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ASYMPTOTIC STABILITY IN NONLINEAR VISCOELASTICITY*

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Abstract. In this paper we present sufficient conditions for asymptotic stability of a homogeneous equilibrium state of a (nonlinear) elastic body with linear viscosity. The body is subject to external conditions of zero displacements on a part of the boundary, zero surface tractions on the remaining part of the boundary and zero body forces in the interior of the body. The meaning and further qualitative consequence of our conditions are also discussed.

1. Introduction. We study in this paper stability of an equilibrium state of a continuous nonlinear elastic body with linear viscosity in a purely mechanical context. Our main concern is asymptotic stability (the reader is referred to the articles by Knops and Wilkes [1] and Gurtin [2] for definitions and a discussion of various types of stability within continuum mechanics). Our results are broadly in line with comments made by Dafermos [3]. Specifically, we consider the body under external conditions which impose zero displacement on part of the boundary, zero traction on the remainder of the boundary and inside the body the body forces vanish. The existence of a homogeneous equilibrium configuration compatible with the external conditions is assumed. Under certain hypotheses this equilibrium configuration is shown to be asymptotically stable with exponential decay within the class of all motions consistent with the external conditions (see Theorem 3 below). From this it can be seen that rate-type viscoelasticity provides a powerful form of dissipation. A special case of Theorem 3 in which both the elastic and viscous responses are linear has been presented by Duvaut and Lions [4]. In this special case the hypotheses made in this paper coincide with those in [4].

We now briefly discuss the nature and origin of our hypotheses. Essentially they follow in part from the first and second laws of thermodynamics. This part consists of assumng the existence of the stored energy function for the static (i.e., elastic) part of the stress. Then, the positive semi-definite nature of the viscosity tensor appears as a restriction on constitutive theory.

The remaining hypotheses contain additional assumptions of a more complex nature. The existence of the stored energy function enables one to define the total stored energy of

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the body as a functional on the kinematically admissible configurations. It is well-known (see [1, 2], and also our Theorem 2) that provided the viscosity tensor is positive semi-definite, the presence of a strong local minimum of the total stored energy at some configuration implies Lyapunov stability of that configuration. Our proof of asymptotic stability requires that the equilibrium configuration be a point of a strong global minimum of the stored energy functional in the class of all kinematically admissible configurations. However, we need a further hypothesis on the static part of the response. To see that the minimum property of the stored energy is not a strong enough assumption to imply the asymptotic stability we note that there exists a simple but important necessary condition for asymptotic stability (see [1] and also Sec. 3 below): an asymptotically stable state of the body under given external conditions must be the only equilibrium state of the body compatible with the external conditions. Clearly the existence of a strong global minimum of the stored energy functional does not guarantee this uniqueness, as the global minimum does not preclude the existence of several local minima leading to further equilibrium states compatible with the external conditions. It is observed that an analogous situation may arise for potential energy in Lagrangian mechanics of point particles. Our additional hypothesis (3.13) ensures positive definiteness of another integral expression which has the physical dimensions of energy. Condition (3.13) implies the desired uniqueness and even more: it ensures; the continuous dependence of equilibrium states of the body upon changes of surface tractions and body forces.

Both the existence of a strong global minimum of the stored energy functional and the additional inequality (3.13) are conditions which combine the static response functions of the body with the external conditions, namely with the constraint that part of the boundary of the body is fixed. We do not pursue in this paper the difficult question of finding sufficient conditions for the validity of these hypotheses in terms of the pointwise inequalities on the response functions. We also note that it is a largely open related problem for which additional constitutive restrictions will have to be imposed on the static response to obtain a sound theory of equilibrium (see Wang and Truesdell [5] and Ball [6] for a discussion of this topic).

Our hypotheses on the static part of the response of the body are finally completed by a technical condition that the stored energy functional is of a quadratic growth.

We also need assumptions on the symmetry of the viscosity tensor. Apart from the symmetries implied by the requirements that the principle of material frame indifference and the symmetry of the Cauchy stress be approximately satisfied (see (4.1) and (4.2)) there is a major symmetry of the viscosity tensor saying (cf. (4.3)) that the bilinear form on the space of all second-order tensors corresponding to the viscosity tensor is symmetric. In several special cases this symmetry is a consequence of the symmetry of the material (such is the case of an isotropic material), but generally it is an independent hypothesis. As a matter of fact, the major symmetry expresses the Onsager reciprocal relations for viscosity. Closing the discussion of the viscous part of the stress we note that our assumption that the Piola-Kirchhoff stress depends linearly on the gradient of velocity with respect to the reference configuration implies that these principles cannot be satisfied exactly (unless the viscosity vanishes). Their exact validity would require that the viscous part of the stress depends also on the deformation gradient in a certain way. Hence, our hypothetical body must be considered to be only an approximation to a true body; an approximation which is suitable in situations when the dependence of the viscous part of the stress on the deformation gradient may be neglected but the non-linearities of the static part of the response cannot be ignored.

2. Basic equations. We consider motions of the continuous body B subject to zero body forces, zero surface tractions on a part of the boundary and zero displacements on the remainder of the boundary. We suppose that there is an equilibrium homogeneous configuration of the body compatible with these external conditions. The purpose of this paper is to investigate the asymptotic stability of this equilibrium configuration within the class of all motions consistent with the external conditions. In this section we record the basic equations describing this situation. We refer the reader to Wang and Truesdell [5] and Truesdell [7] for more details concerning the general mechanical concepts employed below.

