ON THE EQUIPARTITION OF ENERGY IN A LINEAR VISCOELASTIC BODY*

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For the Glory of God

1. Introduction. In this paper, we consider a homogeneous linear viscoelastic filament which has occupied the interval $[0, a]$ at all times prior to time 0. Subsequently one end is held fixed, while the other is subjected to a given traction τ. Let $K(t)$ denote the kinetic energy at time t, and $\Psi(t)$ the free energy at time t defined by

$$\Psi(t) := \int_0^a \int_0^t \int_0^t g(2t - r - s)\dot{\varepsilon}(x, r)\dot{\varepsilon}(x, s)\, dr\, ds\, dx,$$

where g is the relaxation function, and $\dot{\varepsilon}$ is the time derivative of the strain ε. Then, we prove that the mean values of the kinetic and free energies are asymptotically equal, *i.e.*,

$$t^{-1}\int_0^t (K - \Psi) \to 0 \quad \text{as } t \to \infty,$$

provided the traction τ and its derivative $\dot{\tau}$ satisfy

$$\tau(t) \to 0 \quad \text{as } t \to \infty,$$

and $\int_0^\infty |\dot{\tau}(t)|\, dt < \infty$. This result generalizes to viscoelasticity, theorems on the equipartition of energy in elastic bodies due to Day [1] and Levine [2]. Two identities due to Brun [3] play a key role in our proof.

Our hypotheses on the relaxation function for the above result to be true, allow the filament to be elastic. However, if the filament is genuinely viscoelastic in a sense explained later, we are able to derive a stronger conclusion, *viz.*

$$\lim_{t\to\infty} K(t) = \lim_{t\to\infty} \Psi(t) = 0,$$

provided the integral

$$\int_0^\infty \left[\tau(t)^2 + \ddot{\tau}(t)^2\right] dt$$

converges. The proof relies upon estimates due to Day [4].

For simplicity, these results are stated and proved here for a homogeneous filament. But they can be extended with few difficulties to an inhomogeneous three-dimensional body,

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part of whose boundary is clamped while the remainder is subjected to a prescribed surface traction. The body may also be subjected to an external body force.

2. Preliminary results. For convenience, the notation and terminology chosen is almost identical to those of [4].

Consider a homogeneous linear viscoelastic filament, which occupies the interval $[0, a]$ in the reference configuration. The *mass density* ρ: $[0, a] \to (0, \infty)$ is continuous. The *relaxation function* g: $[0, \infty) \to \mathbf{R}$ is a prescribed twice continuously differentiable function which satisfies:

(α) $$g(t) \geq 0, \dot{g}(t) \leq 0, \ddot{g}(t) \geq 0 \quad \text{for all } t \geq 0,$$

(β) $$g(t) \to g(\infty) > 0 \quad \text{and} \quad \dot{g}(t) \to 0 \quad \text{as } t \to \infty.$$

The filament is said to be *genuinely viscoelastic* if g further satisfies:

(γ) the integral

$$\int_0^\infty \dot{g}(t)^2/\ddot{g}(t)\, dt$$

exists as a Lebesgue integral.

Here and elsewhere superimposed dots denote differentiation with respect to time, and primes indicate spatial derivatives.

The condition (γ) was introduced in [4], where it is noted that (γ) follows from the assumption that $t \mapsto \ln[g(t) - g(\infty)]$ is convex. Any relaxation function of the form

$$g(t) = g(\infty) + \sum_{i=1}^{n} \alpha_i \exp(-\mu_i t), \tag{2.1}$$

where $g(\infty)$, α_i and μ_i are strictly positive, obeys (α)–(γ).

