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ON THE BOUNDARY-VALUE PROBLEM ASSOCIATED WITH A GENERAL TWISTED TUBE WITH A SLOWLY VARYING CIRCULAR SECTION*

By

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Abstract. A non-orthogonal curvilinear coordinate system is used to formulate the Dirichlet problem of potential theory associated with the interior of a general twisted tube with a slowly varying circular section. A solution scheme is presented for two cases of a tube of finite length.

Introduction. Recently [1] a method of solution has been presented for the Dirichlet problem of potential theory associated with the interior of a general twisted tube with a uniform non-rotating section. For such a tube it was possible to construct an orthogonal curvilinear coordinate system which could be employed to formulate the boundary-value problem. However, for a tube with a non-uniform section a non-orthogonal curvilinear coordinate system must be used.

It is the purpose of this paper to use the non-orthogonal curvilinear coordinate system established in [2] to formulate the Dirichlet problem of potential theory for the interior of a general twisted tube of finite length and slowly varying circular section. The method of solution presented in the analysis is based on an iterative scheme and involves, respectively, two and three small parameters for the two cases considered.

1.**The coordinate system.** The interior and boundary of a tube of finite length l in R_3 is denoted by D_3 and ∂D_3 respectively and the tube orientation is specified by a curve L (Fig. 1) which has a prescribed unit tangent vector $\mathbf{t}_1(\xi^1)$. The coordinate ξ^1 measures the arc length along L from the origin O to the point O'. The point O' is the centre of the circular section denoted by $D_2 \cup \partial D_2$ which is normal to L and has radius $a(\xi^1)$. If B denotes the curved part of ∂D_3 then ∂D_3 is the union of B, $D_2 \cup \partial D_2$ $(\xi^1 = 0)$, and $D_2 \cup \partial D_2$ $(\xi^1 = l)$. The unit tangent vector $\mathbf{t}_1$ is given by

$$\mathbf{t}_1 \equiv \begin{pmatrix} & \cos\theta \\ \sin\theta & \sin\phi \\ \sin\theta & \cos\phi \end{pmatrix}, \tag{1.1}$$

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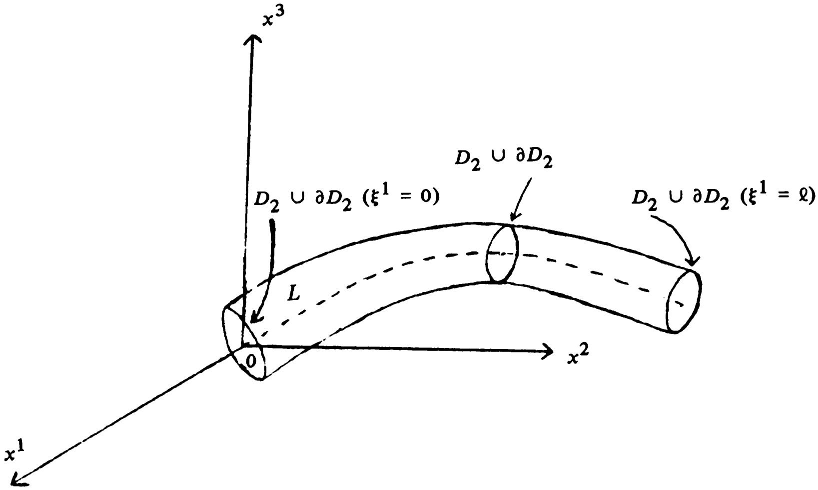

FIG. 1.

where the angles θ and ϕ are prescribed twice-differentiable functions of ξ^1. In what follows we will also need the two unit vectors $\mathbf{t}_2$ and $\mathbf{t}_3$ where

$$\mathbf{t}_2 \equiv \begin{pmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{pmatrix}, \tag{1.2}$$

$$\mathbf{t}_3 \equiv \begin{pmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta \cos\phi \end{pmatrix}, \tag{1.3}$$

respectively. The vectors $\mathbf{t}_i$, $i = 1, 2, 3$, are then mutually orthogonal.

