CRACK SPEEDS IN AN IDEAL FIBER-REINFORCED MATERIAL*

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1. Introduction. Mannion and Pipkin [1] have set up the equations governing dynamic crack propagation in plane stress of sheets or plates composed of two orthogonal families of inextensible fibers. They discussed the crack trajectories in a particularly simple problem, in which the crack has two straight segments connected by a rounded corner. In quasi-static loading the rounded corner is absent and the crack is L-shaped. This has been observed experimentally [2, 3].

For both quasi-static and fully dynamical loading, in the first stage of the motion the crack runs in a straight line with a speed that depends on the initial and boundary conditions. In this stage of the motion the problem reduces to that of solving a free boundary problem for the one-dimensional wave equation. In the present paper we examine this problem in detail, and show how to solve it exactly.

The problem is outlined in Sec. 2. The wave equation is to be satisfied in a domain $0 \leqslant x \leqslant x_0(t)$ that grows as time progresses, $x_0(t)$ being the position of the crack tip. If $x_0(t)$ were known the solution would be straightforward (Sec. 3). However, $x_0(t)$ is not known, but must be determined by using a fracture criterion. The particularly simple fracture criterion that we use leads to a differential-difference inequality to be satisfied by $x_0(t)$ and the functions that arise in the general solution of the wave equation. This inequality is analyzed in Sec. 4. The results in Sec. 3 and 4 give a pair of recursion relations that can be used to determine the deformation and the tip motion in a sequence of intervals determined by the solution (Sec. 5).

Although the purpose of this paper is largely mathematical, we give two examples that may be of physical interest. In Sec. 6 we consider an example of unstable crack growth and crack arrest in an initially over-stressed specimen. The second example, in Sec. 7, illustrates the approach of the crack velocity to a steady-state value when the opening displacement is increased at a constant rate.

Freund [4, 5] has proposed a shear-beam model for double cantilever specimens that are not necessary fiber-reinforced, and has used this model to solve the problem that we

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consider in Sec. 6. The energy release rate fracture criterion used by Freund is different from the critical stress criterion that we use, but very similar to it. With this modification of the fracture criterion, the results in the present paper are applicable to Freund's model.

2. The problem. The problem concerns propagation of a crack into a quarter-infinite sheet $x \geqslant 0$, $y \leqslant H$, composed of inextensible fibers that lie parallel to the x and y directions. The crack starts at $x = y = 0$ and, during an initial stage of its propagation, runs straight along the x-axis. Here we consider only this initial stage. During this stage the inextensibility conditions and the condition of zero displacement at infinity imply that the x-component of displacement is zero and that the y-component is zero except in the region above the crack, where it is a function $u(x, t)$ independent of y. In this region the relation between the shearing stress μu_x and the momentum ρu_t leads to a wave equation:

$$u_{tt} = c^2 u_{xx}, \quad c^2 = \mu/\rho \qquad (0 \leqslant x \leqslant x_0(t), t \geqslant 0). \tag{2.1}$$

The boundary $x = x_0(t)$, which is unknown, is the position of the crack tip. Continuity of u at the tip implies that $u = 0$ there. The value of u at $x = 0$, the opening displacement, is prescribed:

$$u(x_0(t), t) = 0, \qquad u(0, t) = D(t). \tag{2.2}$$

The crack initially has a non-zero length $x_0(0)$, and u and u_t are prescribed in the initial domain:

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = cv_0'(x) \qquad (0 \leqslant x \leqslant x_0(0)). \tag{2.3}$$

The stress and momentum discontinuities across the fiber $x = x_0$ cause a finite force in this fiber, whose value at the crack tip is [1]

$$F = -\mu H u_x(x_0(t), t)\left[1 - U^2(t)/c^2\right]. \tag{2.4}$$

Here U is the crack propagation velocity,

$$U = x_0'(t) \geqslant 0. \tag{2.5}$$

For a fracture criterion, we use the critical force criterion, according to which the fiber $x = x_0$ breaks when F reaches a critical value F_c:

$$F \leqslant F_c, \quad U \geqslant 0, \qquad F = F_c \quad \text{if } U > 0. \tag{2.6}$$

The more realistic critical stress criterion [1] reduces to the same form in the present problem.