A dynamic *viscoelastic process* with a quiescent past is an ordered triplet $[u, \varepsilon, \sigma]$ such that:

(a) u: $[0, a] \times \mathbf{R} \to \mathbf{R}$ is a continuous function whose restriction to $[0, a] \times [0, \infty)$ is of class $\mathcal{C}^2$, and

$$u = 0 \quad \text{on } [0, a] \times (-\infty, 0],$$
$$u(0, \cdot) = 0 \quad \text{on } [0, \infty);$$

(b) $\varepsilon = u'$;

(c) $\sigma(\cdot, t) = g(0)\varepsilon(\cdot, t) + \int_0^t \dot{g}(t - \tau)\varepsilon(\cdot, \tau)\, d\tau$ for every t in $\mathbf{R}$;

(d) the equation of motion

$$\rho \ddot{u} = \sigma',$$

holds on $(0, a) \times (0, \infty)$.

Here u is the *displacement*, ε is the *strain*, and σ is the *stress* in the viscoelastic process. The *traction* τ: $\mathbf{R} \to \mathbf{R}$ corresponding to $[u, \varepsilon, \sigma]$ is defined by $\tau := \sigma(a, \cdot)$.

A viscoelastic process $[u, \varepsilon, \sigma]$ is termed *smooth* if u restricted to $[0, a] \times [0, \infty]$ is of class $\mathcal{C}^3$.

In [2], the displacement problem of linear elastodynamics was considered in an abstract setting. Levine only required that u be piecewise $\mathcal{C}^1$, that the principle of virtual work and conservation of energy hold instead of the equation of motion, and that

$$u(0,\cdot) = u(a,\cdot) = 0 \quad \text{on } [0,\infty).$$

Corresponding to each viscoelastic process $[u, \varepsilon, \sigma]$, it is possible to define the following mappings on $\mathbf{R}$,

$$U(t) := \int_0^a \int_0^t \sigma\,\dot{\varepsilon}, \tag{2.2}$$

$$K(t) := \tfrac{1}{2}\int_0^a \rho\dot{u}(\cdot,t)^2, \tag{2.3}$$

$$E(t) := U(t) + K(t), \tag{2.4}$$

$$I(t) := \tfrac{1}{2}\int_0^a \rho u(\cdot,t)^2, \tag{2.5}$$

$$\Phi(t) := \int_0^a \varphi(\cdot,t), \tag{2.6}$$

$$\Psi(t) := \int_0^a \psi(\cdot,t), \tag{2.7}$$

where

$$\varphi(\cdot,t) := \tfrac{1}{2}g(\infty)\varepsilon(\cdot,t)^2 - \tfrac{1}{2}\int_{-\infty}^t \dot{g}(t-s)[\varepsilon(\cdot,t)-\varepsilon(\cdot,s)]^2\,ds, \tag{2.8}$$

and

$$\psi(\cdot,t) := \tfrac{1}{2}g(\infty)\varepsilon(\cdot,t)^2 + \tfrac{1}{2}\int_0^\infty\int_0^\infty \ddot{g}(r+s)[\varepsilon(\cdot,t)-\varepsilon(\cdot,t-r)][\varepsilon(\cdot,t)-\varepsilon(\cdot,t-s)]\,dr\,ds. \tag{2.9}$$

Here U is the *strain energy*, K is the *kinetic energy*, E is the *total energy*, Φ is *Volterra's free energy*, and Ψ is another *free energy*. It is also convenient to introduce the functions P: $[0,\infty)\times[0,\infty)\to\mathbf{R}$ *and* W: $[0,\infty)\times[0,\infty)\to\mathbf{R}$ by

$$P(r,s) := \tau(r)\dot{u}(a,s), \tag{2.10}$$

$$W(r,s) := \tau(r)u(a,s). \tag{2.11}$$

The function ψ, which appears in [3], can be rewritten as

$$\psi(\cdot,t) = \varepsilon(\cdot,t)\sigma(\cdot,t) - \tfrac{1}{2}g(0)\varepsilon(\cdot,t)^2 + \tfrac{1}{2}\int_0^t\int_0^t \ddot{g}(2t-r-s)\varepsilon(\cdot,r)\varepsilon(\cdot,s)\,dr\,ds, \tag{2.12}$$

$$= \int_0^t\int_0^t g(2t-r-s)\dot{\varepsilon}(\cdot,r)\dot{\varepsilon}(\cdot,s)\,dr\,ds. \tag{2.13}$$