We take the equilibrium configuration as the reference configuration, i.e., we label the typical particle P of the body by the position $\mathbf{X} \in R^3$ it has in this equilibrium configuration. We assume that the region $V_0 \subset R^3$ occupied by the body in the reference configuration has a properly regular boundary ∂V_0 with the unit outward normal $\mathbf{N}$ (see, e.g., Nečas and Hlaváček [8]). A motion of the body is described by the function

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{2.1}$$

giving the position of the particle $\mathbf{X} \in V_0$ at time $t \geqslant 0$. The motion $\mathbf{x}(\mathbf{X}, t)$ is of class C^2 on $\bar{V}_0 \times [0, \infty)$. The displacement $\mathbf{u}(\mathbf{X}, t)$, the velocity $\mathbf{v}(\mathbf{X}, t)$ and the deformation gradient $\mathbf{F}(\mathbf{X}, t)$ are given by

$$\mathbf{v}(\mathbf{X}, t) = \dot{\mathbf{x}}(\mathbf{X}, t) = \dot{\mathbf{u}}(\mathbf{X}, t), \qquad \mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}, \tag{2.2}$$

and

$$\mathbf{F}(\mathbf{X}, t) = \nabla \mathbf{x}(\mathbf{X}, t) = \nabla \mathbf{u}(\mathbf{X}, t) + \mathbf{I}, \qquad \det \mathbf{F} > 0; \tag{2.3}$$

where the superposed dot denotes the material time derivative, ∇ denotes gradient with respect to X and I is the unit tensor. In the absence of body forces the balance of linear momentum has the form

$$\rho_0 \ddot{\mathbf{x}} = \operatorname{Div} \mathbf{S}, \tag{2.4}$$

where $\rho_0 > 0$ is the density of the body in the reference configuration, $\mathbf{S}$ is the Piola-Kirchhoff stress tensor and Div denotes the divergence operator with respect to $\mathbf{X}$. The boundary ∂V_0 is assumed to be divided into two parts $\mathbb{S}_1, \mathbb{S}_2 \subset \partial V_0$ in such a way that $\mathbb{S}_1 \cup \mathbb{S}_2 = \partial V_0$ and that $\mathbb{S}_1$ has a positive area. We do not exclude the case $\mathbb{S}_2 = \varnothing$ and $\mathbb{S}_1 = \partial V_0$. The following boundary conditions are prescribed

$$\mathbf{x}(\mathbf{X}, t) = \mathbf{X}, \qquad \mathbf{X} \in \mathbb{S}_1, t \geqslant 0, \tag{2.5}$$

$$\mathbf{S}(\mathbf{X}, t)\mathbf{N}(\mathbf{X}) = \mathbf{O}, \qquad \mathbf{X} \in \mathbb{S}_2, t \geqslant 0. \tag{2.6}$$

We now proceed to formulate our constitutive hypotheses about the body. We assume that the stress $\mathbf{S} = \mathbf{S}(\mathbf{X}, t)$ corresponding to the motion $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ of the body is a sum of two parts,

$$\mathbf{S} = \mathbf{S}^s + \mathbf{S}^P, \tag{2.7}$$

where $\mathbf{S}^S$ is the static (i.e., elastic) and $\mathbf{S}^P$ the viscous part of the stress. The static part of the stress is given by the constitutive equation

$$\mathbf{S}^S(\mathbf{X}, t) = \mathbf{S}_0(\nabla \mathbf{x}(\mathbf{X}, t)), \tag{2.8}$$

where $\mathbf{S}_0(\cdot)$ is a given continuously differentiable function defined on the set of all deformation tensors $\mathbf{F}$ with positive determinant, while the viscous part of the stress is given by

$$\mathbf{S}^P(\mathbf{X}, t) = \boldsymbol{\Lambda}[\nabla \dot{\mathbf{x}}(\mathbf{X}, t)], \tag{2.9}$$

where $\boldsymbol{\Lambda}[\cdot]$ is a fourth-order tensor of viscosities, interpreted as a linear transformation from the space of all second-order tensors into itself.

There will be additional assumptions on the constitutive functions $\mathbf{S}_0(\cdot)$ and $\boldsymbol{\Lambda}[\cdot]$ which we shall formulate and discuss in the next two sections. Now, we have assembled all that is necessary to introduce the following terminology. We say that a motion $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, $\mathbf{X} \in V_0$, $t \geqslant 0$ is an *admissible motion of the body* if it satisfies the equation of balance of linear momentum (2.4), the boundary conditions (2.5) and (2.6), and the constitutive equations (2.7)–(2.9).

Finally, suppose that the reference configuration is an equilibrium configuration of B compatible with the external conditions, that is to say formally that the rest motion $\mathbf{x}(\mathbf{X}, t)$ given by $\mathbf{x}(\mathbf{X}, t) = \mathbf{X}$, $\mathbf{X} \in V_0$, $t \geqslant 0$, is an admissible motion. We note that for this motion the stress $\mathbf{S}(\mathbf{X}, t)$ is given by $\mathbf{S}(\mathbf{X}, t) = \mathbf{S}^S(\mathbf{X}, t) = \mathbf{S}_0(\mathbf{I})$, so that the balance of linear momentum (2.4) is satisfied. As the boundary condition (2.5) is satisfied, too, the reference configuration is compatible with the external conditions if and only if (2.6) is satisfied. We shall consider two cases. First, if $\mathbb{S}_2 = \varnothing$ then (2.6) places no condition and hence, in this case, the reference configuration is for all time an equilibrium configuration of the body compatible with the external conditions. If $\mathbb{S}_2 \neq \varnothing$ then (2.6) reduces to

$$\mathbf{S}_0(\mathbf{I})\mathbf{N}(\mathbf{X}) = \mathbf{O}, \qquad \mathbf{X} \in \mathbb{S}_2. \tag{2.10}$$

Obviously this is satisfied if

$$\mathbf{S}_0(\mathbf{I}) = \mathbf{O}, \tag{2.11}$$

i.e., if the reference configuration is a stress-free configuration. The condition (2.11) is also a necessary condition in the case when $\mathbb{S}_2$ contains at least three points at which the directions of the normal $\mathbf{N}$ are linearly independent because the validity of (2.10) at these three points amounts to (2.11). Hence, if the body is subjected to the boundary condition of place ($\mathbb{S}_2 = \varnothing$) then the reference configuration need not be a stress-free configuration but if $\mathbb{S}_2 \neq \varnothing$ then almost necessarily the reference configuration is stress-free.

