DETERMINATION OF THE STRETCH AND ROTATION IN THE POLAR DECOMPOSITION OF THE DEFORMATION GRADIENT*

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1. Introduction. Ever since its first use in continuum mechanics by Richter [1] in 1952, the polar decomposition

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \tag{1.1}$$

has played a major role in theoretical studies. Here, the invertible (second-order) tensor $\mathbf{F}$ is the *deformation gradient*; the orthogonal tensor $\mathbf{R}$ is the *rotation tensor*; and the positive definite, symmetric tensors $\mathbf{U}$ and $\mathbf{V}$ are the *right* and *left stretch tensors*, respectively.[1] The stretch tensors are related to the *right* and *left Cauchy-Green strain tensors*, $\mathbf{C}$ and $\mathbf{B}$, by

$$\mathbf{U}^2 = \mathbf{C} = \mathbf{F}^T\mathbf{F}, \qquad \mathbf{V}^2 = \mathbf{B} = \mathbf{F}\mathbf{F}^T. \tag{1.2}$$

Additional formulas implied by Eqs. (1.1) are

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}, \qquad \mathbf{V} = \mathbf{F}\mathbf{U}^{-1}\mathbf{F}^T. \tag{1.3}$$

Presuming that $\mathbf{F}$ is known, we see from Eqs. (1.2) that $\mathbf{C}$ and $\mathbf{B}$ are easy to calculate, but $\mathbf{U}$ and $\mathbf{V}$ have traditionally presented considerable computational difficulty since they are square roots. Of course, once $\mathbf{U}$ is known, $\mathbf{R}$ and $\mathbf{V}$ follow readily from Eqs. (1.3).

In this paper, we point out that by a trivial, but evidently heretofore unnoticed, application of the Cayley-Hamilton theorem, $\mathbf{U}$ can be calculated directly from $\mathbf{C}$ without recourse to tensor square roots, eigenvalues, and principal axes when the underlying vector space has dimension less than five. In the higher-dimensional cases, the eigenvalues of $\mathbf{C}$ are needed—but not the eigenvectors.

In Sec. 2, we determine the inverse of a tensor of the form $\mathbf{C} + c\mathbf{I}$, $c > 0$, which is a result needed throughout. The observation alluded to in the previous paragraph is made in Sec. 3. To facilitate the calculation of $\mathbf{R}$ and $\mathbf{V}$ a formula for $\mathbf{U}^{-1}$ is derived in Sec. 4. Finally, in Sec. 5, we complete our study by giving formulas for the invariants of $\mathbf{U}$ in terms of the readily computed invariants of $\mathbf{C}$ in the two- and three-dimensional cases.

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[1] Reference may be made to the recent text by Gurtin [2] for a clear statement and proof of the polar decomposition theorem and for its use in the present context.

While our results are couched in continuum mechanical terms with particular attention given to the two- and three-dimensional cases, it will be clear that our approach to determining the factors in the polar decomposition without resort to tensor square roots goes through in general for underlying vector spaces of any finite dimension.

2. The inverse of $(\mathbf{C} + c\mathbf{I})$. If $\mathbf{C}$ is positive definite, $\mathbf{I}$ is the identity tensor, and $c > 0$, then the tensor $\mathbf{C} + c\mathbf{I}$ is positive definite and hence invertible; $(\mathbf{C} + c\mathbf{I})^{-1}$ will be needed repeatedly in the following sections. The details of the calculation depend on the dimension of the underlying vector space.

Two-dimensional case. By considering the spectral resolution of $\mathbf{C} + c\mathbf{I}$, one is led to seek $(\mathbf{C} + c\mathbf{I})^{-1}$ in the form

$$(\mathbf{C} + c\mathbf{I})^{-1} = \alpha\mathbf{C} + \beta\mathbf{I}.$$

We can find α and β in terms of c and the invariants of $\mathbf{C}$ from the condition

$$(\mathbf{C} + c\mathbf{I})^{-1}(\mathbf{C} + c\mathbf{I}) = (\alpha\mathbf{C} + \beta\mathbf{I})(\mathbf{C} + c\mathbf{I}) = \mathbf{I},$$

which yields

$$[(c + I_C)\alpha + \beta]\mathbf{C} + (-II_C\alpha + c\beta - 1)\mathbf{I} = \mathbf{0},$$

when $\mathbf{C}^2$ is eliminated *via* the Cayley-Hamilton theorem.[2] Setting the coefficients in the last equation equal to zero, we find that

$$(\mathbf{C} + c\mathbf{I})^{-1} = -[c(c + I_C) + II_C]^{-1}[\mathbf{C} - (c + I_C)\mathbf{I}]. \tag{2.1}$$