It has been shown [2] that a non-orthogonal curvilinear coordinate system can be constructed for the tube when the unit normal to ∂D_2 which lies in the section $D_2 \cup \partial D_2$ is prescribed. These coordinates are denoted by ξ^i, $i = 1, 2, 3$, where $\xi^2 = 0$ on ∂D_2 and $\xi^2 = -\infty$ at O' for all values of ξ^1 and ξ^3. The transformation from Cartesian coordinates x^i, $i = 1, 2, 3$, to the curvilinear coordinates ξ^i, $i = 1, 2, 3$, is given by

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = T_1(\phi)^{-1} T_2(\theta)^{-1} \begin{pmatrix} 0 \\ v \\ u \end{pmatrix} + \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}, \tag{1.4}$$

where

$$T_1(\phi) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \tag{1.5}$$

$$T_2(\theta) \equiv \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{1.6}$$

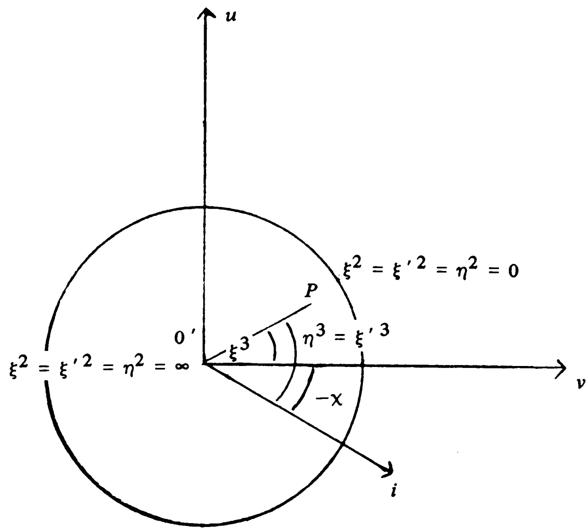

FIG. 2.

$$\begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix} \equiv \int_0^{\xi^1} \mathbf{t}_1(\bar{\xi}^1)\, d\bar{\xi}^1. \tag{1.7}$$

The point O' is represented by the vector in (1.7) and the functions $u(\xi^1, \xi^2, \xi^3)$ and $v(\xi^1, \xi^2, \xi^3)$ are given by

$$v + iu = a(\xi^1)e^{\xi^2 + i\xi^3}. \tag{1.8}$$

Moreover $O'v$, $O'u$ are the axes of a Cartesian frame of reference and coincide with the unit vectors $\mathbf{t}_2$ and $\mathbf{t}_3$ respectively and the coordinate ξ^3 measures the angle between $O'v$ and $O'P$ (Fig. 2). Additional properties of the coordinate system are given in [2] and will be employed in the analysis which follows.

2. Formulation of the boundary-value problem. Laplace's equation in the non-orthogonal curvilinear coordinate system ξ^i, $i = 1, 2, 3$, has the form

$$\frac{1}{J}\frac{\partial}{\partial \xi^i}\left(J g^{ij}\frac{\partial V}{\partial \xi^j}\right) = 0, \tag{2.1}$$

where J is the Jacobian of the transformation given by Eqs. (1.4)–(1.7) and g^{ij}, i, $j = 1, 2, 3$, are the components of the contravariant metric tensor. Employing the expressions for J and g^{ij} which were established in [2] together with the relation $\dot{\gamma} = -\tau - \omega$ [1], we find that Eq. (2.1) can be written in the form

$$\begin{aligned} & b_1 \frac{\partial^2 V}{\partial \xi^{1^2}} + b_1 a^{-2}\left(e^{-2\xi^2} b_1^2 + \dot{a}^2\right) \frac{\partial^2 V}{\partial \xi^{2^2}} + b_1\left(a^{-2} e^{-2\xi^2} b_1^2 + \omega^2\right) \frac{\partial^2 V}{\partial \xi^{3^2}} \\ & \quad -2a^{-1}\dot{a} b_1 \frac{\partial^2 V}{\partial \xi^1 \partial \xi^2} - 2a^{-1}\dot{a}\omega b_1 \frac{\partial^2 V}{\partial \xi^2 \partial \xi^3} + 2\omega b_1 \frac{\partial^2 V}{\partial \xi^3 \partial \xi^1} + a e^{\xi^2} b_2 \frac{\partial V}{\partial \xi^1} \\ & \quad + \left(a^{-2}(\dot{a}^2 - a\ddot{a}) b_1 - \dot{a} e^{\xi^2} b_2 - a^{-1} e^{-\xi^2} b_1^2 \kappa \sin\beta\right) \frac{\partial V}{\partial \xi^2} \\ & \quad + \left(\dot{\omega} b_1 + a e^{\xi^2} \omega b_2 - a^{-1} e^{-\xi^2} b_1^2 \kappa \cos\beta\right) \frac{\partial V}{\partial \xi^3} = 0, \end{aligned} \tag{2.2}$$

where $b_1 = 1 - \kappa a e^{\xi^2} \sin\beta$, $b_2 = \dot{\kappa} \sin\beta - \kappa\tau\cos\beta$, $\beta = \gamma + \xi^3$, $\dot{\theta} = \kappa\cos\gamma$, $\dot{\phi}\sin\theta = \kappa\sin\gamma$ and $\omega = \dot{\phi}\cos\theta$. The curvature and torsion of L are represented by $\kappa(\xi^1)$ and $\tau(\xi^1)$ respectively and the dot notation denotes the operation $d/d\xi^1$.