3. Hypotheses on the static part of the stress. Uniqueness. In this section we lay down the hypotheses on the constitutive function $\mathbf{S}_0$ and discuss their consequences. We show in particular that our hypotheses imply that the reference configuration is the only equilibrium configuration of the body compatible with the external conditions.

Our first hypothesis may be regarded as a consequence of the first and second laws of thermodynamics.

H1. (The existence of the stored energy function.) *There exists a twice continuously* *differentiable function* $\psi = \psi(\mathbf{F})$ *such that*

$$\mathbf{S}_0(\mathbf{F}) = \rho_0 \partial_{\mathbf{F}} \psi(\mathbf{F}) \tag{3.1}$$

for all second-order tensors $\mathbf{F}$ *with positive determinant.*

(Here $\partial_{\mathbf{F}}$ denotes differentiation with respect to $\mathbf{F}$). The function ψ is called the *stored energy function* of the body and we use the normalization

$$\psi(\mathbf{I}) = 0. \tag{3.2}$$

A standard consequence of H1 is the following variant of the power theorem.

PROPOSITION 1. For any admissible motion of the body we have

$$\dot{K} + \dot{P} = -D, \tag{3.3}$$

where

$$K = K(t) = \frac{1}{2} \int_{V_0} \mathbf{v}(\mathbf{X}, t)^2 \rho_0 \, dV_0 \tag{3.4}$$

is the kinetic energy of the body at time t,

$$P = P(t) = \int_{V_0} \psi(\mathbf{F}(\mathbf{X}, t)) \rho_0 \, dV_0 \tag{3.5}$$

is the total stored energy of the body at time t and

$$D = D(t) = \int_{V_0} \mathbf{\Lambda}[\nabla \mathbf{v}(\mathbf{X}, t)] \cdot \nabla \mathbf{v}(\mathbf{X}, t) \, dV_0 \tag{3.6}$$

is the power of viscious forces in the body at time t.

It is convenient to call any function $\mathbf{x} = \mathbf{x}(\mathbf{X})$, $\mathbf{X} \in V_0$ of class C^2 on $\bar{V}_0$, satisfying

$$\det \nabla \mathbf{x}(\mathbf{X}) > 0, \qquad \mathbf{X} \in V_0,$$

and

$$\mathbf{x}(\mathbf{X}) = \mathbf{X}, \mathbf{X} \in \mathcal{S}_1 \tag{3.7}$$

an *admissible configuration* of the body, and to define functionals P and I on the set of all admissible configurations by

$$P(\mathbf{x}(\cdot)) = \int_{V_0} \psi(\nabla \mathbf{x}(\mathbf{X})) \rho_0 \, dV_0 \tag{3.8}$$

and

$$I(\mathbf{x}(\cdot)) = \int_{V_0} \mathbf{S}_0(\nabla \mathbf{x}(\mathbf{X})) \cdot (\nabla \mathbf{x}(\mathbf{X}) - \mathbf{I}) \, dV_0 \tag{3.9}$$

for any admissible configuration $\mathbf{x}(\cdot)$. The value $P(\mathbf{x}(\cdot))$ is the total stored energy of the body in the configuration $\mathbf{x}(\cdot)$, while the meaning of $I(\mathbf{x}(\cdot))$ will appear shortly. We notice that

$$P(\mathbf{x}_0(\cdot)) = I(\mathbf{x}_0(\cdot)) = 0, \tag{3.10}$$

where $\mathbf{x}_0$ is the reference configuration given by

$$\mathbf{x}_0(\mathbf{X}) = \mathbf{X}, \qquad \mathbf{X} \in V_0 \tag{3.11}$$

and where in $(3.10)_1$ the normalisation condition (3.2) has been used. With this notation we now state the following hypothesis.

H2. (positiveness of P and I). *There exist positive constants c_1 and c_2 such that*

$$P(\mathbf{x}(\cdot)) \geqslant c_1 \|\nabla \mathbf{x} - \mathbf{I}\|^2, \tag{3.12}$$

and

$$I(\mathbf{x}(\cdot)) \geqslant c_2 \|\nabla \mathbf{x} - \mathbf{I}\|^2 \tag{3.13}$$

for each admissible configuration $\mathbf{x}(\cdot)$ *of the body.*

Here $\|\cdot\|$ denotes the L_2-norm on V_0, i.e., if f is any measurable scalar-, vector- and tensor-valued function defined on V_0, then

$$\|f\| = \left(\int_{V_0} |f|^2 \, dV_0\right)^{1/2}$$

where $|\cdot|$ denotes any norm on scalars, vectors or tensors. Note that (3.12) together with $(3.10)_1$ implies that P attains a strong global minimum at the admissible configuration $\mathbf{x}_0$ in the set of all admissible configurations. The relevance of a condition of this type to the stability of the body is well-known (see, e.g., [1, 2]). Nevertheless, it is the second inequality of $H2$, (3.13), that is in a certain sense more important for our proof of the asymptotic stability. We start the discussion of (3.13) by showing that it almost implies (3.12) More precisely, we have