Two scale factors are important:

$$M = F_c/\mu H, \qquad V = Mc/2. \tag{2.7}$$

According to (2.4) and (2.6), M is the maximum amount of shear $(-u_x)$ that can be maintained statically. M must be small in order for infinitesimal deformation theory to be valid as assumed. Problems can be treated quasistatically when the opening rate D' is small in comparison to V [1].

3. Solution when the tip motion is known. The general solution of (2.1) is

$$u = Vf(t - x/c) - Vg(t + x/c), \tag{3.1}$$

where the factor V is introduced for convenience later. The initial conditions (2.3) are satisfied if we take

$$f(t) = (2V)^{-1}[u_0(-ct) - v_0(-ct)] \qquad (-t^* \leq t \leq 0) \tag{3.2}$$

and

$$g(t) = -(2V)^{-1}[u_0(ct) + v_0(ct)] \qquad (0 \leq t \leq t^*), \tag{3.3}$$

where t^* is the time required for a wave with speed c to traverse the initial length of the crack:

$$t^* = x_0(0)/c. \tag{3.4}$$

The condition $u = 0$ at the crack tip gives the reflected wave g in terms of the wave f incident on the tip:

$$g(t + x_0(t)/c) = f(t - x_0(t)/c). \tag{3.5}$$

The forward wave f at $x = 0$ is determined by the displacement $u = D$ prescribed there, as modified by the returning wave g:

$$f(t) = D(t)/V + g(t). \tag{3.6}$$

In particular, with (3.3), f is given on $[0, t^*]$ by

$$f(t) = D(t)/V - (2V)^{-1}[u_0(ct) + v_0(ct)] \qquad (0 \leq t \leq t^*). \tag{3.7}$$

Thus from (3.2) and (3.7), the forward wave f is known on the initial interval $[-t^*, t^*]$.

Consider a wave that starts from the end $x = 0$ at time t. Let $t_0(t)$ be the time at which it reaches the crack tip:

$$x_0(t_0(t)) = c[t_0(t) - t]. \tag{3.8}$$

Let $t_1(t)$ be the time at which its reflection from the tip returns to $x = 0$:

$$t_1(t) = t_0(t) + x_0(t_0(t))/c = t + 2[t_0(t) - t]. \tag{3.9}$$

When the tip trajectory $x_0(t)$ is known, $t_0(t)$ and $t_1(t)$ are easily determined graphically. In fact, any one of these three functions determines the other two.

The relations (3.5) and (3.6) can be written more concisely in terms of t and $t_1(t)$. Replacing t by t_0 in (3.5) gives

$$g(t_1) = f(t). \tag{3.10}$$

With this, (3.6) gives a recursion relation for f:

$$f(t_1) = D(t_1)/V + f(t). \tag{3.11}$$

The forward wave f can now be found for all t by beginning with the initial data (3.2) and (3.7). With f known, g is given by (3.10), except on the initial interval where it is specified by (3.3). With f and g known, (3.1) gives the displacement.

4. Analysis of the tip condition. To determine the tip trajectory, or thus the function $t_1(t)$ to be used in (3.11), we use the fracture criterion (2.6). The relation to be derived actually connects $t_0(t)$ to $f(t)$, but as remarked earlier, the functions $t_0(t)$, $t_1(t)$ and $x_0(t)$ all contain equivalent information about the trajectory.

We first reduce the fracture criterion to a relation between the tip velocity U and the stress carried by the forward wave f. With (2.4), (2.7) and (3.1), the fracture criterion (2.6) yields

$$\left[f'(t - x_0/c) + g'(t + x_0/c)\right]\left[1 - (U/c)^2\right] \leqslant 2. \tag{4.1}$$

To eliminate the reflected wave g, we differentiate (3.5) to obtain

$$g'(t + x_0/c)(1 + U/c) = f'(t - x_0/c)(1 - U/c). \tag{4.2}$$

Then substitution in (4.1) gives

$$f'(t - x_0/c)(1 - U/c) \leqslant 1, \tag{4.3}$$

with equality when $U > 0$.