ψ is the density of a free energy in the sense that

$$\psi(\cdot,t) \geqslant \tfrac{1}{2}g(\infty)\varepsilon(\cdot,t)^2, \tag{2.14}$$

and

$$\int_s^t \sigma(\cdot, r)\dot{\varepsilon}(\cdot, r)\, dr \geq \psi(\cdot, t) - \psi(\cdot, s),$$

for every s and t in $\mathbf{R}$, provided that for $k = 1, 2$,

$$(-1)^k \int_0^\infty \int_0^\infty g^{(k)}(r + s) f(r) f(s)\, dr\, ds \geq 0,$$

for every f in $\mathcal{C}[0, \infty)$ with compact support. These inequalities certainly hold if g has the form (2.1) with $g(\infty)$, α_i and μ_i positive. The requirement (2.14) for ψ to be a free energy is akin to Definition 3 of Day [5]. A mechanical interpretation of $\Psi(t)$ is given in the following result, whose proof is trivial.

PROPOSITION. Let $[u, \varepsilon, \sigma]$ be a viscoelastic process with

$$\varepsilon(\cdot, s) = \begin{cases} 0, & s \in (-\infty, 0] \cup [2t, \infty), \\ \varepsilon(\cdot, 2t - s), & s \in (0, 2t), \end{cases}$$

for some $t > 0$. Then the work done on the body is

$$\int_0^a \int_0^{2t} \sigma\, \dot{\varepsilon} = 2[U(t) - \Psi(t)].$$

Thus, $U(t) - \Psi(t)$ is half the work dissipated in the closed viscoelastic process obtained by retracing the displacement followed over $(-\infty, t)$. Compatibility with thermodynamics, whose implications for a linear viscoelastic body are investigated in [5], would demand that $U(t) - \Psi(t)$ be positive.

We now cite two important inequalities. The first is a special case of Theorem 256 of Hardy, Littlewod and Pôlya [6], while the second is a consequence of the Cauchy-Schwarz inequality.

LEMMA 1. Let f be in $\mathcal{C}^1[0, a]$ with $f(0) = 0$. Then

$$\int_0^a f^2 \leq \frac{4a^2}{\pi^2} \int_0^a f'^2, \tag{2.15}$$

and

$$f(a)^2 \leq a \int_0^s f'^2. \tag{2.16}$$

For the sake of conciseness in our later proofs, we record, as in [4], the following elementary inequality.

LEMMA 2. If $\theta \geq 0$ and $\theta \leq \xi + \eta\theta^{1/2}$, then $\theta \leq 2\xi + \eta^2$.

Next we present two important identities, which form the basis of a strong uniqueness theorem in elastodynamics that requires no positivity assumption on the elasticity tensor (cf., e.g., [7], Sec. 63). The second identity can be found in [3].

Henceforth, we shall usually surpress the spatial argument of functions appearing in integrands.

LEMMA 3. Let $[u, \varepsilon, \sigma]$ be a viscoelastic process. Then, for every $t \geqslant 0$,

$$\int_0^t [\Psi - K] = \tfrac{1}{4}\int_0^t [W(t+s, t-s) + 2W(s,s) - W(t-s, t+s)]\, ds + \int_0^a u(2t)\dot{u}(0), \tag{2.17}$$

and

$$\Psi(t) - K(t) = \tfrac{1}{2}\int_0^t [P(t+s, t-s) - P(t-s, t+s)]\, ds - \tfrac{1}{2}\int_0^a \rho\dot{u}(0)\dot{u}(2t). \tag{2.18}$$

Proof. The equation of motion (d) implies that

$$\rho\ddot{u}(\cdot, r)u(\cdot, s) = \sigma'(\cdot, r)u(\cdot, s),$$

for all r, s in $[0, \infty)$. Integrate this over $[0, a]$ and then integrate by parts, to see that

$$W(r, s) = \int_0^a [\rho\ddot{u}(r)u(s) + \sigma(r)\varepsilon(s)]. \tag{2.19}$$