The quantity ω represents the rate at which the frame $O'v$, $O'u$ rotates about L as ξ^1 varies and can be eliminated from Eq. (2.2) by employing the transformation

$$\xi'^1 = \xi^1, \quad \xi'^2 = \xi^2, \quad \xi'^3 = \xi^3 - \chi, \tag{2.3}$$

where $\chi \equiv \int_0^{\xi^1} \omega(\bar{\xi}^1)\, d\bar{\xi}^1$. Eq. (2.2) now reduces to

$$\begin{aligned} & b_1 \frac{\partial^2 V}{\partial \xi'^{1^2}} + b_1 a^{-2}\left(e^{-2\xi'^2} b_1^2 + \dot{a}^2\right) \frac{\partial^2 V}{\partial \xi'^{2^2}} + b_1^3 a^{-2} e^{-2\xi'^2} \frac{\partial^2 V}{\partial \xi'^{3^2}} \\ & \quad - 2a^{-1}\dot{a} b_1 \frac{\partial^2 V}{\partial \xi'^1 \partial \xi'^2} + a e^{\xi'^2} b_2 \frac{\partial V}{\partial \xi'^1} \\ & \quad + \left(a^{-2}(\dot{a}^2 - a\ddot{a}) b_1 - \dot{a} e^{\xi'^2} b_2 - a^{-1} e^{-\xi'^2} b_1^2 \kappa \sin\beta\right) \frac{\partial V}{\partial \xi'^2} \\ & \quad - a^{-1} e^{-\xi'^2} b_1^2 \kappa \cos\beta \frac{\partial V}{\partial \xi'^3} = 0. \end{aligned} \tag{2.4}$$

The coordinate ξ'^3 now measures the angle between $O'P$ and $\mathbf{i}$ ($\equiv \cos\chi \mathbf{t}_2 + \sin\chi \mathbf{t}_3$) and the vector $\mathbf{i}$ does not rotate about L as ξ'^1 varies (see [1]).

The boundary conditions satisfied by V on ∂D_3 have the form

$$V = V_0(\xi'^1, \xi'^3) \quad \text{on } B, \tag{2.5}$$

$$V = V_1(\xi'^2, \xi'^3) \quad \text{on } D_2 \cup \partial D_2 \ (\xi'^1 = 0), \tag{2.6}$$

$$V = V_2(\xi'^2, \xi'^3) \quad \text{on } D_2 \cup \partial D_2 \ (\xi'^1 = l), \tag{2.7}$$

where V_0, V_1 and V_2 are prescribed functions. This completes the formulation of the boundary-value problem.

3. Solution scheme. If $Q \equiv \max_{0 \leqslant \xi'^1 \leqslant l} a$, $\varepsilon\ (< 1) \equiv \max_{0 \leqslant \xi'^1 \leqslant l} \kappa Q$ and $\eta^1 = \xi'^1/Q$ we can write $\kappa Q = \varepsilon_1 f_1(\eta^1)$ where $f_1(\eta^1)$ is $O(1)$. Also if τQ is $O(1)$ for $0 \leqslant \xi'^1 \leqslant l$ we can write $\tau Q = f_2(\eta^1)$ where $f_2(\eta^1)$ is $O(1)$. Moreover, since the tube section is slowly varying we can write $a = Q(1 + \varepsilon_3 f_3(\eta^1))$ where $0 < \varepsilon_3 < 1$ and $f_3(\eta^1)$ is $O(1)$. With $\eta^2,\ \eta^3 \equiv \xi'^2$, ξ'^3 equation (2.4) can be written in the form