This condition is expressed more concisely in terms of $t_0(t)$. From (3.8) we have

$$t_0'(t)\left[1 - U(t_0)/c\right] = 1. \tag{4.4}$$

Then by replacing t by t_0 in (4.3) and using (3.8) and (4.4) we obtain

$$f'(t) \leqslant t_0'(t), \tag{4.5}$$

with equality when $U > 0$.

Let f_0 be the value of f at the crack tip. Then (4.3) is simply the condition that $df_0/dt \leqslant 1$. When the stress carried by the incident wave is small, specifically $f' \leqslant 1$, (4.3) is satisfied with $U = 0$ and the crack does not propagate, so $t_0' = 1$. But when the stress carried by the incoming wave is large, so that $f' > 1$, the tip moves forward at a rate sufficient to reduce df_0/dt to unity, and then $t_0' = f'$. In summary, we have

$$\begin{aligned} f'(t) \leqslant t_0'(t) &= 1 \quad \text{if } U(t_0) = 0, \\ f'(t) = t_0'(t) &> 1 \quad \text{if } U(t_0) > 0. \end{aligned} \tag{4.6}$$

These results allow us to express t_0' directly in terms of f':

$$t_0'(t) = \max\left[f'(t), 1\right]. \tag{4.7}$$

With t^* as defined in (3.4), the initial value of t_0 is

$$t_0(-t^*) = 0. \tag{4.8}$$

Then (4.7) and (4.8) allow us to determine the crack trajectory, in terms of $t_0(t)$, when the incident wave $f(t)$ is known.

5. Solution by recursion. As explained in Sec. 3, the displacement u can be determined when the forward wave f is known, so our object is to determine f for all time. This can now be done recursively. First, the initial conditions (3.2) and (3.7) specify f on the interval $[-t^*, t^*]$. The forward wave determines the tip motion, so from (4.7) and (4.8) we can determine $t_0(t)$ on the same interval, and (3.9) gives $t_1(t)$. Then (3.11) gives $f(t)$ on the new interval $[t_1(-t^*), t_1(t^*)]$. The process is now repeated, beginning with the values of f on the new interval. Since $t_1' \geqslant 1$, each succeeding interval is at least as long as the preceding one, so eventually any chosen time t is reached.

Certain qualitative conclusions about the motion are derived more easily by using a differential form of the recursion relation. By differentiating (3.11) we obtain

$$f'(t_1) = D'(t_1)/V + f'(t)/t_1'(t), \tag{5.1}$$

and the final term here is, from (3.9) and (4.7),

$$f'(t)/t_1'(t) = f'(t)/[2\max(f', 1) - 1]. \tag{5.2}$$

One immediate conclusion is that if the opening displacement $D(t)$ ceases to increase, the crack stops propagating as soon as the news reaches the tip. For, the term (5.2) does not exceed unity, so (5.1) gives $f' \leqslant 1$ when $D' \leqslant 0$. But when a wave with $f' \leqslant 1$ reaches the tip, the crack stops, according to (4.6). We note in passing that the crack may also stop even while $D' > 0$, because a reduction in the opening rate creates an unloading wave.

If D grows arbitrarily large the crack must eventually propagate, but it does not necessarily do so immediately. Propagation during the initial interval $[0, t^*]$ depends on the initial conditions rather than on D. Let us suppose that $f' \leqslant 1$ at first, so that the crack does not propagate. Then (5.1) reduces to

$$f'(t_1) = D'(t_1) + f'(t). \tag{5.3}$$

Each round trip of a wave from $x = 0$ to the tip and back increases f' by the amount D', so if $D' \geqslant k > 0$ then f' must eventually exceed unity. However, if D' is small there can be a long initial interval in which the crack does not propagate.