Three-dimensional case. Here, we seek $(\mathbf{C} + c\mathbf{I})^{-1}$ in the form

$$(\mathbf{C} + c\mathbf{I})^{-1} = \alpha\mathbf{C}^2 + \beta\mathbf{C} + \gamma\mathbf{I}.$$

The outcome is

$$(\mathbf{C} + c\mathbf{I})^{-1} = \{c[c(c + I_C) + II_C] + III_C\}^{-1} \times \{\mathbf{C}^2 - (c + I_C)\mathbf{C} + [c(c + I_C) + II_C]\mathbf{I}\}. \tag{2.2}$$

3. Determination of U. While our scheme of using the Cayley-Hamilton theorem to determine $\mathbf{U}$ in terms of $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ goes through in general, the specific details depend on the dimension of the underlying vector space.

Two-dimensional case. In this case, the Cayley-Hamilton theorem states that

$$\mathbf{U}^2 - I_U\mathbf{U} + II_U\mathbf{I} = \mathbf{0}. \tag{3.1}$$

Here, I_U and II_U denote the *fundamental invariants* of $\mathbf{U}$, *i.e.*,

$$I_U = \operatorname{tr}\mathbf{U}, \qquad II_U = \det\mathbf{U}. \tag{3.2}$$

Since $\mathbf{U}^2 = \mathbf{C}$, we have immediately from Eq. (3.1) that

$$\mathbf{U} = I_U^{-1}(\mathbf{C} + II_U\mathbf{I}). \tag{3.3}$$

Simple formulas for I_U and II_U in terms of the invariants of $\mathbf{C}$ will be derived in Sec. 5.

[2] The invariants of a tensor and the Cayley-Hamilton theorem are considered more fully in the next section.

Three-dimensional case. Here, the *fundamental invariants* of **U** are

$$I_U = \operatorname{tr}\mathbf{U},\quad II_U = \tfrac{1}{2}\left[(\operatorname{tr}\mathbf{U})^2 - \operatorname{tr}\mathbf{U}^2\right],\quad III_U = \det\mathbf{U}, \tag{3.4}$$

and the Cayley-Hamilton theorem is

$$\mathbf{U}^3 - I_U\mathbf{U}^2 + II_U\mathbf{U} - III_U\mathbf{I} = \mathbf{0}. \tag{3.5}$$

With $\mathbf{U}^2 = \mathbf{C}$, Eq. (3.5) can be written as

$$(\mathbf{C} + II_U\mathbf{I})\mathbf{U} = I_U\mathbf{C} + III_U\mathbf{I}.$$

Thus,

$$\mathbf{U} = (\mathbf{C} + II_U\mathbf{I})^{-1}(I_U\mathbf{C} + III_U\mathbf{I}). \tag{3.6}$$

Substituting for $(\mathbf{C} + II_U\mathbf{I})^{-1}$ from Eq. (2.2) and using the Cayley-Hamilton theorem to reduce the degree of the resulting polynomial, we get

$$\begin{aligned}\mathbf{U} = {}&\{II_U[II_U(II_U + I_C) + II_C] + III_C\}^{-1}\\ &\times\{-(I_U II_U - III_U)\mathbf{C}^2 + (I_U II_U - III_U)(II_U + I_C)\mathbf{C}\\ &+\{I_U III_C + III_U[II_U(II_U + I_C) + II_C]\}\mathbf{I}\}.\end{aligned} \tag{3.7}$$

Formulas for the invariants of **U** in terms of the invariants of **C** will be given in Sec. 5.

4. Determination of $\mathbf{U}^{-1}$. As noted earlier, knowledge of $\mathbf{U}^{-1}$ leads us directly to **R** and **V** *via* Eqs. (1.3).

Two-dimensional case. Eqs. (3.3) and (2.1) imply

$$\mathbf{U}^{-1} = -I_U[II_U(II_U + I_C) + II_C]^{-1}[\mathbf{C} - (II_U + I_C)\mathbf{I}]. \tag{4.1}$$

The invariants I_U and II_U will be given in terms of I_C and II_C in Sec. 5.