$$
\begin{aligned}
V_{,11} + \nabla^2 V = {} & \epsilon_1 f_1 e^{\eta^2}(s_\beta V_{,11} + 3s_\beta \nabla^2 V + (f_2 c_\beta - M_1(|f_1|)s_\beta)V_{,1} \\
& + M_2(V)) - \epsilon_3 f_3(2V_{,11} - 2M_1(|f_3|)e^{\beta^2} V_{,12} - M_1(|f_3|)M_1(|f_{3,1}|)e^{\eta^2} V_{,2}) \\
& - \epsilon_1^2 f_1^2 e^{2\eta^2} s_\beta(3s_\beta \nabla^2 V + 2M_2(V)) - \epsilon_3^2 f_3^2(V_{,11} + M_1^2(|f_3|)e^{2\eta^2} V_{,22} \\
& - 2M_1(|f_3|)e^{\eta^2} V_{,12} + M_1(|f_3|)(2M_1(|f_3|) - M_1(|f_{3,1}|))e^{\eta^2} V_{,2}) \\
& + \epsilon_1 \epsilon_3 f_1 f_3 e^{\eta^2}(3s_\beta V_{,11} + 3s_\beta \nabla^2 V - 2M_1(|f_3|)s_\beta e^{\eta^2} V_{,12} \\
& + 3(f_2 c_\beta - M_1(|f_1|)s_\beta)V_{,1} - M_1(|f_3|)((M_1(|f_{3,1}|) - M_1(|f_1|))s_\beta \\
& \qquad + f_2 c_\beta)e^{\eta^2} V_{,2} + M_2(V)) \\
& + \varepsilon_1^3 f_1^3 e^{3\eta^2} s_\beta^2\big(s_\beta \nabla^2 V + M_2(V)\big) - 2\varepsilon_1^2 \varepsilon_3 f_1^2 f_3 e^{2\eta^2} s_\beta\big(3s_\beta \nabla^2 V + 2M_2(V)\big) \\
& + \varepsilon_1 \varepsilon_3^2 f_1 f_3^2 e^{\eta^2}\big(3s_\beta V_{,11} + M_1^2(|f_3|)s_\beta e^{2\eta^2} V_{,22} - 4M_1(|f_3|)s_\beta e^{\eta^2} V_{,12} \\
& \qquad + 3\big(f_2 c_\beta - M_1(|f_1|)s_\beta\big)V_{,1} \\
& + 2M_1(|f_3|)\big(s_\beta\big(M_1(|f_3|) - M_1(|f_{3,1}|) + M_1(|f_1|)\big) - f_2 c_\beta\big)e^{\eta^2} V_{,2}\big) \\
& + 3\varepsilon_1^3 \varepsilon_3 f_1^3 f_3 e^{3\eta^2} s_\beta^2\big(s_\beta \nabla^2 V + M_2(V)\big) \\
& + \varepsilon_1 \varepsilon_3^3 f_1 f_3^3 e^{\eta^2}\big(s_\beta\big(V_{,11} + M_1^2(|f_3|)e^{2\eta^2} V_{,22} - 2M_1(|f_3|)e^{\eta^2} V_{,12}\big) \\
& \qquad + \big(f_2 c_\beta - M_1(|f_1|)s_\beta\big)V_{,1} \\
& - M_1(|f_3|)\big(s_\beta\big(M_1(|f_{3,1}|) - 2M_1(|f_3|) - M_1(|f_1|)\big) + f_2 c_\beta\big)e^{\eta^2} V_{,2} \\
& - \varepsilon_1^2 \varepsilon_3^2 f_1^2 f_3^2 e^{2\eta^2} s_\beta\big(3s_\beta \nabla^2 V + 2M_2(V)\big) \\
& + 3\varepsilon_1^3 \varepsilon_3^2 f_1^3 f_3^2 e^{3\eta^2} s_\beta^2\big(s_\beta \nabla^2 V \\
& + M_2(V)\big) + \varepsilon_1^3 \varepsilon_3^3 f_1^3 f_3^3 e^{3\eta^2} s_\beta^2\big(s_\beta \nabla^2 V + M_2(V)\big),
\end{aligned} \tag{3.1}
$$

We will seek a solution of (3.1) in the form

$$
V = \sum_{n=0}^{\infty} \varepsilon_1^i \varepsilon_3^k V_{ik}^{(n)}, \qquad i + k = n. \tag{3.2}
$$

The system of equations for $V_{ik}^{(n)}$, $n \geqslant 0$, is

$$
V_{00,11}^{(0)} + \nabla^2 V_{00}^{(0)} = 0, \tag{3.3}
$$

together with

$$
V_{ik,11}^{(n)} + \nabla^2 V_{ik}^{(n)} = U_{ik}^{(n-1)}, \qquad i + k = n, n \geqslant 1, \tag{3.4}
$$