Another integration by parts shows that

$$\int_0^a \rho \int_0^t [\ddot{u}(t+s)u(t-s) - \ddot{u}(t-s)u(t+s)]\, ds = \int_0^a \rho[\dot{u}(0)u(2t) - 2u(t)\dot{u}(t)]. \tag{2.20}$$

Also a short calculation using Fubini's Theorem demonstrates that

$$\int_0^t [\sigma(t+s)\varepsilon(t-s) - \sigma(t-s)\varepsilon(t+s)]\, ds = \int_0^t \int_0^t \dot{g}(2t - r - s)\varepsilon(r)\varepsilon(s)\, dr\, ds. \tag{2.21}$$

By integrating (2.12) over $(0, t)$, it is quickly established that

$$4\int_0^t \psi - 2\int_0^t \varepsilon\,\sigma = \int_0^t \int_0^t \dot{g}(2t - r - s)\varepsilon(r)\varepsilon(s)\, dr\, ds. \tag{2.22}$$

It now follows from (2.19)–(2.22) that

$$\begin{aligned} 4\int_0^t \Psi - 2\int_0^t \int_0^a \varepsilon\sigma = {} & \int_0^t [W(t+s, t-s) - W(t-s, t+s)]\, ds \\ & - \int_0^a \rho[u(2t)\dot{u}(0) - 2u(t)\dot{u}(t)]. \end{aligned} \tag{2.23}$$

Next we infer from (2.19) that

$$\int_0^t W(s, s)\, ds = \int_0^t \int_0^a [\rho\ddot{u}\, u + \sigma\,\varepsilon],$$

which, on integrating by parts becomes

$$\int_0^a \rho u(t)\dot{u}(t) = \int_0^t W(s, s)\, ds + \int_0^t \left[2K - \int_0^a \varepsilon\,\sigma\right]. \tag{2.24}$$

(2.17) now follows by substituting the right-hand side of (2.24) into (2.23).

The proof of (2.18) is similar. An argument similar to that which established (2.19) shows that

$$P(r, s) = \int_0^a [\rho\ddot{u}(r)\dot{u}(s) + \sigma(r)\dot{\varepsilon}(s)]. \tag{2.25}$$

Using Fubini's Theorem and (2.3), we see that

$$\int_0^t \int_0^a \rho[\ddot{u}(t+s)\dot{u}(t-s) - \dot{u}(t+s)\ddot{u}(t-s)]\,ds = \int_0^a \rho \int_0^t \frac{d}{ds}\dot{u}(t+s)\dot{u}(t-s)\,ds,$$
$$= \int_0^a \rho\dot{u}(0)\dot{u}(2t) - 2K(t).$$

This equation and (2.25) imply that

$$\begin{aligned}\int_0^a \int_0^t [\sigma(t+s)\dot{\varepsilon}(t-s) - \sigma(t-s)\dot{\varepsilon}(t+s)]\,ds - 2K(t) \\ = \int_0^t [P(t+s, t-s) - P(t-s, t+s)]\,ds - \int_0^a \rho\dot{u}(0)\dot{u}(2t).\end{aligned} \tag{2.26}$$

Integration by parts of (c) shows that the stress is also given by

$$\sigma(\cdot, t) = \int_0^t g(t-s)\dot{\varepsilon}(\cdot, s)\,ds.$$

This expression and Fubini's Theorem allow us to calculate that

$$2\Psi(t) = \int_0^a \int_0^t [\sigma(t+s)\dot{\varepsilon}(t-s) - \sigma(t-s)\dot{\varepsilon}(t+s)]\,ds. \tag{2.27}$$

To complete the proof notice that (2.26) and (2.27) imply the desired result.

(2.17) can be established from the principle of virtual work by mimicking the proof of Lemma 3.1 in [2]. ■

We state without proof the conversation of energy.

