STABILITY CONDITIONS FOR LINEAR NON-AUTONOMOUS DELAY DIFFERENTIAL EQUATIONS*

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Abstract. We derive new sufficient conditions for uniform asymptotic stability of the zero solution of linear non-autonomous delay differential equations. The equations considered include scalar equations of the form

$$x'(t) = -c(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t - T_i)$$

where $c(t)$, $b_i(t)$ are continuous for $t \geqslant 0$ and T_i is a positive number ($i = 1, 2, \ldots, n$), and also systems of the form

$$x'(t) = B(t)x(t - T) - C(t)x(t)$$

where $B(t)$ and $C(t)$ are $n \times n$ matrices. The results are found by using the method of Lyapunov functionals.

1. Scalar equations with a single delay. The purpose of this paper is to derive some new sufficient conditions for stability of linear delay differential equations. We first consider the scalar equation

$$x'(t) = b(t)x(t - T) - c(t)x(t) \tag{1}$$

where b and c are given continuous functions and T is a positive constant. Extensions to scalar equations with several delays and to systems of equations are given in Secs. 2 and 3.

The simplest available sufficient condition for asymptotic stability is contained in the following theorem of Hale [5, page 108].

THEOREM 1. Suppose that b and c are bounded continuous functions on $\mathbf{R}$ and satisfy
(i) $c(t) \geqslant \delta > 0$ for all t, and
(ii) $|b(t)| \leqslant \theta\delta$ for all t, and for some θ, $0 \leqslant \theta < 1$.
Then, the zero solution of (1) is uniformly asymptotically stable.

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In this result, the function c is required to dominate the function $|b|$ in the very strong sense that the supremum of $|b|$ must be less than the infimum of c. Some such condition is needed, since if b and c are constants and $b \geqslant 0$, then $b < c$ is necessary for stability. In the theorems that we give here the hypotheses on b and c are less stringent. For example, when b and c are periodic with period T, the hypothesis $|b(t)| < c(t)$ suffices. This can also be shown to hold in more general circumstances by applying a stability theorem of Dyson and Villella-Bressan [4].

Our results are obtained by using certain simple Lyapunov functionals $V(t, \phi)$ rather than the autonomous functionals $V(\phi)$ used in proving Thm. 1. Although the theory of Lyapunov functionals has been extensively developed for autonomous equations, for example by Carvalho, Infante and Walker [3], a similar development is still lacking for non-autonomous equations.

Our first result for Eq. (1) is contained in the following theorem.

THEOREM 2. Suppose that b and c are continuous and assume that the following conditions are satisfied:

(a) Given $\eta > 0$ there exists $\tau > 0$ such that

$$\int_t^{t+\tau} |b(s)|\,ds < \eta \quad \text{for } t \geqslant 0$$

(and consequently for some $B > 0$

$$\int_{-T}^{0} |b(t + T + \theta)|\,d\theta \leqslant B < \infty,$$

$t \geqslant 0$).

(b) There exist $a > 0$ and $q > 0$ such that

$$2c(t) - a|b(t)| - |b(t + T)|/a \geqslant q \quad \text{for } t \geqslant 0.$$

Then the zero solution of (1) is uniformly asymptotically stable.

Proof. The proof consists in applying the Lyapunov theorem for functional differential equations given in Sec. 4 with a Lyapunov function $V\colon \mathbf{R} \times C \to C$ of the form

$$V(t, \phi) = a\phi^2(0) + \int_{-T}^{0} K(t + \theta)\phi^2(\theta)\,d\theta$$

where K is a continuous function, $K\colon \mathbf{R} \to \mathbf{R}$, to be chosen later. Let $x(s, \phi)$ denote the solution of (1) satisfying $x_s = \phi$ and, for simplicity, let $x(t)$ denote the value of $x(s, \phi)$ at t. Then

$$\begin{aligned}\dot{V}(t, \phi) &= \overline{\lim_{h \downarrow 0}}\ \frac{1}{h}\left[V(t + h, x_{t+h}(t, \phi)) - V(t, \phi)\right]\\ &= \frac{d}{dt} a x^2(t)\\ &\quad + \overline{\lim_{h \downarrow 0}}\ \frac{1}{h}\left\{\int_{-T+h}^{h} K(t + \theta)x^2(t + \theta)\,d\theta - \int_{-T}^{0} K(t + \theta)x^2(t + \theta)\,d\theta\right\}\\ &= 2ax(t)x'(t) + K(t)x^2(t) - K(t - T)x^2(t - T).\end{aligned}$$

