical value of $t^{(1)}$ is

$$t_c^{(1)} = J_c^{*1/2}\frac{\partial W^*}{\partial J^*}(1, J_c^*, \lambda_3). \tag{45}$$

With the help of (44) it is easy to show that (26) reduces to

$$\frac{\partial^2 W^*}{\partial \lambda^* \partial J^*}(1, J_c^*, \lambda_3) = 0. \tag{46}$$

This is automatically satisfied since, from (42), $\partial W^*/\partial\lambda^*(1, J^*, \lambda_3) \equiv 0$.

Differentiation of (41) three times shows that

$$\frac{\partial^3 W^*}{\partial \lambda^{*3}}(1, J_c^*, \lambda_3) = -3\frac{\partial^2 W^*}{\partial \lambda^{*2}}(1, J_c^*, \lambda_3) \tag{47}$$

and, from (42), by means of a limiting process and use of (47), we obtain

$$dJ^*/d\lambda^* = d^2J^*/d\lambda^{*2} = 0$$

at the critical point. Also, from (43) with the help of (47), it can be shown that $\partial t^{(1)}/\partial\lambda^* = 0$ and

$$\frac{\partial^2 t^{(1)}}{\partial \lambda^{*2}} = J_c^{*-1/2}\left\{\frac{1}{6}\frac{\partial^4 W^*}{\partial \lambda^{*4}}(1, J_c^*, \lambda_3) - 2\frac{\partial^2 W^*}{\partial \lambda^{*2}}(1, J_c^*, \lambda_3)\right\} \tag{48}$$

at the critical point.

The Taylor expansion of (43) near the critical point therefore yields

$$t^{(1)} = t_c^{(1)} + \tfrac{1}{2}(\lambda^* - 1)^2\frac{\partial^2 t^{(1)}}{\partial \lambda^{*2}} \tag{49}$$

to the second order in $|\lambda^* - 1|$, showing that two deformation branches emanate from the bifurcation point. If the expression (48) is positive (negative) then $t^{(1)} > t_c^{(1)}(t^{(1)} < t_c^{(1)})$ and the branches are neutrally stable (unstable).

Corresponding results for incompressible materials given in [1] are recovered by setting $J^* = \lambda_3^{-1}$ in (43) and (48) in particular.

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