LEMMA 4. Let $[u, \varepsilon, \sigma]$ be a viscoelastic process. Then, for all $t \geqslant 0$,

$$E(t) = E(0) = \int_0^t \tau\dot{u}(a, \cdot). \tag{2.28}$$

LEMMA 5. Let $[u, \varepsilon, \sigma]$ be a viscoelastic process. Then, for each $t \geqslant 0$,

$$\int_0^a \varepsilon(\cdot, t)^2 \leqslant \frac{2}{g(\infty)} U(t). \tag{2.29}$$

Proof. An easy calculation using (2.8) and (a) shows that

$$\dot{\varphi}(\cdot, t) = \dot{\varepsilon}(\cdot, t)\sigma(\cdot, t) - \tfrac{1}{2}\int_{-\infty}^t \int_{-\infty}^t \ddot{g}(t-s)[\varepsilon(\cdot, t) - \varepsilon(\cdot, s)]^2\,ds.$$

Since $\ddot{g} \geqslant 0$, we deduce that

$$\dot{\varphi}(\cdot, t) \leqslant \dot{\varepsilon}(\cdot, t)\sigma(\cdot, t).$$

It follows from this inequality (2.2), (2.6) and the fact that $\varphi(\cdot, 0) = 0$ that

$$\Phi(t) = \int_0^a \varphi(\cdot, t) \leqslant \int_0^a \int_0^t \dot{\varepsilon}\,\sigma = U(t). \tag{2.30}$$

Due to (α),

$$\Phi(t) \geqslant \tfrac{1}{2} g(\infty) \int_0^a \varepsilon(\cdot, t)^2. \tag{2.31}$$

The desired result is an immediate consequence of (2.30) and (2.31). ■

The following result is almost identical to Lemma 5 of [4].

LEMMA 6. Let $[u, \varepsilon, \sigma]$ be a viscoelastic process. Then, for each $t \geqslant 0$,

$$I(t) \leqslant \gamma \int_0^t E. \tag{2.32}$$

where

$$\gamma := \max\left\{1, \frac{4a^2}{g(\infty)\pi^2} \max_{[0,a]} \rho\right\}. \tag{2.33}$$

Proof. It is easily seen using (2.5), (2.15) and (2.29) that

$$\begin{aligned} I(t) &= \tfrac{1}{2} \int_0^a \rho u(\cdot, t)^2, \\ &\leqslant \frac{2a^2}{\pi^2} \max \rho \int_0^a \varepsilon(\cdot, t)^2, \\ &\leqslant \frac{4a^2}{g(\infty)\pi^2} \max \rho \, U(t). \end{aligned} \tag{2.34}$$

The derivative of I is given by

$$\dot{I}(t) = \int_0^a \rho u(t) \dot{u}(t),$$

so that

$$I(t) = \int_0^t \int_0^a \rho u \, \dot{u}.$$

The Cauchy-Schwarz inequality, the arithmetic-geometric mean inequality, (2.34) and (2.5) enable us to see that

$$\begin{aligned} I(t) &\leqslant \int_0^t \left| \int_0^a \rho u \, \dot{u} \right| \leqslant 2 \int_0^t I^{1/2} K^{1/2}, \\ &\leqslant \int_0^t (I + K) \leqslant \gamma \int_0^t (U + K), \\ &= \gamma \int_0^t E, \end{aligned}$$

where γ is given by (2.33). ■

The final result of this section can be found as (2.10) and (3.6) of Day [4].

LEMMA 7. Suppose that the filament is genuinely viscoelastic. Let $[u, \varepsilon, \sigma]$ be a smooth viscoelastic process. Then there is a material constant κ, such that

$$\int_0^t \int_0^a \dot{\varepsilon}^2 \leqslant \kappa \left[1 + \int_0^t (\tau^2 + \dot{\tau}^2 + \ddot{\tau}^2)\right], \tag{2.35}$$

for all $t \geq 0$. Moreover, if

$$\int_0^\infty (\tau^2 + \ddot{\tau}^2) < \infty,$$

then

$$K(t) \to 0 \quad \text{as } t \to \infty. \tag{2.36}$$

Lastly we record an inequality which can be found in [6] as Theorem 260.