where $U_{ik}^{(n-1)}$ is given by

$$
\begin{aligned}
U_{ik}^{(n-1)} = f_1 e^{\eta^2}\Big(& s_\beta V_{i-1k,11}^{(n-1)} + 3s_\beta \nabla^2 V_{i-1k}^{(n-1)} + \big(f_2 c_\beta - M_1(|f_1|)s_\beta\big) \\
& \times V_{i-1k,1}^{(n-1)} + M_2\big(V_{i-1k}^{(n-1)}\big)\Big) - f_3\big(2V_{ik-1,11}^{(n-1)} - 2M_1(|f_3|)e^{\eta^2}V_{ik-1,12}^{(n-1)} \\
& - M_1(|f_3|)M_1(|f_{3,1}|)e^{\eta^2}V_{ik-1,2}^{(n-1)}\big) \\
& - f_1^2 e^{2\eta^2} s_\beta\big(3s_\beta \nabla^2 V_{i-2k}^{(n-2)} + 2M_2\big(V_{i-2k}^{(n-2)}\big)\big) \\
& - f_3^2\big(V_{ik-2,11}^{(n-2)} + M_1^2(|f_3|)e^{2\eta^2}V_{ik-2,22}^{(n-2)} \\
& - 2M_1(|f_3|)e^{\eta^2}V_{ik-2,12}^{(n-2)} + M_1(|f_3|)\big(2M_1(|f_3|) - M_1(|f_{3,1}|)\big)e^{\eta^2}V_{ik-2,2}^{(n-2)}\big) \\
& + f_1 f_3 e^{\eta^2}\big(3s_\beta V_{i-1k-1,11}^{(n-2)} + 3s_\beta \nabla^2 V_{i-1k-1}^{(n-2)} - 2M_1(|f_3|)s_\beta e^{\eta^2} V_{i-1k-1,12}^{(n-2)} \\
& + 3\big(f_2 c_\beta - M_1(|f_1|)s_\beta\big)V_{i-1k-1,1}^{(n-2)} \\
& - M_1(|f_3|)\big(\big(M_1(|f_{3,1}|) - M_1(|f_1|)\big)s_\beta + f_2 c_\beta\big)e^{\eta^2}V_{i-1k-1,2}^{(n-2)} + M_2\big(V_{i-1k-1}^{(n-2)}\big)\big) \\
& + f_1^3 e^{3\eta^2} s_\beta^2\big(s_\beta \nabla^2 V_{i-3k}^{(n-3)} + M_2\big(V_{i-3k}^{(n-3)}\big)\big) \\
& - 2f_1^2 f_3 e^{2\eta^2} s_\beta\big(3s_\beta \nabla^2 V_{i-2k-1}^{(n-3)} + 2M_2\big(V_{i-2k-1}^{(n-3)}\big)\big) \\
& + f_1 f_3^2 e^{\eta^2}\big(3s_\beta V_{i-1k-2,11}^{(n-3)} + M_1^2(|f_3|)s_\beta e^{2\eta^2} V_{i-1k-2,22}^{(n-3)} \\
& - 4M_1(|f_3|)s_\beta e^{\eta^2} V_{i-1k-2,12}^{(n-3)} + 3\big(f_2 c_\beta - M_1(|f_1|)s_\beta\big)V_{i-1k-2,1}^{(n-3)} \\
& 2M_1(|f_3|)\big(s_\beta\big(M_1(|f_3|) - M_1(|f_{3,1}|) + M_1(|f_1|)\big) - f_2 c_\beta\big)e^{\eta^2}V_{i-1k-2,2}^{(n-3)}\big) \\
& + 3f_1^3 f_3 e^{3\eta^2} s_\beta^2\big(s_\beta \nabla^2 V_{i-3k-1}^{(n-4)} + M_2\big(V_{i-3k-1}^{(n-4)}\big)\big) \\
& + f_1^3 f_3^3 e^{\eta^2}\big(s_\beta\big(V_{i-1k-3,11}^{(n-4)} + M_1^2(|f_3|)e^{2\eta^2}V_{i-1k-3,22}^{(n-4)} - 2M_1(|f_3|)e^{\eta^2}V_{i-1k-3,12}^{(n-4)}\big) \\
& + \big(f_2 c_\beta - M_1(|f_1|)s_\beta\big)V_{i-1k-3,1}^{(n-4)} \\
& M_1(|f_3|)\big(s_\beta\big(M_1(|f_{3,1}|) - 2M_1(|f_3|) - M_1(|f_1|)\big) + f_2 c_\beta\big)e^{\eta^2}V_{i-1k-3,2}^{(n-4)}\big) \\
& - f_1^2 f_3^2 e^{2\eta^2} s_\beta\big(3s_\beta \nabla^2 V_{i-2k-2}^{(n-4)} + 2M_2\big(V_{i-2k-2}^{(n-4)}\big)\big) \\
& + 3f_1^3 f_3^2 e^{3\eta^2} s_\beta^2\big(s_\beta \nabla^2 V_{i-3k-2}^{(n-5)} + M_2\big(V_{i-3k-2}^{(n-5)}\big)\big) \\
& + f_1^3 f_3^3 e^{3\eta^2} s_\beta^2\big(s_\beta \nabla^2 V_{i-3k-3}^{(n-6)} + M_2\big(V_{i-3k-3}^{(n-6)}\big)\big),
\end{aligned}
\tag{3.5}
$$