The recursion process is simplified if it is known from the outset that the crack is always propagating. To be specific, let us consider problems in which $u = u_t = 0$ initially. Then $f = 0$ for $t < 0$ and $f = D/V$ on $[0, t^*]$. Use of the recursion relations, beginning with $f = 0$ for $t < 0$, shows that f is still equal to D/V on $[t^*, 2t^*]$:

$$f(t) = D(t)/V \qquad (0 \leqslant t \leqslant 2t^*). \tag{5.4}$$

Then if $D' > V$, the crack starts to propagate as soon as the first wave reaches the tip (at $t = t^*$). For the crack to continue to propagate for all time it is sufficient (but not necessary) that D' remains greater than V for all time. In such cases (4.7) and (4.8) give

$$t_0(t) = t^* + f(t) \qquad (t \geqslant 0), \tag{5.5}$$

and (3.9) yields

$$t_1(t) = 2t^* + 2f(t) - t \qquad (t \geqslant 0). \tag{5.6}$$

Then the recursion to find f and t_1 can be done with (3.11) and (5.6), with the initial condition (5.4).

The recursive method of solution requires knowledge of f on an initial interval of non-zero length. If the length of the crack is zero initially, we know only the single value $f(0) = 0$ so the recursion relation cannot be used immediately. To determine f on a finite interval, we suppose that $D(t)$ has a power-series expansion for small t and we seek a power-series representation of f. Assuming that $D'(0) > 0$, a displacement discontinuity would develop if the crack did not extend, so $f'(0) > 1$ according to (4.6). Also $t_1(0) = 0$ if the crack has zero length initially. Then (5.1) and (5.2) give

$$[f'(0) - D'(0)/V][2f'(0) - 1] - f'(0) = 0. \tag{5.7}$$

This is a quadratic equation for $f'(0)$, whose larger root exceeds unity if $D'(0) > 0$ as assumed. Then f is approximately $f'(0)t$ on a sufficiently short interval. Higher-order terms in the power-series expansion can be computed from (5.1) if a longer initial interval is wanted.

6. Example: Slow wedging. As an example, suppose that a crack of initial length L is wedged open so that initially

$$u = D_0(1 - x/L) \quad \text{and} \quad u_t = 0, \tag{6.1}$$

with an amount of shear D_0/L that exceeds the largest value that can be maintained statically, M (see (2.7)). Let K be the overstress factor defined by

$$K = D_0/LM > 1. \tag{6.2}$$

At $t = 0$ the constraint preventing the crack from running is released, and for $t > 0$ the opening displacement is held fixed at the value D_0. From Sec. 5 we know that the crack will run only in the interval $0 \leqslant t \leqslant t^*$, where $t^* = L/c$ is the time at which the signal that D is not increasing reaches the tip.

The forward wave on the initial interval is found by using the preceding data in (3.2) and (3.7):

$$f(t) = K(t^* + t) \qquad (-t^* \leqslant t \leqslant t^*). \tag{6.3}$$

Then $f' = K$. If K were not greater than unity the crack would not propagate, according to (4.6). With $K > 1$ as assumed, (4.7) and (3.9) give

$$t_0(t) = K(t + t^*), \quad t_1(t) = (2K - 1)t + 2Kt^* \qquad (-t^* \leqslant t \leqslant t^*). \tag{6.4}$$

The values of t_1 span the interval from t^* to $(4K - 1)t^*$. then from (3.11), the forward wave is given on this new interval by

$$f(t) = 3Kt^* + K(t - 2Kt^*)/(2K - 1). \tag{6.5}$$

But then $f' < 1$, so the crack ceases to propagate. It then follows from (4.7), with $t_0(t^*) = 2Kt^*$, that

$$t_0(t) = (2K - 1)t^* + t, \tag{6.6}$$

and (3.9) gives

$$t_1(t) = 2(2K - 1)t^* + t. \tag{6.7}$$

The relations (6.5) to (6.7) remain valid for all $t \geqslant t^*$.