LEMMA 8. Let $\zeta\colon (0, \infty) \to \mathbf{R}$ be of class $\mathcal{C}^2$. Then

$$\int_0^\infty \dot{\zeta}^2 \leq \int_0^\infty (\zeta^2 + \ddot{\zeta}^2). \tag{2.37}$$

3. Main results.

THEOREM 1. Let $[u, \varepsilon, \sigma]$ be a viscoelastic process where associated traction τ satisfies

$$\tau(t) \to 0 \quad \text{as } t \to \infty, \tag{3.1}$$

and

$$\int_0^\infty |\dot{\tau}| < \infty. \tag{3.2}$$

Then the total energy E remains bounded, the limit

$$\lim_{t \to \infty} E(t) \tag{3.3}$$

exists, and

$$t^{-1}\int_0^t (K - \Psi) \to 0 \quad \text{as } t \to 0. \tag{3.4}$$

Proof. It will firstly be shown that E is bounded. Integration by parts of (2.28) gives

$$E(t) = E(0) + \tau(t)u(a, t) - \int_0^t \dot{\tau}(s)u(a, s)\, ds. \tag{3.5}$$

Hence by (2.16), (2.29) and (2.4),

$$\begin{aligned} E(t) &\leq E(0) + a^{1/2}|\tau(t)|\left(\int_0^a \varepsilon(\cdot, t)^2\right)^{1/2} + a^{1/2}\int_0^t |\dot{\tau}(s)|\left(\int_0^a \varepsilon(\cdot, s)^2\right)^{1/2} ds \\ &\leq E(0) + \left(\frac{2a}{g(\infty)}\right)^{1/2} |\tau(t)|E(t)^{1/2} + \left(\frac{2a}{g(\infty)}\right)^{1/2} \int_0^t |\dot{\tau}(s)|E(s)^{1/2}\, ds. \end{aligned}$$

Now we apply Lemma 2 and obtain

$$\begin{aligned} E(t) &\leq 2E(0) + \frac{2a}{g(\infty)}\tau(t)^2\left(\frac{8a}{g(\infty)}\right)^{1/2} \int_0^t |\dot{\tau}(s)|E(s)^{1/2}\, ds \\ &\leq 2E(0) + \frac{2a}{g(\infty)}\tau(t)^2 + \left(\frac{8a}{g(\infty)}\right)^{1/2} \left(\max_{[0, t]} E\right)^{1/2} \int_0^t |\dot{\tau}|, \end{aligned}$$

and therefore,

$$\max_{[0,t]} E \leqslant 2E(0) + \frac{2a}{g(\infty)} \max_{[0,t]} \tau^2 + \left(\frac{8a}{g(\infty)}\right)^{1/2} \left(\max_{[0,t]} E\right)^{1/2} \int_0^t |\dot{\tau}|.$$

Now use Lemma 2 again to see that

$$\max_{[0,t]} E \leqslant 4E(0) + \frac{8a}{g(\infty)} \left\{ \max_{[0,t]} \tau^2 + \left[\int_0^t |\dot{\tau}|\right]^2 \right\}.$$

Due to (3.1) and (3.2), this implies that

$$\sup_{[0,\infty)} E < \infty. \tag{3.6}$$

Because E is positive, this establishes that E is bounded.

Note that (2.16), (2.29) and (3.6) show that

$$\sup_{[0,\infty)} |u(a,\cdot)| \leqslant \nu, \tag{3.7}$$

where $\nu^2 := 2ag(\infty)^{-1} \sup_{[0,\infty)} E$. Let $\eta > 0$. By (3.1) and (3.2), there is a time $t_\eta > 0$ such that

$$|\tau(t)| < \eta/3\nu, \tag{3.8}$$

for all $t > t_\eta$, and

$$\int_s^t |\dot{\tau}| < \eta/3\nu, \tag{3.9}$$

for all s, $t > t_\eta$. In view of (3.5),

$$E(t) - E(s) = \tau(t)u(a,t) - \tau(s)u(a,s) - \int_s^t \dot{\tau} u(a,\cdot).$$

thus (3.7)–(3.9) imply that

$$\begin{aligned} |E(t) - E(s)| &\leqslant |\tau(t)|\,|u(a,t)| + |\tau(s)|\,|u(a,s)| + \int_s^t |\dot{\tau}|\,|u(a,\cdot)| \\ &\leqslant \nu\left[|\tau(t)| + |\tau(s)| + \int_s^t |\dot{\tau}|\,|u(a,\cdot)|\right] \\ &< \eta, \end{aligned}$$

whenever $t > s > t_\eta$. This establishes that the limit (3.3) exists.