with $V_{i_1,i_2}^{(m)}$, $i_1 + i_2 = m$, and all derivatives of $V_{i_1i_2}^{(m)}$ identically zero when $i_1 < 0$ or $i_2 < 0$. Eq. (3.3) can be considered as Laplace's equation in the *cylindrical* coordinate system (η^1, η^2, η^3) with scaling factors 1, e^{η^2}, e^{η^2} and is solved subject to the Dirichlet boundary conditions (2.5)–(2.7). Eqs. (3.4) are Poisson-type and are solved subject to homogeneous boundary conditions. The method of solution of these boundary-value problems is the same as that employed in [1] and the details can be omitted for brevity.

When $|\tau| Q < 1$ for $0 \leqslant \xi'^1 \leqslant l$ we can write $\tau Q = \varepsilon_2 f_2(\eta^1)$ where $\varepsilon_2(<1) \equiv \max_{0<\xi'^1<l} |\tau| Q$ and $f_2(\eta^1)$ is again $O(1)$. Eq. (2.4) can now be written in the form

$$\begin{aligned}
V_{,11} + \nabla^2 V = {} & \varepsilon_1 f_1 e^{\eta^2}\big(s_\beta V_{,11} + 3s_\beta \nabla^2 V - M_1(|f_1|)s_\beta V_{,1} + M_2(V)\big) \\
& -\varepsilon_3 f_3\big(2V_{,11} - 2M_1(|f_3|)e^{\eta^2}V_{,12} - M_1(|f_3|)M_1(|f_{3,1}|)e^{\eta^2}V_{,2}\big) \\
& -\varepsilon_1^2 f_1^2 e^{2\eta^2} s_\beta\big(3s_\beta \nabla^2 V + 2M_2(V)\big) \\
& -\varepsilon_3^2 f_3^2\big(V_{,11} + M_1^2(|f_3|)e^{2\eta^2}V_{,22} \\
& \quad -2M_1(|f_3|)e^{\eta^2}V_{,12} + M_1(|f_3|)\big(2M_1(|f_3|) - M_1(|f_{3,1}|)\big)e^{\eta^2}V_{,2}\big) \\
& +\varepsilon_1\varepsilon_2 f_1 f_2 e^{\eta^2} c_\beta V_{,1} + \varepsilon_1\varepsilon_3 f_1 f_3 e^{\eta^2}\big(3s_\beta V_{,11} + 3s_\beta\nabla^2 V - 2M_1(|f_3|)s_\beta e^{\eta^2}V_{,12} - 3M_1(|f_1|)s_\beta V_{,1} \\
& \quad -M_1(|f_3|)\big(M_1(|f_{3,1}|) - M_1(|f_1|)\big)s_\beta e^{\eta^2}V_{,2} + M_2(V)\big) \\
& +\varepsilon_1^3 f_1^3 e^{3\eta^2} s_\beta^2\big(s_\beta \nabla^2 V + M_2(V)\big) - 2\varepsilon_1^2\varepsilon_3 f_1^2 f_3 e^{2\eta^2} s_\beta\big(3s_\beta\nabla^2 V + 2M_2(V)\big) \\
& +\varepsilon_1\varepsilon_3^2 f_1 f_3^2 e^{\eta^2} s_\beta\big(3V_{,11} + M_1^2(|f_3|)e^{2\eta^2}V_{,22} - 4M_1(|f_3|)e^{\eta^2}V_{,12} - 3M_1(|f_1|)V_{,1} \\
& \quad +2M_1(|f_3|)\big(M_1(|f_3|) - M_1(|f_{3,1}|) + M_1(|f_1|)\big)e^{\eta^2}V_{,2}\big) \\
& +\varepsilon_1\varepsilon_2\varepsilon_3 f_1 f_2 f_3 e^{\eta^2} c_\beta\big(3V_{,1} - M_1(|f_3|)e^{\eta^2}V_{,2}\big) \\
& +3\varepsilon_1^3\varepsilon_3 f_1^3 f_3 e^{3\eta^2} s_\beta^2\big(s_\beta\nabla^2 V + M_2(V)\big) \\
& +\varepsilon_1\varepsilon_3^3 f_1 f_3^3 e^{\eta^2} s_\beta\big(V_{,11} + M_1^2(|f_3|)e^{2\eta^2}V_{,22} - 2M_1(|f_3|)e^{\eta^2}V_{,12} \\
& \quad -M_1(|f_1|)V_{,1} - M_1(|f_3|)\big(M_1(|f_{3,1}|) - 2M_1(|f_3|) - M_1(|f_1|)\big)e^{\eta^2}V_{,2}\big) \\
& -\varepsilon_1^2\varepsilon_3^2 f_1^2 f_3^2 e^{2\eta^2} s_\beta\big(3s_\beta\nabla^2 V + 2M_2(V)\big) \\
& +\varepsilon_1\varepsilon_2\varepsilon_3^2 f_1 f_2 f_3^2 e^{\eta^2} c_\beta\big(3V_{,1} - 2M_1(|f_3|)e^{\eta^2}V_{,2}\big) \\
& +3\varepsilon_1^3\varepsilon_3^2 f_1^3 f_3^2 e^{3\eta^2} s_\beta^2\big(s_\beta\nabla^2 V + M_2(V)\big) \\
& +\varepsilon_1\varepsilon_2\varepsilon_3^3 f_1 f_2 f_3^3 e^{\eta^2} c_\beta\big(V_{,1} - M_1(|f_3|)e^{\eta^2}V_{,2}\big) \\
& +\varepsilon_1^3\varepsilon_3^3 f_1^3 f_3^3 e^{3\eta^2} s_\beta^2\big(s_\beta\nabla^2 V + M_2(V)\big).
\end{aligned} \tag{3.6}$$