PROPOSITION 2. Assume that the inequality (3.13) holds in the set C_0 of all admissible configurations $\mathbf{x}$ of the body which satisfy

$$\det[\mathbf{I} + \lambda(\nabla \mathbf{x}(\mathbf{X}) - \mathbf{I})] > 0 \tag{3.14}$$

for all $\mathbf{X} \in V_0$ and all $\lambda \in [0, 1]$. Then also (3.12) holds for all $\mathbf{X} \in V_0$ with $c_1 = \frac{1}{2}c_2$. If the set of all tensors with positive determinant were to form a star-shaped set with respect to the identity tensor then the set C_0 would coincide with the set of all admissible configurations and Proposition 2 would say that (3.13) does imply (3.12). This, however, is not the case as, for instance, the centre of the segment in the space of all second order tensors with endpoints $I = \operatorname{diag}(1, 1, 1)$ and $\operatorname{diag}(-\frac{1}{2}, -3, 1)$ has negative determinant.

Proof of Proposition 2. If $\mathbf{x} \in C_0$, then for each $\lambda \in [0, 1]$ the configuration $\mathbf{x}_\lambda$ given by

$$\mathbf{x}_\lambda(\mathbf{X}) = \mathbf{x}_0(\mathbf{X}) + \lambda(\mathbf{x}(\mathbf{X}) - \mathbf{x}_0(\mathbf{X}))$$

(where $\mathbf{x}_0$ is given by (3.11)) satisfies

$$\det \nabla \mathbf{x}_\lambda(X) = \det[\mathbf{I} + \lambda(\nabla x(X) - \mathbf{I})] > 0$$

by (3.14) and also $\mathbf{x}_\lambda(\mathbf{X}) = \mathbf{X}$, $\mathbf{X} \in \mathcal{S}_1$ so that $\mathbf{x}_\lambda$ is an admissible configuration. One may hence define a function $g(\lambda)$, $\lambda \in [0, 1]$ by setting

$$g(\lambda) = P(\mathbf{x}_\lambda(\cdot)) = \int_{V_0} \psi(\mathbf{I} + \lambda(\nabla \mathbf{x}(\mathbf{X}) - I))\rho_0 \, dV_0.$$

The derivative of g with respect to λ is then given by

$$g'(\lambda) = \int_{V_0} \partial_{\mathbf{F}} \psi(\mathbf{I} + \lambda(\nabla \mathbf{x} - \mathbf{I})) \cdot (\nabla \mathbf{x} - \mathbf{I}) \rho_0 \, dV_0$$

$$= \int_{V_0} \mathbf{S}_0(\mathbf{I} + \lambda(\nabla \mathbf{x} - \mathbf{I})) \cdot (\nabla \mathbf{x} - \mathbf{I}) \, dV_0$$

$$= I(\mathbf{x}_\lambda(\cdot))/\lambda,$$

see (3.1) and (3.9). Applying (3.13) to the configuration $\mathbf{x}_\lambda$ yields

$$I(\mathbf{x}_\lambda(\cdot)) \geqslant c_2 \|\nabla \mathbf{x}_\lambda - \mathbf{I}\|^2 = c_2 \lambda^2 \|\nabla \mathbf{x} - \mathbf{I}\|^2$$

so

$$g'(\lambda) \geqslant c_2 \lambda \|\nabla \mathbf{x} - \mathbf{I}\|^2.$$

Integrate this inequality with respect to λ over [0, 1] to find that

$$g(1) - g(0) \geqslant \tfrac{1}{2} c_2 \|\nabla \mathbf{x} - I\|^2.$$

Noting that $g(1) = P(\mathbf{x}_1(\cdot)) = P(\mathbf{x}(\cdot))$ and $g(0) = P(\mathbf{x}_0(\cdot)) = 0$ completes the proof.

We next establish a simple static consequence of (3.13). Namely, we shall consider the equilibrium admissible configurations of the body corresponding to non-zero body forces and surface tractions on $\mathcal{S}_2$ and prove continuous dependence of such configurations on the body forces and surface tractions. An immediate corollary of this continuous dependence is that the reference configuration $\mathbf{x}_0$ is the only equilibrium configuration corresponding to zero body forces and zero surface tractions on $\mathcal{S}_2$. We note that this type of uniqueness is an important necessary condition for the asymptotic stability of the configuration $\mathbf{x}_0$. Indeed, if there were another equilibrium configuration $\mathbf{x}'$ of the body corresponding to the external conditions, then the rest process $\mathbf{x}'(\mathbf{X}, t)$ given by

$$\mathbf{x}'(\mathbf{X}, t) = \mathbf{x}'(\mathbf{X}), \qquad \mathbf{X} \in V_0, t \geqslant 0,$$

would be an admissible process of the body and yet it would not tend to $\mathbf{x}_0$ as $t \to \infty$.

Any admissible equilibrium configuration $\mathbf{x}(\cdot)$ of the body satisfies

$$\operatorname{Div} \mathbf{S}_0(\nabla \mathbf{x}) + \rho_0 \mathbf{b} = \mathbf{O} \quad \text{in } V_0, \tag{3.15}$$
$$\mathbf{x}(\mathbf{X}) = \mathbf{X}, \qquad \mathbf{X} \in \mathcal{S}_1,$$

and

$$\mathbf{S}_0(\nabla \mathbf{x}) \cdot \mathbf{N} = \mathbf{s} \quad \text{on } \mathcal{S}_2 \tag{3.16}$$

where $\mathbf{b}$ is the body force and $\mathbf{s}$ is the surface traction which holds the body in the configuration $\mathbf{x}$.