Since x satisfies (1), we have

$$\dot{V}(t,\phi) = [K(t) - 2ac(t)]\phi^2(0) + 2ab(t)\phi(0)\phi(-T) - K(t-T)\phi^2(-T). \quad (2)$$

Letting $K(t) = |b(t+T)|$ in (2), we note that the discriminant of the resulting quadratic form is

$$4a^2b^2(t) + 4|b(t)|[|b(t+T)| - 2ac(t)] = 4|b(t)|[a^2|b(t)| + |b(t+T)| - 2ac(t)] \leqslant -4aq|b(t)|, \quad (3)$$

the inequality following from condition (b). Now, whenever $|b(t)| \geqslant q/8a$, we see from (3) that the quadratic form (2) is negative definite (uniformly for all such t). Hence, there exists a constant $\alpha_1 > 0$, such that $\dot{V}(t,\phi) \leqslant -\alpha_1\phi^2(0)$ for all t where $|b(t)| \geqslant q/8a$. However, if $|b(t)| < q/8a$, we have from (2) with $K(t) = |b(t+T)|$:

$$\begin{aligned}\dot{V}(t,\phi) &= [|b(t+T)| - 2ac(t)]\phi^2(0) + 2ab(t)\phi(0)\phi(-T) - |b(t)|\phi^2(-T)\\ &\leqslant -aq\phi^2(0) + 2ab(t)\phi(0)\phi(-T) - |b(t)|\phi^2(-T) \quad (4)\end{aligned}$$

since from (b) we have $|b(t+T)| - 2ac(t) \leqslant -aq$. Now, if $2ab(t)\phi(0)\phi(-T) - |b(t)|\phi^2(-T) > 0$, then $2a|\phi(0)| > |\phi(-T)|$, hence

$$\begin{aligned}2ab(t)\phi(0)\phi(-T) - |b(t)|\phi^2(-T)| &< 4a^2|b(t)|\phi^2(0) - |b(t)|\phi^2(-T)\\ &< 4a^2|b(t)|\phi^2(0) < \frac{qa}{2}\phi^2(0).\end{aligned}$$

Using this in (4) we obtain

$$\dot{V}(t,\phi) \leqslant -\frac{aq}{2}\phi^2(0), \quad \text{whenever } |b(t)| < q/8a.$$

Letting $\alpha = \min(\alpha_1, aq/2)$, we see that $\dot{V}(t,\phi) \leqslant -\alpha\phi^2(0)$ for all $t \geqslant 0$ and all $\phi \in C$. Moreover, the inequalities

$$a\phi^2(0) \leqslant V(t,\phi) \leqslant (B+a)|\phi|_\infty^2$$

follow directly from (a) and the definition of V; and the zero solution is asymptotically stable. This completes the proof of the theorem.

Remark. The condition (a) can hold even when $c(t) - b(t) \geqslant q > 0$ and $c(t) - b(t+T) \geqslant q > 0$ fail to hold. In fact if we take $a = 1$ and $q = 1/2$ in condition (a), we see that it holds for the special case

$$T = 3, \qquad c(t) \equiv 1,$$

$$b(t) = \begin{cases} \frac{3}{2}[1 - |6n + 3 - t|], & t \in (6n+2, 6n+4), n = 0, \pm 1, \pm 2, \ldots,\\ 0, & \text{otherwise.}\end{cases}$$

However, $c(6n+3) - |b(6n+3)| = -1/2 < 0$, and $c(6n) - |b(6n+3)| = -1/2 < 0$. Note that the stability conditions of Dyson and Villella-Bressan [4] when applied to Eq. (1) require that $c(t) - b(t) \geqslant q > 0$.

Theorem 2 has some immediate corollaries that are worth stating because they deal with situations that are frequently encountered in applications.

COROLLARY 1. Suppose that c is continuous and b is continuous and periodic of period T. Then, if there exists $q > 0$ such that

$$c(t) - |b(t)| \geqslant q, \qquad t > 0,$$

the zero solution of (1) is uniformly asymptotically stable.