Finally we proceed to prove (3.4). By (2.11) and (3.7),

$$\begin{aligned} \left| t^{-1} \int_0^t W(t+s, t-s)\, ds \right| &\leqslant t^{-1} \int_0^t |\tau(t+s)|\,|u(a, t-s)|\,ds \\ &\leqslant \nu t^{-1} \int_0^t |\tau(t+s)|\,ds = \nu t^{-1} \int_t^{2t} |\tau| \\ &= \nu\left(2\frac{1}{2t} \int_0^{2t} |\tau| - \frac{1}{t} \int_0^t |\tau|\right) \to 0 \end{aligned} \tag{3.10}$$

as $t \to \infty$.The right-hand side tends to zero, since (3.1) implies that

$$\frac{1}{t}\int_0^t |\tau| \to 0 \quad \text{as } t \to \infty .$$

Similarly, it is easy to show that

$$t^{-1}\int_0^t W(s, s)\, ds \to 0 \quad \text{as } t \to 0, \tag{3.11}$$

and

$$t^{-1}\int_0^t W(t - s, t + s)\, ds \to 0 \quad \text{as } t \to \infty . \tag{3.12}$$

Also, the Cauchy-Schwarz inequality, (2.15), (2.29), (2.4) and (3.6) enable us to deduce that

$$\begin{aligned}\left| t^{-1}\int_0^a \rho\dot{u}(\cdot,0)u(\cdot,2t)\right| &\leqslant t^{-1}\left(\int_0^a \rho^2\dot{u}(\cdot,0)^2\right)^{1/2}\left(\int_0^a u(\cdot,2t)^2\right)^{1/2} \\ &\leqslant \frac{2a}{\pi t}\left(\int_0^a \rho^2\dot{u}(\cdot,0)^2\right)^{1/2}\left(\int_0^a \varepsilon(\cdot,2t)^2\right)^{1/2} \\ &\leqslant \frac{1}{t}\left(\frac{8a^2}{\pi^2 g(\infty)}\right)^{1/2}\left(\int_0^a \rho^2\dot{u}(\cdot,0)^2\right)^{1/2} E(2t)^{1/2} \to 0 \end{aligned} \tag{3.13}$$

as $t \to \infty$. (3.4) now follows from (3.10)–(3.13) and the identity (2.17). ■

Part of the conclusion of Theorem 1 is that E is bounded. Theorem 2 tells us that (3.4) holds merely if the total energy is *subbounded* in $\mathfrak{L}^1$ *i.e.*,

$$\sup_{t\geqslant 0} t^{-1}\int_0^t |E| < \infty . \tag{3.14}$$

THEOREM 2. Let $[u, \varepsilon, \sigma]$ be a viscoelastic process whose total energy is subbbounded, and

$$t^{-1}\int_0^t |\tau|^2 \to 0 \tag{3.15}$$

as $t \to \infty$. Then (3.4) holds.

Proof. We show that each of the limits (3.10)–(3.13) hold under the hypotheses stated above.

By (2.11), (2.16), (2.29), (2.4) and the Cauchy-Schwarz inequality,

$$\begin{aligned}\left| t^{-1}\int_0^t W(t + s, t - s)\, ds\right| &\leqslant t^{-1}\int_0^1\int_0^t |\tau(t + s)|\, |u(a, t - s)| ds \\ &\leqslant a^{1/2}t^{-1}\int_0^t |\tau(t + s)|\left(\int_0^a \varepsilon(t - s)^2\right)^{1/2} ds \\ &\leqslant t^{-1}(2a/g(\infty))^{1/2}\int_0^t |\tau(t + s)| E(t - s)^{1/2}\, ds \\ &\leqslant (2a/g(\infty))^{1/2}\left(t^{-1}\int_0^t E\right)^{1/2}\left(t^{-1}\int_t^{2t} |\tau|^2\right)^{1/2}. \end{aligned}$$

The first two terms in parentheses are bounded, while the last term tends to zero due to (3.15). Thus (3.10) holds.