THEOREM 1. There exists a constant $c_3 > 0$ such that any admissible equilibrium configuration $\mathbf{x}$ of the body corresponding to the body force $\mathbf{b}$ and surface traction $\mathbf{s}$ on $\mathcal{S}_2$ satisfies

$$\|\mathbf{x} - \mathbf{x}_0\| + \|\nabla(\mathbf{x} - \mathbf{x}_0)\| \leqslant c_3(\|\mathbf{b}\| + \| |s| \|), \tag{3.17}$$

where

$$\| |\mathbf{s}| \| = \left(\int_{\mathbb{S}_2} |\mathbf{s}|^2 \, dA_0 \right)^{1/2},$$

and dA_0 denotes the element of the surface measure on ∂V_0.

PROOF. The definition of I and the divergence theorem yield

$$I(\mathbf{x}(\cdot)) = -\int_{\mathcal{V}_0} \operatorname{Div} \mathbf{S}_0(\nabla \mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0) \, dV_0 + \int_{\partial \mathcal{V}_0} \mathbf{S}_0(\nabla \mathbf{x})\mathbf{N} \cdot (\mathbf{x} - \mathbf{x}_0) \, dA_0$$

which in view of (3.15) and (3.16) implies

$$I(\mathbf{x}(\cdot)) = \int_{\mathcal{V}_0} \mathbf{b} \cdot (\mathbf{x} - \mathbf{x}_0) \rho_0 \, dV_0 + \int_{\mathbb{S}_2} \mathbf{s} \cdot (\mathbf{x} - \mathbf{x}_0) \, dA_0.$$

Applying the Schwarz inequality to the integrals of the right hand side of the equality shows that

$$I(\mathbf{x}(\cdot)) \leqslant \rho_0 \|b\| \, \|\mathbf{x} - \mathbf{x}_0\| + \| |s| \| \, \| |\mathbf{x} - \mathbf{x}_0| \| \tag{3.18}$$

where

$$\| |\mathbf{x} - \mathbf{x}_0| \| = \left(\int_{\mathbb{S}_2} |\mathbf{x} - \mathbf{x}_0|^2 \, dA_0 \right)^{1/2}.$$

Since $\mathbb{S}_1$ is of non-zero area, the Poincaré inequality (see Morrey [9]) assures the existence of a positive constant c_4 such that

$$\|\mathbf{x} - \mathbf{x}_0\| \leqslant c_4 \| \nabla(\mathbf{x} - \mathbf{x}_0)\| \tag{3.19}$$

for each admissible configuration $\mathbf{x}$ while the trace theorem (Kufner, John, Fucík [10]) assures the existence of a constant $c_5 > 0$ such that

$$\| |\mathbf{x} - \mathbf{x}_0| \| \leqslant c_5 \| \nabla(\mathbf{x} - \mathbf{x}_0)\| \tag{3.20}$$

for each admissible configuration $\mathbf{x}$. By (3.18)–(3.20) then

$$I(\mathbf{x}(\cdot)) \leqslant \left(\rho_0 c_4 \|\mathbf{b}\| + c_5 \| |\mathbf{s}| \| \right) \| \nabla(\mathbf{x} - \mathbf{x}_0)\|.$$

On combining this with (3.13) we see that

$$c_2 \| \nabla(\mathbf{x} - \mathbf{x}_0)\|^2 \leqslant \left(\rho_0 c_4 \|\mathbf{b}\| + c_5 \| |s| \| \right) \| \nabla(\mathbf{x} - \mathbf{x}_0)\| \tag{3.21}$$

The inequalities (3.21) and (3.19) yield (3.17) with

$$c_3 = (\rho_0 c_4 + c_5)(1 + c_4)/c_2,$$

and the proof is complete.

Setting $\mathbf{b} = \mathbf{O}$, $\mathbf{s} = \mathbf{O}$ in (3.17) we obtain the following

COROLLARY. The reference configuration $\mathbf{x}_0$ is the only admissible equilibrium configuration of the body compatible with zero body forces and zero surface tractions on $\mathbb{S}_2$.

It is worth mentioning that the corollary remains valid under the weaker assumption that $I(\mathbf{x}(\cdot)) > 0$ for each admissible configuration $\mathbf{x}$ different from $\mathbf{x}_0$. We close this section by imposing another condition on P.

H3. (Growth of P.) *There exists a constant* $c_6 > 0$ *such that*

$$P(\mathbf{x}(\cdot)) \leqslant c_6 \|\nabla \mathbf{x} - \mathbf{I}\|^2 \tag{3.22}$$

for each admissible configuration $\mathbf{x}$.

4. Hypotheses on the viscous part of the stress and Lyapunov stability. In this section we lay down and discuss the hypotheses on the viscosity tensor $\Lambda[\cdot]$ which are relevant to our goal. We also establish Lyapnov stability of the reference configuration. The first hypothesis on $\Lambda[\cdot]$ is

H4. (Symmetries of $\Lambda[\cdot]$.) *If* $\mathbf{G}$ *and* $\mathbf{H}$ *are two second order tensors, the*

$$\Lambda[\mathbf{H}] = \Lambda[\mathbf{H}^T], \tag{4.1}$$

$$\Lambda[\mathbf{H}] = \Lambda[\mathbf{H}]^T, \tag{4.2}$$

and

$$\Lambda[\mathbf{G}] \cdot \mathbf{H} = \Lambda[\mathbf{H}] \cdot \mathbf{G}. \tag{4.3}$$

It can be shown that the first requirement (4.1) is satisfied if and only if the viscous part of the stress satisfies approximately the principle of material frame indifference for small rotations and small values of the gradient of velocity. The second requirement (4.2) is satisfied if and only if the viscous part of the Cauchy stress is approximately symmetric for small deformation gradients. We also note that the viscous part of the stress as given by (2.9) can never satisfy the above principles exactly. The symmetry (4.3) is crucial to our proof as it implies an important identity (4.4) stated below. As a matter of fact (4.3) expresses the Onsager reciprocity relations for viscosity. In certain special cases (4.3) is a consequence of the symmetry of the material. Such is the case in an isotropic body.