In this case we seek a solution for V in the form

$$V = \sum_{n=0}^{\infty} \varepsilon_1^i \varepsilon_2^j \varepsilon_3^k V_{ijk}^{(n)}, \qquad i + j + k = n.$$

The system of equations for $V_{ijk}^{(n)}$, $n \geqslant 0$, is

$$V_{000,11}^{(0)} + \nabla^2 V_{000}^{(0)} = 0, \tag{3.7}$$

and

$$V_{ijk,11}^{(n)} + \nabla^2 V_{ijk}^{(n)} = U_{ijk}^{(n-1)}, \qquad i + j + k = n, n \geqslant 1, \tag{3.8}$$

where $U_{ijk}^{(n-1)}$ is given by

$$
\begin{aligned}
U_{ijk}^{(n-1)} = {} & f_1 e^{\eta^2}\Big(s_\beta V^{(n-1)}_{i-1jk,11} + 3s_\beta \nabla^2 V^{(n-1)}_{i-1jk} - M_1\big(|f_1|\big) s_\beta V^{(n-1)}_{i-1jk,1} + M_2\big(V^{(n-1)}_{i-1jk}\big)\Big) \\
& -f_3\Big(2V^{(n-1)}_{ijk-1,11} - 2M_1\big(|f_3|\big)e^{\eta^2}V^{(n-1)}_{ijk-1,12} - M_1\big(|f_3|\big)M_1\big(|f_{3,1}|\big)e^{\eta^2}V^{(n-1)}_{ijk-1,2}\Big) \\
& -f_1^2 e^{2\eta^2} s_\beta \Big(3s_\beta\nabla^2 V^{(n-2)}_{i-2jk} + 2M_2\big(V^{(n-2)}_{i-2jk}\big)\Big) \\
& -f_3^2\Big(V^{(n-2)}_{ijk-2,11} + M_1^2\big(|f_3|\big)e^{2\eta^2}V^{(n-2)}_{ijk-2,22} \\
& \qquad -2M_1\big(|f_3|\big)e^{\eta^2}V^{(n-2)}_{ijk-2,12} + M_1\big(|f_3|\big)\big(2M_1\big(|f_3|\big) - M_1\big(|f_{3,1}|\big)\big)e^{\eta^2}V^{(n-2)}_{ijk-2,2}\Big) \\
& +f_1 f_2 e^{\eta^2} c_\beta V^{(n-2)}_{i-1j-1k,1} \\
& +f_1 f_3 e^{\eta^2}\Big(3s_\beta V^{(n-2)}_{i-1jk-1,11} + 3s_\beta\nabla^2 V^{(n-2)}_{i-1jk-1} \\
& \qquad -2M_1\big(|f_3|\big)s_\beta e^{\eta^2}V^{(n-2)}_{i-1jk-1,12} - 3M_1\big(|f_1|\big)s_\beta V^{(n-2)}_{i-1jk-1,1} \\
& \qquad -M_1\big(|f_3|\big)\big(M_1\big(|f_{3,1}|\big) - M_1\big(|f_1|\big)\big)s_\beta e^{\eta^2}V^{(n-2)}_{i-1jk-1,2} + M_2\big(V^{(n-2)}_{i-1jk-1}\big)\Big) \\
& +f_1^3 e^{3\eta^2}s_\beta^2\Big(s_\beta\nabla^2V^{(n-3)}_{i-3jk} + M_2\big(V^{(n-3)}_{i-3jk}\big)\Big) \\
& -2f_1^2f_3e^{2\eta^2}s_\beta\Big(3s_\beta\nabla^2V^{(n-3)}_{i-2jk-1} + 2M_2\big(V^{(n-3)}_{i-2jk-1}\big)\Big) \\
& +f_1f_3^2e^{\eta^2}s_\beta\Big(3V^{(n-3)}_{i-1jk-2,11} + M_1^2\big(|f_3|\big)e^{2\eta^2}V^{(n-3)}_{i-1jk-2,22} \\