Note that if b and c are constants, then condition (a) with $a = 1$ reduces to $c - |b| > 0$. This is the best possible stability condition regardless of the size of the delay T in this case ([5], page 108). So, in this sense, the condition (a) is also the best possible condition of this type.

COROLLARY 2. Assume that b and c are continuous and that:

(b) There exists $\lambda \in (0,1)$ such that $|b(t)| \leqslant \lambda c(t)$, $t \geqslant 0$,

(c) $c(t) \geqslant c_1 > 0$, and either $c(t)$ is non-increasing or $|b(t)|$ is non-increasing.

Then the zero solution of (1) is uniformly asymptotically stable.

The above results were obtained by choosing $K(t) = |b(t+T)|$ in (2). If different choices of K are taken, then other stability conditions can be obtained. For example, we shall prove the following theorem by choosing $K(t) = b^2(t+T)$.

THEOREM 3. The results of Theorem 2 hold provided that

(a′) $2ac(t) - b^2(t+T) - a^2 \geqslant q$, for some $a > 0$, $q > 0$, and

(b′) $\int_t^{t+T} b^2(s)\,ds$ is bounded and given $\eta > 0$ there exists $\tau > 0$ such that

$$\int_t^{t+\tau} |b(s)|\,ds < \eta$$

for $t \geqslant 0$.

Proof. If $K(t) = b^2(t+T)$, then (2) has the form

$$\dot V(t,\phi) = \left[b^2(t+T) - 2ac(t)\right]\phi^2(0) + 2ab(t)\phi(0)\phi(-T) - b^2(t)\phi^2(-T).$$

Using (a′), we obtain

$$\begin{aligned}\dot V(t,\phi) &\leqslant -(a^2+q)\phi^2(0) + 2ab(t)\phi(0)\phi(-T) - b^2(t)\phi^2(-T)\\ &\leqslant -q\phi^2(0) - \left[a\phi(0) - b(t)\phi(-T)\right]^2\\ &\leqslant -q\phi^2(0),\end{aligned}$$

for all $\phi \in C$. Moreover, $V(t,\phi) \geqslant a\phi^2(0)$ and

$$\begin{aligned}V(t,\phi) &\leqslant a\phi^2(0) + |\phi|_\infty^2 \int_{-T}^{0} b^2(t+T+\theta)\,d\theta\\ &\leqslant a\phi^2(0) + |\phi|_\infty^2 \int_t^{t+T} b^2(s)\,ds.\end{aligned}$$

By condition (b′), there is a constant B such that

$$V(t,\phi) \leqslant B|\phi|_\infty^2.$$

As for Thm. 2, uniform asymptotic stability follows from Theorem 8, and the theorem is proved.

A special case occurs again when b is periodic of period T. Then conditions (a′) and (b′) are implied by the single condition

$$2ac(t) - b^2(t) - a^2 > 0, \qquad 0 \leqslant t \leqslant T.$$

As a final example, we examine the consequences of choosing

$$K(t) = b^2(t+T)/c(t+T),$$

as was done in [2].

THEOREM 4. Assume that b and c are continuous and that the following conditions hold.

(a″) There is a constant λ such that $b^2(t+T)/c(t)c(t+T) \leqslant \lambda < 1$ for $t \geqslant 0$,

(b″) $\int_t^{t+T} b^2(s)\,ds$ is bounded and given $\eta > 0$ there exists $\tau > 0$ such that

$$\int_t^{t+\tau} |b(s)|\,ds < \eta$$

for $t \geqslant 0$.

(c″) There is a constant c_1 such that $c(t) \geqslant c_1 > 0$ for $t \geqslant 0$.

Then the zero solution of (1) is uniformly asymptotically stable.