PROPOSITION 3. If $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is any admissible motion of the body, then

$$\ddot{U} - 2K + I + \dot{L} = 0, \tag{4.4}$$

where K is the kinetic energy (see (3.4)), I is defined by

$$I = I(t) = \int_{v_0} \mathbf{S}^s \cdot (\mathbf{F} - \mathbf{I})\, dV_0$$

(cf. (3.9)), $U = U(t)$ is a measure of deformation given by

$$U(t) = \tfrac{1}{2}\int_{v_0} |\mathbf{x} - \mathbf{x}_0|^2 \rho_0\, dV_0 = \tfrac{1}{2}\rho_0 \|\mathbf{x} - \mathbf{x}_0\|^2, \tag{4.5}$$

and

$$L = L(t) = \tfrac{1}{2}\int_{v_0} \Lambda[\nabla \mathbf{x} - \mathbf{I}] \cdot (\nabla \mathbf{x} - \mathbf{I})\, dV_0. \tag{4.6}$$

PROOF. On forming the inner product of (2.4) with the displacement $\mathbf{x} - \mathbf{x}_0$, integrating over V_0, using the divergence theorem and the boundary conditions (2.5), (2.6) leads to

$$\ddot{U} - 2K = -\int_{v_0} \mathbf{S} \cdot (\nabla \mathbf{x} - \mathbf{I})\, dV_0.$$

On invoking the constitutive equations (2.7)–(2.9) we obtain

$$\ddot{U} - 2K = -\int_{v_0} \mathbf{S}_0(\nabla \mathbf{x}) \cdot (\nabla \mathbf{x} - \mathbf{I})\, dV_0 - \int_{v_0} \Lambda[\nabla \mathbf{v}] \cdot (\nabla \mathbf{x} - \mathbf{I})\, dV_0.$$

The first integral on the right-hand side of the last equality is obviously identified with I while the symmetry condition (4.3) enables one to identify the second integral with $\dot{L}$ and the proof is complete.

Our last hypothesis is

H5. (Positive Definiteness of $\Lambda[\cdot]$.) *If* $\mathbf{H}$ *is any non-zero second order symmetric tensor, then*

$$\Lambda[\mathbf{H}] \cdot \mathbf{H} > 0. \tag{4.7}$$

We note that the symmetries (4.1), (4.2) together with (4.7) imply that

$$\Lambda[\mathbf{G}] \cdot \mathbf{G} \geqslant 0 \tag{4.8}$$

for a general (not necessarily symmetric) second order tensor $\mathbf{G}$. It is well-known that the inequality (4.8) is a consequence of the second law of thermodynamics. The strict inequality (4.7), however, is what we need. It is pertinent to notice that one cannot assume that

$$\Lambda[\mathbf{G}] \cdot \mathbf{G} > 0$$

be satisfied for each non-zero second order tensor $\mathbf{G}$ since the symmetry (4.1) or (4.2) implies that

$$\Lambda[\mathbf{G}] \cdot \mathbf{G} = 0$$

whenever G is skew-symmetric.

LEMMA. There exist positive constants c_7, c_8, c_9 and c_{10} such that

$$c_7\|\nabla \mathbf{v}(\cdot, t)\|^2 \leqslant D(t) \leqslant c_8\|\nabla \mathbf{v}(\cdot, t)\|^2, \tag{4.9}$$

$$\tfrac{1}{2}c_7\|\nabla \mathbf{u}(\cdot, t)\|^2 \leqslant L(t) \leqslant \tfrac{1}{2}c_8\|\nabla \mathbf{u}(\cdot, t)\|^2, \tag{4.10}$$

$$U(t) \leqslant c_9\|\nabla \mathbf{u}(\cdot, t)\|^2, \tag{4.11}$$

and

$$K(t) \leqslant c_{10} D(t) \tag{4.12}$$

for each admissible motion of the body.

PROOF. We first observe that it follows from H5 that there exists a constant $\bar{c}_7 > 0$ such that $\bar{c}_7|\mathbf{H}|^2 \leqslant \Lambda[\mathbf{H}] \cdot \mathbf{H}$ for each symmetric second-order tensor $\mathbf{H}$. The symmetries (4.1) and (4.2) imply that

$$\Lambda[\nabla \mathbf{v}] \cdot \nabla \mathbf{v} = \Lambda[\mathbf{H}] \cdot \mathbf{H}$$

where $\mathbf{H} = \frac{1}{2}[\nabla \mathbf{v} + (\nabla \mathbf{v})^T]$ is the symmetric part of $\nabla \mathbf{v}$. Hence

$$\bar{c}_7|\mathbf{H}|^2 \leqslant \Lambda[\nabla \mathbf{v}] \cdot \nabla \mathbf{v}$$

and integrating this inequality over V_0 leads to

$$\bar{c}_7 \int_{v_0} |\mathbf{H}|^2 \, dV_0 \leqslant D(t). \tag{4.13}$$

Since $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is an admissible motion of the body, it follows from (2.5) that