& \qquad -4M_1\big(|f_3|\big)e^{\eta^2}V^{(n-3)}_{i-1jk-2,12} - 3M_1\big(|f_1|\big)V^{(n-3)}_{i-1jk-2,1} \\
& \qquad +2M_1\big(|f_3|\big)\big(M_1\big(|f_3|\big) - M_1\big(|f_{3,1}|\big) + M_1\big(|f_1|\big)\big)e^{\eta^2}V^{(n-3)}_{i-1jk-2,2}\Big) \\
& +f_1f_2f_3e^{\eta^2}c_\beta\Big(3V^{(n-3)}_{i-1j-1k-1,1} - M_1\big(|f_3|\big)e^{\eta^2}V^{(n-3)}_{i-1j-1k-1,2}\Big) \\
& +3f_1^3f_3e^{3\eta^2}s_\beta^2\Big(s_\beta\nabla^2V^{(n-4)}_{i-3jk-1} + M_2\big(V^{(n-4)}_{i-3jk-1}\big)\Big) \\
& +f_1f_3^3e^{\eta^2}s_\beta\Big(V^{(n-4)}_{i-1jk-3,11} + M_1^2\big(|f_3|\big)e^{2\eta^2}V^{(n-4)}_{i-1jk-3,22} \\
& \qquad -2M_1\big(|f_3|\big)e^{\eta^2}V^{(n-4)}_{i-1jk-3,12} - M_1\big(|f_1|\big)V^{(n-4)}_{i-1jk-3,1} \\
& \qquad M_1\big(|f_3|\big)\big(M_1\big(|f_{3,1}|\big) - 2M_1\big(|f_3|\big) - M_1\big(|f_1|\big)\big)e^{\eta^2}V^{(n-4)}_{i-1jk-3,2}\Big) \\
& -f_1^2f_3^2e^{2\eta^2}s_\beta\Big(3s_\beta\nabla^2V^{(n-4)}_{i-2jk-2} + 2M_2\big(V^{(n-4)}_{i-2jk-2}\big)\Big) \\
& +f_1f_2f_3^2e^{\eta^2}c_\beta\Big(3V^{(n-4)}_{i-1j-1k-2,1} - 2M_1\big(|f_3|\big)e^{\eta^2}V^{(n-4)}_{i-1j-1k-2,2}\Big) \\
& +3f_1^3f_3^2e^{3\eta^2}s_\beta^2\Big(s_\beta\nabla^2V^{(n-5)}_{i-3jk-2} + M_2\big(V^{(n-5)}_{i-3jk-2}\big)\Big) \\
& +f_1f_2f_3^3e^{\eta^2}c_\beta\Big(V^{(n-5)}_{i-1j-1k-3,1} - M_1\big(|f_3|\big)e^{\eta^2}V^{(n-5)}_{i-1j-1k-3,2}\Big) \\
& +f_1^3f_3^3e^{3\eta^2}s_\beta^2\Big(s_\beta\nabla^2V^{(n-6)}_{i-3jk-3} + M_2\big(V^{(n-6)}_{i-3jk-3}\big)\Big),
\end{aligned}
\tag{3.9}
$$

with $V^{(m)}_{i_1 i_2 i_3}$, $i_1 + i_2 + i_3 = m$, and all derivatives of $V^{(m)}_{i_1 i_2 i_3}$ identically zero when $i_1 < 0$, $i_2 < 0$ or $i_3 < 0$. The solution of equations (3.7)–(3.9) subject to the given Dirichlet boundary condition is formally the same as the case when τQ is 0(1).

The above analysis can readily be extended to include the cases of a semi-infinite, infinite and closed tube as was done in [1].

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