Proof. If $K(t) = b^2(t+T)/c(t+T)$, then (2) has the form

$$\dot V(t,\phi) = \left[\frac{b^2(t+T)}{c(t+T)} - 2ac(t)\right]\phi^2(0) + 2ab(t)\phi(0)\phi(-T) - \frac{b^2(t)}{c(t)}\phi^2(-T).$$

From (a″) and (c″) we get

$$\begin{aligned}\dot V(t,\phi) &\leqslant (\lambda - 2a)c(t)\phi^2(0) + 2ab(t)\phi(0)\phi(-T) - \frac{b^2(t)}{c(t)}\phi^2(-T)\\ &= -\left[(2a-\lambda)c^2(t)\phi^2(0) - 2ab(t)c(t)\phi(0)\phi(-T) + b^2(t)\phi^2(-T)\right]/c(t).\end{aligned}$$

Choosing $a = 1$, we have, for all $\phi \in C$,

$$\begin{aligned}\dot V(t,\phi) &\leqslant -(1-\lambda)c(t)\phi^2(0) - \left[c(t)\phi(0) - b(t)\phi(-T)\right]^2/c(t)\\ &\leqslant -(1-\lambda)c(t)\phi^2(0) \leqslant -(1-\lambda)c_1\phi^2(0).\end{aligned}$$

Moreover,

$$\phi^2(0) \leqslant V(t,\phi) \leqslant |\phi|_\infty^2\left(1 + \int_t^{t+T}\frac{b^2(s)}{c(s)}\,ds\right) \leqslant B|\phi|_\infty^2$$

and the proof is completed.

2. Scalar equations with several delays. The analysis of the previous section can be directly generalized to cover equations with several delays of the form

$$x'(t) = -c(t)x(t) + \sum_{i=1}^{N} b_i(t)x(t-T_i) \tag{5}$$

where $T_i > 0$ is a positive constant ($i = 1,2,\dots,N$). We use the functional

$$V(t,\phi) = \phi^2(0) + \sum_{i=1}^{N}\int_{-T_i}^{0} K_i(t+\theta)\phi^2(\theta)\,d\theta,$$

where K_i are continuous functions to be chosen below. A calculation of the same sort as in Sec. 1 yields

$$\dot{V}(t,\phi) = \left[-2c(t) + \sum_{i=1}^{N} K_i(t)\right]\phi^2(0) + 2\phi(0)\sum_{i=1}^{N} b_i(t)\phi(-T_i) - \sum_{i=1}^{N} K_i(t-T_i)\phi^2(-T_i). \tag{6}$$

When $-\dot{V}$ is viewed as a quadratic form in $\phi(0)$ and $\phi(-T_i)$, $i = 1,2,\ldots,N$, it has the following associated symmetric matrix

$$M = \begin{bmatrix} 2c(t) - \sum_{i=1}^{N} K_i(t) & -b_1(t) & -b_2(t) & \cdots & -b_N(t) \\ -b_1(t) & K_1(t-T_1) & 0 & \cdots & 0 \\ -b_2(t) & 0 & K_2(t-T_2) & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ -b_N(t) & 0 & 0 & \cdots & K_N(t-T_N) \end{bmatrix}.$$

We now choose

$$K_i(t) = |b_i(t+T_i)|/a_i, \qquad i = 1,2,\ldots,N, \tag{7}$$

and note that the principal minors of M are

$$2c(t) - \sum |b_i(t+T_i)|/a_i$$

$$\frac{1}{a_1}|b_1(t)|\left[2c(t) - a_1|b_1(t)| - \sum_{i=1}^{N}\frac{1}{a_i}|b_i(t+T_i)|\right]$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{1}{a_1\cdots a_N}|b_1(t)|\cdots|b_N(t)|\left[2c(t) - \sum_{i=1}^{N} a_i|b_i(t)| - \sum_{i=1}^{N}\frac{1}{a_i}|b_i(t+T_i)|\right].$$

If $|b_i(t)| \geqslant \varepsilon > 0$ for all i, then the quadratic form $-\dot{V}$ is positive definite whenever there exists $q > 0$ such that

$$2c(t) - \sum_{i=1}^{N} a_i|b_i(t)| - \sum_{i=1}^{N}\frac{1}{a_i}|b_i(t+T_i)| \geqslant q > 0. \tag{8}$$

Using the arguments of Thm. 2, we can conclude that, if (8) holds, then there exists $\alpha > 0$ such that

$$\dot{V}(t,\phi) \leqslant -\alpha\phi^2(0).$$

Clearly,

$$|\phi(0)|^2 \leqslant V(t,\phi) \leqslant |\phi|_\infty^2\left\{1 + \sum_{i=1}^{N}\int_{-T_i}^{0}\frac{1}{a_i}|b_i(t+T_i+\theta)|\,d\theta\right\}$$

$$= |\phi|_\infty^2\left\{1 + \sum_{i=1}^{N}\int_{t}^{t+T_i}\frac{1}{a_i}|b_i(s)|\,ds\right\},$$

and we have established the following result.