$$\mathbf{v}(\mathbf{X}, t) = \mathbf{O} \quad \text{on } \mathbb{S}_1$$

and hence the Korn inequality (see Necas and Hlavácek [8]), says that

$$\bar{\bar{c}}_7 \| \nabla \mathbf{v}(\cdot, t) \|^2 \leqslant \int_{V_0} |\mathbf{H}|^2 \, dV_0 \tag{4.14}$$

where $\bar{\bar{c}}_7 > 0$ is a constant independent of the motion. Inequalities (4.13) and (4.14) then yield $(4.9)_1$ with $c_7 = \bar{c}_7 \bar{\bar{c}}_7$. The same argument leads to $(4.10)_1$. Observing that $\mathbf{\Lambda}[\mathbf{G}] \cdot \mathbf{G} \leqslant c_8 |\mathbf{G}|^2$ for some $c_8 > 0$ and all second order tensors leads to $(4.9)_2$ and $(4.10)_2$. Finally, since $\mathbf{x} - \mathbf{x}_0 = \mathbf{O}$ and $\mathbf{v} = \mathbf{O}$ on $\mathbb{S}_1$, the Poincaré inequality, the definitions of U, K, and D, and the inequality $(4.9)_1$ lead to (4.11) and (4.12). The proof is complete.

Our next result establishes Lyapunov stability of the reference configuration (cf [2]).

THEOREM 2. There exists a constant $c > 0$ such that for each admissible motion of the body and for each $t \geqslant 0$ we have

$$\|\mathbf{u}(\cdot, t)\| \leqslant c\big(\|\nabla \mathbf{u}(\cdot, 0)\| + \|\mathbf{v}(\cdot, 0)\|\big), \tag{4.15}$$

$$\|\nabla \mathbf{u}(\cdot, t)\| \leqslant c\big(\|\nabla \mathbf{u}(\cdot, 0)\| + \|\mathbf{v}(\cdot, 0)\|\big), \tag{4.16}$$

$$\|\mathbf{v}(\cdot, t)\| \leqslant c\big(\|\nabla \mathbf{u}(\cdot, 0)\| + \|\mathbf{v}(\cdot, 0)\|\big), \tag{4.17}$$

and

$$\left(\int_0^t \|\nabla \mathbf{v}(\cdot, \tau)\|^2 \, d\tau\right)^{1/2} \leqslant c\big(\|\nabla \mathbf{u}(\cdot, 0)\| + \|\mathbf{v}(\cdot, 0)\|\big). \tag{4.18}$$

PROOF. Integrating the identity (3.6) over $[0, t]$ yields

$$K(t) + P(t) + \int_0^t D(\tau) \, d\tau = K(0) + P(0) \tag{4.19}$$

which in view of the identity

$$K(t) = \tfrac{1}{2}\rho_0 \|(\cdot, t)\|^2, \tag{4.20}$$

and inequalities (3.12), $(4.9)_1$, and (3.22) yields

$$\begin{aligned} \tfrac{1}{2}\rho_0 \|\mathbf{v}(\cdot, t)\|^2 + c_1 \|\nabla \mathbf{u}(\cdot, t)\|^2 + c_7 \int_0^t \|\nabla \mathbf{v}(\cdot, \tau)\|^2 \, d\tau \\ \leqslant \tfrac{1}{2}\rho_0 \|\mathbf{v}(\cdot, 0)\|^2 + c_6 \|\nabla \mathbf{u}(\cdot, 0)\|^2 \\ \leqslant \big(\tfrac{1}{2}\rho_0 + c_6\big)\big(\|\nabla \mathbf{u}(\cdot, 0)\| + \|v(\cdot, 0)\|\big)^2. \end{aligned}$$

The last inequality implies (4.16), (4.17), and (4.18) with the constant c replaced by $[c_1^{-1}(\tfrac{1}{2}\rho_0 + c_6)]^{1/2}$. It follows that (4.15)–(4.18) are satisfied with

$$c = \big(\tfrac{1}{2}\rho_0 + c_6\big)^{1/2} \max\left\{c_1^{-1/2}, \big(2\rho_0^{-1}\big)^{1/2}, c_7^{-1/2}, c_4 c_1^{-1/2}\right\}.$$

The proof is complete.

Setting $\mathbf{u}(\cdot,0) = \mathbf{O}$ and $\mathbf{v}(\cdot,0) = \mathbf{O}$, we obtain the following uniqueness results (cf. [2]).

COROLLARY. If $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ is an admissible motion of the body such that $\mathbf{x}(\mathbf{X},0) = \mathbf{X}$ and $\mathbf{v}(\mathbf{X},0) = \mathbf{O}$ for all $\mathbf{X} \in V_0$, then

$$\mathbf{x}(\mathbf{X}, t) = \mathbf{X}, \qquad \mathbf{X} \in V_0, t \geqslant 0.$$

It is worth mentioning that this corollary can be established under weaker hypotheses, namely that (4.8) is satisfied for all second order tensors and

$$P(\mathbf{x}(\cdot)) \geqslant 0 \tag{4.21}$$

for each admissible configuration $\mathbf{x}$. Indeed, (4.8) yields $D(t) \geqslant 0$ for each admissible motion and the identity (4.19) implies

$$K(t) + P(t) \leqslant K(0) + P(0) = 0 \tag{4.22}$$

where $(4.22)_2$ follows from the intitial conditions of the special motion about which the corollary speaks. By (4.21) we have $P(t) \geqslant 0$ and since also $K(t) \geqslant 0$, we see from (4.22) that $K(t) = P(t) = 0$. But $K(t) = 0$ implies $\mathbf{v}(\cdot, t) = \mathbf{O}$ in V_0 and the result follows.