The velocity of the crack while it is running is given by

$$U/c = 1 - 1/t_0'(t) = 1 - K^{-1}. \tag{6.8}$$

Thus U approaches the shear-wave speed c when the overstress factor K is very large. The final length of the crack is, from (6.6),

$$c(t_0 - t) = ct^*(2K - 1) = L(2K - 1), \tag{6.9}$$

and the time required for the crack to reach this length is $2KL/c$. As soon as the crack reaches its final length the material becomes motionless, in a state of simple shear with the amount of shear equal to

$$D_0/L(2K-1) = MK/(2K-1) < M, \tag{6.10}$$

where M is the static limit. Thus if the initial overstress K is very large, the crack runs until the amount of shear is only half the static limit.

Freund [4, 5] has solved the present problem by using an energy release rate fracture criterion that is quadratic in the amount of shear but otherwise the same as the criterion we have used. Freund's results differ from ours only in their form of dependence on the overstress factor K. These results are discussed much more fully in Freund's papers [4, 5].

7. Example: Fast wedging. As a second example, let us suppose that $u = u_t = 0$ initially and that the crack is opened at a constant rate S. If $S < V$ (see (2.7)) there is some initial interval during which waves propagate back and forth from $x = 0$ to the tip of the crack but the crack does not propagate. To simplify the example, let us suppose that $S > V$, so that the crack begins to extend as soon as the first wave reaches the tip.

The present problem belongs to a class considered in Sec. 5, for which the crack propagates continually. In the present case, the initial condition (5.4) becomes

$$f(t) = at, \quad a = S/V > 1 \qquad (0 \leqslant t \leqslant 2t^*), \tag{7.1}$$

and the recursion relations (3.11) and (5.6) are

$$t_1(t) = 2t^* + 2f(t) - t \qquad (t \geqslant 0) \tag{7.2}$$

and

$$f(t_1) = at_1 + f(t) \qquad (t \geqslant 0). \tag{7.3}$$

With the initial condition (7.1), the recursion relations (7.2) and (7.3) imply that f is piecewise linear. The slope of f can change at the times 0, $t_1(0) = 2t^*$, $t_1(t_1(0))$, and so on, a sequence of times to be determined by recursion. These times and the corresponding values of f can be generated from (7.2) and (7.3) without calculating f at the intermediate times. The first few values in these sequences are shown in Table 1.

TABLE 1

t/t^*	$f(t)/t^*$	$\Delta f/t^*$	$\Delta t/t^*$	f'
0	0			
2	$2a$	$2a$	2	a
$4a$	$4a^2 + 2a$	$4a^2$	$4a - 2$	$a(2a)/(2a-1)$
$8a^2 + 2$	$8a^3 + 4a^2 + 4a$	$8a^3 + 2a$	$8a^2 - 4a + 2$	$a(4a^2+1)/(4a^2-2a+1)$

Since f is piecewise linear, the difference quotient $\Delta f/\Delta t$ is the derivative f', shown in the last column. The values of f' converge rapidly to a value $f'(\infty)$. This value can be found directly by assuming that f and t_1 are linear for large t, and using (7.2) and (7.3). The value found in this way is

$$f'(\infty) = (1/2)\left[a + 1 + (a^2 + 1)^{1/2}\right]. \tag{7.4}$$

From (4.4) and (7.2), the crack velocity is given by

$$U(t_0)/c = 1 - 1/f'(t). \tag{7.5}$$

Thus the speed is constant in each interval. Since the approach of f' to its limit is oscillating, the same is true for the convergence of the crack speed to its limiting value. The crack speed is smallest during the first interval:

$$U_1/c = 1 - a^{-1}. \tag{7.6}$$

It is largest in the second interval:

$$U_2/c = 1 - a^{-1} + (2a^2)^{-1}. \tag{7.7}$$

Successive values of U for a few values of the parameter $a = S/V$ are given in Table 2.

TABLE 2

a	U_1/c	U_2/c	U_3/c	...	U_∞/c
1	0	0.5	0.4		0.414
2	0.5	0.625	0.618		0.618
3	0.667	0.722	0.721		0.721

Convergence to the steady-state velocity is very rapid at all opening rates, so although the approach to the limit is oscillatory, the main change in velocity is a sudden increase from the initial to the final value.

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