THEOREM 5. Let $c(t)$ and $b_i(t)$ be continuous functions satisfying the following conditions:

(i) Given $\eta > 0$ there exists $\tau > 0$ such that

$$\int_t^{t+\tau} |b_i(s)| ds < \eta$$

for $i = 1, 2, \ldots, n$ and $t \geqslant 0$.

(ii) $2c(t) - \Sigma_{i=1}^N a_i |b_i(t)| - \Sigma_{i=1}^N |b_i(t + T_i)|/a_i \geqslant q > 0$, for some constants $q > 0$, $a_i > 0$, $i = 1, 2, \ldots, N$ and for $t \in [0, \infty)$.

Then, the zero solution of (5) is uniformly asymptotically stable.

It is easy to derive analogues of Corollaries 1 and 2 of Sec. 1. We only mention one of these.

COROLLARY 3. If $c(t)$ and $b_i(t)$ are continuous, and $b_i(t)$ is periodic of period T_i, $i = 1, 2, \ldots, N$; a sufficient condition for the uniform asymptotic stability of the zero solution of (5) is that there exist $q > 0$, with

$$c(t) - \sum_{i=1}^N |b_i(t)| \geqslant q, \qquad t \in [0, \infty).$$

Other results follow from different choices of the K_i. For example, the choice

$$K_i(t) = \frac{1}{a_i} b_i^2(t + T_i)$$

yields the following form for $\dot{V}(t, \phi)$

$$\dot{V}(t, \phi) = \left[-2c(t) + \sum_{i=1}^N \frac{1}{a_i} b_i^2(t + T_i) \right] \phi^2(0) + 2\phi(0) \sum_{i=1}^N b_i(t) \phi(-T_i)$$
$$- \sum_{i=1}^N \frac{1}{a_i} b_i^2(t) \phi^2(-T_i), \tag{9}$$

and we have the following result.

THEOREM 6. The zero solution of Eq. (5) is uniformly asymptotically stable if $c(t)$ and $b_i(t)$, $i = 1, 2, \ldots, N$, are continuous and

(i′) there exist constants $q > 0$, $a_i > 0$, $i = 1, 2, \ldots, N$ with

$$2c(t) - \sum_{i=1}^N a_i - \sum_{i=1}^N \frac{1}{a_i} b_i^2(t + T_i) \geqslant q,$$

(ii′) $\Sigma_{i=1}^N \int_t^{t+T_i} b_i^2(s)\, ds \leqslant B < \infty$, and given $\eta > 0$ there exists $\tau > 0$ such that

$$\int_t^{t+\tau} |b_i(s)| ds < \eta$$

for $i = 1, 2, \ldots, n$ and $t \geqslant 0$.

Proof. Using condition (i′) in (9) we note that

$$\dot V(t,\phi) \leqslant -q\phi^2(0) + \sum_{i=1}^{N} \frac{1}{a_i}\left[-a_i^2\phi^2(0) + 2a_i b_i(t)\phi(0)\phi(-T_i) - b_i^2(t)\phi^2(-T_i)\right]$$

$$\leqslant -q\phi^2(0) - \sum_{i=1}^{N} \frac{1}{a_i}\left[a_i\phi(0) - b_i(t)\phi(-T_i)\right]^2 \leqslant -q\phi^2(0).$$

The condition (ii′) immediately implies that

$$|\phi(0)|^2 \leqslant V(t,\phi) \leqslant |\phi|_\infty^2\left[1 + \sum_{i=1}^{N}\frac{1}{a_i}\int_t^{t+T_i} b_i^2(s)\,ds\right]$$

$$\leqslant |\phi|_\infty^2[1 + BL],$$

where $L = N\sum_{i=1}^{N} 1/a_i$, and the proof is completed.

An immediate corollary is the following.