5. Asymptotic stability. We are now able to prove the main result of this paper. We employ the following terminology: a positive function $f(t)$, $t \geqslant 0$, is said to decay exponentially with exponent $\nu > 0$ as $t \to \infty$ if there exists a positive constant A such that $f(t) \leqslant Ae^{-\nu t}$. It is possible to prove

THEOREM 3. There exists a constant $\nu > 0$ such that for each admissible solution $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ the functions $\|\mathbf{u}(\cdot, t)\|$, $\|\nabla\mathbf{u}(\cdot, t)\|$, $\|\mathbf{v}(\cdot, t)\|$ and $\{\int_t^\infty \|\nabla\mathbf{v}(\cdot, \tau)\|\, d\tau\}^{1/2}$ decay exponentially with exponent ν as $t \to \infty$.

PROOF. First we show that our hypotheses imply that we can choose the positive constants ν_1, ν_2 and ν_3 small enough to satisfy the following inequalities:

$$\nu_1\left(1 + \tfrac{3}{2}\nu_2\right)K \leqslant D \tag{5.1}$$

$$-\tfrac{1}{2}I + \nu_2 P + \tfrac{1}{2}\nu_1\nu_2 L + \tfrac{1}{2}\nu_1^2\nu_2 U \leqslant 0 \tag{5.2}$$

and

$$P + \tfrac{1}{2}\nu_1 L - \tfrac{1}{2}\nu_1^2 U \geqslant \nu_3\|\nabla\mathbf{u}(\cdot, t)\|^2 \tag{5.3}$$

for each admissible solution of the body. Note that from inequalities (3.12) and (4.11) there exists a positive constant ν_1 such that

$$P - \tfrac{1}{2}\nu_1^2 U \geqslant 0 \tag{5.4}$$

Then, from inequality (4.10), result (5.3) follows with $\nu_3 = \frac{1}{4}\nu_1 c_7$. Moreover, from inequalities (3.13), (3.22), $(4.10)_2$ and (4.11) for sufficiently small ν_1 and ν_2 inequality (5.2) is satisfied. Finally, on comparing inequalities (4.12) with (5.1) it is clear that (5.1) is valid provided ν_1 and ν_2 are small enough.

Forming (3.3) plus $\frac{1}{2}\nu_1$ of identity (4.4) reveals that

$$\dot{K} + \dot{P} + \tfrac{1}{2}\nu_1\ddot{U} - \nu_1 K + \tfrac{1}{2}\nu_1 I + \tfrac{1}{2}\nu_1\dot{L} = -D. \tag{5.5}$$

Multiply this equation at $t = \tau$ by $e^{\nu_1 \nu_2 \tau}$ and integrate over $[0, t]$ to obtain

$$\left[\left(K + P + \frac{\nu_1}{2}\dot{U} + \frac{\nu_1}{2}L\right)e^{\nu_1\nu_2\tau}\right]_0^t$$
$$= \int_0^t \left\{-D + \nu_1(1+\nu_2)K - \tfrac{1}{2}\nu_1 I + \nu_1\nu_2 P + \tfrac{1}{2}\nu_1^2\nu_2\dot{U} + \tfrac{1}{2}\nu_1^2\nu_2 L\right\} e^{\nu_1\nu_2\tau}\, d\tau. \tag{5.6}$$

Next, observe the weighted arithmetic-geometric mean inequality

$$\pm \dot{U} \leqslant K/w + wU \tag{5.7}$$

where the positive weight w is constant. With $w = \nu_1$ in (5.7) then the right-hand side of equation (5.6) in combination with inequalities (5.1) and (5.2) yields

$$K + P + \nu_1\dot{U}/2 + \nu_1 L/2 \leqslant Ae^{-2\nu t} \tag{5.8}$$

where

$$A = K(0) + P(0) + \tfrac{1}{2}\nu_1\dot{U}(0) + \tfrac{1}{2}\nu_1 L(0) \quad \text{and} \quad 2\nu = \nu_1\nu_2.$$

A further application of (5.7) with $w = \nu_1$ provides

$$\tfrac{1}{2}K + P + \tfrac{1}{2}\nu_1 L - \tfrac{1}{2}\nu_1^2 U \leqslant Ae^{-2\nu t}. \tag{5.9}$$

Thus, from (5.3) and (4.20) it follows that

$$\tfrac{1}{4}\rho_0\|\mathbf{v}(\cdot, t)\|^2 + \nu_3\|\nabla\mathbf{u}(\cdot, t)\|^2 \leqslant Ae^{-2\nu t}. \tag{5.10}$$

Hence, both $\|\mathbf{v}(\cdot, t)\|$ and $\|\nabla\mathbf{u}(\cdot, t)\|$ decay exponentially as $t \to \infty$ with exponent ν. The Poincaré inequality (3.19) then leads to $\|\mathbf{u}(\cdot, t)\|$ decaying exponentially with exponent ν. Finally, integrating (3.3) over $[t, T]$ with $t < T$ gives

$$K(T) + P(T) + \int_t^T D(\tau)\, d\tau = K(t) + P(t). \tag{5.11}$$

On letting $T \to \infty$ and using the established exponential decay of K and P then we obtain

$$\int_t^\infty D(\tau)\, d\tau = K(t) + P(t). \tag{5.12}$$

The inequalities (4.9), (3.22) and the identity (4.2) provides

$$c_7\int_t^\infty \|\nabla\mathbf{v}(\cdot, \tau)\|^2\, d\tau \leqslant \left(\tfrac{1}{2}\rho_0 + c_6\right)\left\{\|\mathbf{v}(\cdot, t)\|^2 + \|\nabla\mathbf{u}(\cdot, t)\|^2\right\}. \tag{5.13}$$

Then, the exponential decay already established on the right hand side of (5.13) implies the desired result.

The theorem is proved.

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