Similar arguments establish the inequalities

$$\left|t^{-1}\int_0^t W(s,s)\,ds\right| \leq (2a/g(\infty))^{1/2}\left(t^{-1}\int_0^t E\right)^{1/2}\left(t^{-1}\int_0^t |\tau|^2\right)^{1/2},$$

$$\left|t^{-1}\int_0^t W(t-s,t+s)\,ds\right| \leq (2a/g(\infty))^{1/2}\left(t^{-1}\int_0^{2t} E\right)^{1/2}\left(t^{-1}\int_0^t |\tau|^2\right)^{1/2}.$$

We conclude from these that (3.11) and (3.12) are true using (3.14) and (3.15).

Finally, we apply the Cauchy-Schwarz inequality and (2.32) to conclude that

$$\begin{aligned} t^{-1}\left|\int_0^s \rho\dot{u}(0)u(2t)\right| &\leq 2t^{-1}K(0)^{1/2}I(2t)^{1/2} \\ &\leq 2\gamma^{1/2}K(0)t^{-1/2}\left(t^{-1}\int_0^{2t} E\right)^{1/2} \to 0. \end{aligned}$$

as $t \to \infty$ using (3.14). ■

The assumptions (α), (β) on the relaxation function allow the filament to be elastic, and therefore cannot exclude resonance. It is not surprising that the dampening in a genuinely viscoelastic filament enables us to strengthen Theorem 1.

THEOREM 3. Suppose that the filament is genuinely viscoelastic. Let $[u, \varepsilon, \sigma]$ be a smooth viscoelastic process whose associated traction τ satisfies

$$\int_0^\infty (\tau^2 + \ddot{\tau}^2) < \infty. \tag{3.16}$$

Then

$$\lim_{t\to\infty} K(t) = \lim_{t\to\infty} \Psi(t) = 0.$$

Proof. The theorem will be proved by demonstrating that each of the terms in (2.18) tends to zero.

It follows from (2.10) and (2.16) that

$$\begin{aligned} \left|\int_0^t P(t+s,t-s)\,ds\right| &\leq \int_0^t |\tau(t+s)|\,|\dot{u}(a,t-s)|ds \\ &\leq \left(\int_0^t \tau(t+s)^2\,ds\right)^{1/2}\left(\int_0^t |\dot{u}(a,t-s)|^2\,ds\right)^{1/2} \\ &\leq a^{1/2}\left(\int_t^{2t}\tau^2\right)^{1/2}\left(\int_0^t\int_0^a \dot{\varepsilon}^2\right)^{1/2}. \end{aligned} \tag{3.17}$$

But due to (2.35), (2.37) and (3.16),

$$\int_0^\infty \int_0^a \dot{\varepsilon}^2 < \infty, \tag{3.18}$$

and, by (3.16),

$$\int_t^{2t} \tau^2 \to 0 \quad \text{as } t \to \infty. \tag{3.19}$$

It is an easy consequence of (3.17)–(3.19) that

$$\int_0^t P(t+s, t-s)\, ds \to 0 \quad \text{as } t \to \infty. \tag{3.20}$$

Similarly, it can be shown that

$$\int_0^t P(t-s, t+s)\, ds \to 0 \quad \text{as } t \to \infty. \tag{3.21}$$

The Cauchy-Schwarz inequality and (2.36) imply that

$$\frac{1}{2}\left|\int_0^a \rho \dot{u}(0)\dot{u}(2t)\right| \leq K(0)K(2t) \to 0 \quad \text{as } t \to \infty. \tag{3.22}$$

(2.18), (3.20), (3.21) and (3.22) now yield the desired result. ■

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