COROLLARY 4. If $c(t)$ and $b_i(t)$ are continuous and if $b_i(t)$ is periodic with period T_i, $i = 1,2,\ldots,N$, then a sufficient condition for the uniform asymptotic stability of the zero solution of (5) is that there exist $q > 0$, $a_i > 0$, $i = 1,2,\ldots,N$, such that

$$2c(t) - \sum_{i=1}^{N} a_i - \sum_{i=1}^{N}\frac{1}{a_i}b_i^2(t) \geqslant q.$$

We note that all of these results can be generalized, at the expense of complicating the stability conditions, by choosing Lyapunov functions of the form

$$V(t,\phi) = \alpha(t)\phi^2(0) + \sum_{i=1}^{N}\int_{-T_i}^{0} K_i(t+\theta)\phi^2(\theta)\,d\theta,$$

with $\alpha(t) \geqslant \alpha_0 > 0$, a continuously differentiable function. The proofs of the corresponding results proceed in the same manner as before with obvious changes in the stability conditions. For example, the conclusions of Thm. 5 hold if condition (ii) of that result is replaced by

$$2c(t) - \alpha(t)\sum_{i=1}^{N} a_i|b_i(t)| - \frac{1}{\alpha(t)}\sum_{i=1}^{N}\frac{1}{a_i}|b_i(t+T_i)| - \frac{\alpha'(t)}{\alpha(t)} \geqslant q > 0,$$

for any function α of the type described above. All of our results have analogous extensions.

3. Some simple stability criteria for systems. Consider the system

$$x'(t) = B(t)x(t-T) - C(t)x(t) \tag{10}$$

where x is an n-dimensional vector and B and C are continuous functions whose range is in the set of $n \times n$ matrices. Introducing the functional (the superscript T denotes the transpose of a matrix):

$$V(t,\phi) = \phi(0)^T D\phi(0) + \int_{-T}^{0}\phi(\theta)^T K(t+\theta)\phi(\theta)\,d\theta \tag{11}$$

where $K(t)$ and D are $n \times n$ matrices to be chosen below, and assuming that K is continuous, we obtain

$$\begin{aligned}\dot{V}(t,\phi) &= x'(t)^T D x(t) + x(t)^T D x'(t) + x(t)^T K(t) x(t)\\ &\quad - x(t-T)^T K(t-T) x(t-T)\\ &= \left[x(t-T)^T B(t)^T - x(t)^T C(t)^T\right] D x(t)\\ &\quad + x(t)^T D\left[B(t)x(t-T) - C(t)x(t)\right]\\ &\quad + x(t)^T K(t) x(t) - x(t-T)^T K(t-T) x(t-T)\\ &= -\phi(0)^T\left[C(t)^T D + DC(t) - K(t)\right]\phi(0)\\ &\quad + \phi(-T)^T B(t)^T D\phi(0) + \phi(0)^T D B(t)\phi(-T)\\ &\quad - \phi(-T)^T K(t-T)\phi(-T).\end{aligned}$$

If $D = D^T$, we have

$$\begin{aligned}\dot{V}(t,\phi) = &-\phi(0)^T\left[C(t)^T D + DC(t) - K(t)\right]\phi(0)\\ &+ 2\phi(0)^T D B(t)\phi(-T) - \phi(-T)^T K(t-T)\phi(-T).\end{aligned} \tag{12}$$

This quadratic form $-\dot{V}$ has the associated symmetric matrix

$$\begin{bmatrix} C(t)^T D + DC(t) - K(t) & \tfrac{1}{2}\left(DB(t) + B(t)^T D\right)\\ \tfrac{1}{2}\left(DB(t) + B(t)^T D\right) & K(t-T)\end{bmatrix}.$$

Several tests can be applied to establish that this is a positive definite matrix.