Three-dimensional case. Eq. (3.6) implies

$$\mathbf{U}^{-1} = I_U^{-1}\left(\mathbf{C} + \frac{III_U}{I_U}\mathbf{I}\right)^{-1}(\mathbf{C} + II_U\mathbf{I}).$$

Substituting for $(\mathbf{C} + III_U\mathbf{I}/I_U)^{-1}$ from Eq. (2.2) and using the Cayley-Hamilton theorem to eliminate $\mathbf{C}^3$, we get

$$\begin{aligned}\mathbf{U}^{-1} = {}&\{III_U^2(III_U + I_U I_C) + I_U^2(I_U III_C + III_U II_C)\}^{-1}\\ &\times\{I_U(I_U II_U - III_U)\mathbf{C}^2 - (I_U II_U - III_U)(III_U + I_U I_C)\mathbf{C}\\ &+[II_U III_U(III_U + I_U I_C) + I_U^2(II_U II_C + III_C)]\mathbf{I}\}.\end{aligned} \tag{4.2}$$

The invariants of **U** are given in terms of the invariants of **C** in the next section.

5. Invariants of U in terms of the invariants of C. Our Eqs. (3.3), (3.7) for **U** and Eqs. (4.1), (4.2) for $\mathbf{U}^{-1}$ involve the invariants of **U**. Here, we provide explicit formulas for the invariants of **U** in terms of the invariants of **C** for the two- and three-dimensional cases so that **U** and $\mathbf{U}^{-1}$ may be written entirely in terms of **C** and its invariants. Procedures for higher-dimensional cases are discussed briefly.

Two-dimensional case. In terms of the *principal stretches* λ_1 and λ_2 (the eigenvalues of U), we have

$$I_U = \lambda_1 + \lambda_2, \qquad II_U = \lambda_1\lambda_2$$

and

$$I_C = \lambda_1^2 + \lambda_2^2, \qquad II_C = \lambda_1^2\lambda_2^2.$$

Obviously,[3]

$$II_U = \sqrt{II_C}, \tag{5.1}$$

and

$$I_U^2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2,$$

so

$$I_U = \sqrt{I_C + 2\sqrt{II_C}}\,. \tag{5.2}$$

Three-dimensional case. In this case,

$$I_U = \lambda_1 + \lambda_2 + \lambda_3, \quad II_U = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad III_U = \lambda_1\lambda_2\lambda_3$$

and

$$I_C = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad II_C = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \quad III_C = \lambda_1^2\lambda_2^2\lambda_3^2.$$

Thus,

$$III_U = \sqrt{III_C}, \tag{5.3}$$

$$I_U^2 = I_C + 2II_U,$$

and

$$II_U^2 = II_C + 2\sqrt{III_C}\,I_U. \tag{5.4}$$

Elimination of II_U^2 between the last two equations leaves us with

$$I_U^4 - 2I_CI_U^2 - 8\sqrt{III_C}\,I_U + \left(I_C^2 - 4II_C\right) = 0.$$

The invariants of **C** determine the squares of the λ's *via* the characteristic equation of **C**; consequently, the invariants of **C** uniquely determine the invariants of **U**. Therefore, the above quartic for I_U can have but one positive root (possibly repeated). Forearmed with this knowledge, the usual procedure for solving quartics leads us to the following algorithm for the determination of I_U through the invariants of **C**.

Let

$$\xi = \frac{2^5}{27}\left(2I_C^3 - 9I_CII_C + 27III_C\right),$$

$$\eta = \frac{2^{10}}{27}\left(4II_C^3 - I_C^2II_C^2 + 4I_C^3III_C - 18I_CII_CIII_C + 27III_C^2\right),$$

$$\zeta = -\frac{2}{3}I_C + \left(\xi + \sqrt{\eta}\right)^{1/3} + \left(\xi - \sqrt{\eta}\right)^{1/3}.$$

[3] Here and in the sequel all square roots are taken as positive.

Then

$$I_U = \begin{cases} \frac{1}{2}\left(\sqrt{2I_C + \zeta} + \sqrt{2I_C - \zeta + 16\sqrt{III_C}/\sqrt{2I_C + \zeta}}\right), & \zeta \neq -2I_C, \\ \sqrt{I_C + 2\sqrt{II_C}}, & \zeta = -2I_C. \end{cases} \tag{5.5}$$

Of course, once I_U has been found, II_U is given by Eq. (5.4).

Higher-dimensional cases. In general, one can proceed by solving the characteristic equation of **C** for the squares of the λ's. Then the invariants of **U** can be constructed directly. In the four-dimensional case, the characteristic equation of **C** is a quartic and can be solved algebraically.

References

[1] H. Richter, *Zur Elastizitätstheorie endlicher Verformungen*, Math. Nachr. **8**, 65–73 (1952); English transl., Continuum mechanics III, Foundations of elasticity theory, C. Truesdell, ed., Gordon and Breach, New York, 1965

[2] M. E. Gurtin, *An introduction to continuum mechanics*, Academic Press, New York, 1981