As a specific example, choose D to be positive definite and symmetric, and let

$$K(t) = B(t+T)^T B(t+T).$$

Then

$$\begin{aligned}\dot{V}(t,\phi) = &-\phi(0)^T\left[C(t)^T D + DC(t) - B(t+T)^T B(t+T)\right]\phi(0)\\ &+ 2\phi(0)^T D B(t)\phi(-T) - \phi(-T)^T B(t)^T B(t)\phi(-T).\end{aligned} \tag{13}$$

and if we impose the condition

$$C(t)^T D + DC(t) - B(t+T)^T B(t+T) - D^2 \geqslant \gamma I,$$

where $\gamma > 0$ and I is the identity, we obtain from (13)

$$\begin{aligned}\dot{V}(t,\phi) &\leqslant -\gamma\phi(0)^T\phi(0) - \left(D\phi(0) + B\phi(-T)\right)^T\left(D\phi(0) + B\phi(-T)\right)\\ &\leqslant -\gamma\phi(0)^T\phi(0).\end{aligned}$$

Moreover, since D is positive definite, there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$ with

$$\alpha_1\|\phi(0)\|^2 \leqslant \phi(0)^T D\phi(0) \leqslant \alpha_2\|\phi(0)\|^2,$$

hence, if $\int_t^{t+T}\|B(s)\|^2\,ds \leqslant \beta < \infty$, we have

$$\begin{aligned}\alpha_1\|\phi(0)\|^2 \leqslant V(t,\phi) &\leqslant \alpha_2\|\phi(0)\|^2 + \|\phi\|_\infty^2\int_t^{t+T}\|B(s)\|^2\,ds\\ &\leqslant \|\phi\|_\infty^2(\alpha_2 + \beta).\end{aligned}$$

Applying Theorem 8 in Sec. 4, we have

THEOREM 7. Consider the system (10) and assume B and C are continuous matrix valued functions satisfying the conditions

(i) $C(t)^T D + DC(t) - B(t+T)^T B(t+T) - D^2 \geqslant \gamma I$, for some $\gamma > 0$ and some positive definite matrix D, and

(ii) Given $\eta > 0$ there exists $\tau > 0$ such that

$$\int_t^{t+\tau} \|B(s)\| ds < \eta \quad \text{for } t \geqslant 0.$$

Then the zero solution of (10) is uniformly asymptotically stable.

Theorem 7 is an exact extension to systems of Thm. 3 for scalar equations. It is clear that Thm. 6 also has an analogous extension to systems of the form

$$x'(t) = -C(t)x(t) + \sum_{i=1}^{N} B_i(t)x(t-T_i). \tag{14}$$

The stability conditions in this case take the form

$$\sum_{i=1}^{N} \int_t^{t+T} \|B_i(s)\|^2 ds \leqslant \beta < \infty$$

and

$$C(t)^T D + DC(t) - \sum_{i=1}^{N} a_i D^2 - \sum_{i=1}^{N} \frac{1}{a_i} B(t+T_i)^T B(t+T_i) \geqslant \gamma I,$$

for some positive definite symmetric matrix D and some constants $\gamma > 0$, $a_i > 0$, $i = 1, 2, \dots, N$. If D is taken to be non-constant: D: $[0, \infty) \to$ positive definite $n \times n$ matrices, $D(t)$ continuously differentiable and $D(t) \geqslant D_0$, where D_0 is a constant positive definite matrix, then the second stability condition changes to

$$C(t)^T D(t) + D(t)C(t) - \sum_{i=1}^{N} a_i D^2(t) - D'(t)$$
$$- \sum_{i=1}^{N} \frac{1}{a_i} B(t+T_i)^T B(t+T_i) \geqslant \gamma I.$$

We finally note that our results are intrinsically different from those of Lewis and Anderson [6] because we allow the possibility that the matrices $B_i(t)$ in (14) have non-zero diagonal terms. The hypotheses in [6] require that all diagonal terms in the $B_i(t)$ be equal to zero. The techniques in [6] can be extended to cover the situation where the $B_i(t)$ have non-zero diagonal terms, and stability criteria which differ from those presented here are obtainable in that way. This has been done and will be described in a forthcoming paper of R. Volz.

4. A Lyapunov functional result. In proving the results of the previous sections we have used a version of the Lyapunov asymptotic stability result of Krasovskii that does not require the standard restriction that the right-hand side of the functional equation map

$\mathbf{R} \times$ (bounded sets of C) into bounded sets of $\mathbf{R}^n$. This can be done because of the special form of the functional differential equations that we are considering, namely

$$\frac{dx}{dt}(t) = F(t, x_t) - G(t, x(t)) = f(t, x_t). \tag{15}$$

We assume that $f(t, 0) = 0$ and f: $\mathbf{R} \times C \to \mathbf{R}^n$ is continuous and is smooth enough to ensure that the solution of (15) through $(s, \phi) \in \mathbf{R} \times C$ is continuous in (s, ϕ, t) in the domain of definition of f. In this section we shall use the notation of Hale [15, Chapter 5]. The result we need is the following:

THEOREM 8. Suppose that there exist continuous nondecreasing functions u, v, w: $\mathbf{R}^+ \to \mathbf{R}^+$ with $u(0) = v(0) = 0$ and $u(s) > 0$, $v(s) > 0$, $w(s) > 0$ for $s > 0$. Suppose also that there exists a continuous function V: $\mathbf{R} \times C \to \mathbf{R}$ such that

$$u(|\phi(0)|) \leqslant V(t, \phi) \leqslant v(|\phi|),$$
$$\dot{V}(t, \phi) \leqslant -w(|\phi(0)|).$$

Finally, assume that given positive $\eta > 0$, $\gamma > 0$ there exists $\tau > 0$ such that $\int_t^{t+\tau} |F(s, \phi)|\, ds < \eta$ for all $t > 0$ and $|\phi| \leqslant \gamma$, and $x^T D G(t, x) \geqslant 0$ for some positive definite symmetric matrix D and for $(t, x) \in \mathbf{R}^+ \times \mathbf{R}^n$. Then the solution $x = 0$ of (15) is uniformly asymptotically stable.

Proof. The proof proceeds in the same way as that of Theorem 2.1 in [5, page 105] with the exception of the part of that proof which uses the added assumption that f in (15) take $\mathbf{R} \times$ (bounded sets of C) into bounded sets of $\mathbf{R}^n$. So, we shall present only that part of the proof.

Let $\delta_0 > 0$ be such that $|\phi| < \delta_0$ implies $|x(t, \phi)| < 1$ for all $t \geqslant \sigma$. Assume that there exists a sequence $\{t_k\}$ such that

$$\sigma + (2k - 1)r \leqslant t_k \leqslant \sigma + 2k(r), \qquad k: 1, 2, \dots$$

and, with $|x|_D = (x^T D x)^{1/2}$,

$$|x(t_k)|_D \geqslant \delta, \quad \text{for some } \delta > 0.$$

The proof can be completed as in [5], if we can show that there exists $\tau > 0$ such that $|x(t)|_D > \delta/2$ for

$$t \in [t_k - \tau, t_k + \tau].$$

Now, for $|\phi| \leqslant \varepsilon$ choose $\tau > 0$ so that

$$\int_t^{t+\tau} |F(s, \phi)| ds < \frac{3\delta^2}{8d}, \quad \text{where } d = \|D\|.$$

Note that the continuity of $x(t)$ implies that there exist $\tau_k > 0$ with $|x(t)|_D > \delta/2$ for $t \in [t_k - \tau_k, t_k + \tau_k] = I_k$, and for each k let $\tau_k > 0$ be the maximal such τ_k. We shall show that $\tau_k \geqslant \tau$. For, supposing that $\tau_k < \tau$ we have

$$\begin{aligned}\frac{d}{dt}(x^T(t) D x(t)) &= x^T(t) D F(t, x_t) + F^T(t, x_t) D x(t) \\ &\quad - [x^T(t) D G(t, x(t)) + G^T(t, x(t)) D x(t)] \\ &\leqslant x^T(t) D F(t, x_t) + F^T(t, x_t) D x(t).\end{aligned}$$

So, for $t \in [t_k - \tau, t_k + \tau]$

$$|x(t_k)|_D^2 - |x(t)|_D^2 \leqslant 2d\left|\int_{t_k}^{t_k+\tau} |F(s, x_s)|ds\right| < 3\delta^2/4,$$

that is $|x(t)|_D > \delta/2$, which contradicts the maximalities of τ_k. So $\tau_k \geqslant \tau$, and the proof is completed.

Remark. Burton [1] gives extensions of the Lyapunov theorem of Krasovskii for more general equations than (15). In the scalar case ($n = 1$), the condition on $G(t, x)$ reduces to the requirement that $xG \geqslant 0$.

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