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BUCKLING PROBLEMS IN FINITE PLANE ELASTICITY-HARMONIC MATERIALS*

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Abstract. Bucklings of biaxially deformed annular, rectangular and arbitrary regions are considered. It is found that for many different configurations the buckling conditions are governed by the same equation $\chi = 0$, where χ is merely a material function. Furthermore, the buckling solutions are completely unrelated to the buckling loads.

1. Introduction. The buckling of a half space subjected to free-surface-parallel compression was first solved by Biot [1]. The buckling of an infinite space containing a crack and subjected to crack-parallel compression has recently been studied by Wu [2, 3]. Both problems turn out to have the same buckling condition and hence the same buckling load. One of the objectives of this investigation is to find an explanation for this seemingly peculiar coincidence. In the process, we found that the same buckling condition also applies to several other cases. For the rather large class of problems that shares the same buckling condition, there is another anomalous phenomenon. There are infinitely many solutions associated with each one of the finite number of buckling loads. This is not quite the same as that for a standard linear eigenvalue problem involving an infinite region for which the Fourier-transform parameter may be interpreted as the eigenvalue and, for the sake of argument, one may make the convenient statement that the eigenvalue can take on any value but the associated eigensolution is fixed. The class of problems studied in this paper is governed by linear equations derived from a small-superposed-on-large analysis. The eigenvalue of the physical problem, however, is not coupled with the "Fourier-transform parameter". Thus one may arbitrarily form Fourier sums to obtain infinitely many eigensolutions for a given eigenvalue. This immediately leads to the uncertainty about the post-buckling solution which, in an ordinary case, would be just the associated eigensolution with a specific amplitude. But since there is no *the eigensolution* to speak about, what would be the form of the post-buckling solution? If no post-buckling solution could be found, what would be the meaning of the buckling condition of the class of problems, including the one first solved by Biot? Attempts have been made to clarify these last two points, but no conclusive answers have been obtained so far.

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A brief description is now given to the problems studied in this paper. We consider plane deformations so that a point initially at (Z_1, Z_2) moves to (z_1, z_2), and

$$z_1 = \lambda_1 Z_1 + u_1(\mathbf{Z}), \qquad z_2 = \lambda_2 Z_2 + u_2(\mathbf{Z})$$

where λ_1 and λ_2 are two constants characterizing a primary deformation, and u_1 and u_2 are assumed to be small. The objective is to assert the existence of a buckling condition of λ_1 and λ_2 under which nontrivial u_1 and u_2 exist. The specific cases considered include:

(A) Circular Regions with $\lambda_1 = \lambda_2$;
- (A-1) Annular Region,
- (A-2) Circular Hole in Infinite Region,
- (A-3) Circular Disk;

(B) Arbitrary Regions with $\lambda_1 = \lambda_2$;

(C) Rectangular Regions:
- (C-1) Finite Rectangular Region,
- (C-2) Semi-infinite Strip,
- (C-3) Half Space.

The buckling conditions for cases (A-2), (A-3), (C-2) and (C-3), together with analogous cases of (B), are found to be the same.

2. Finite plane elastostatic strain for compressible harmonic materials. Let D be the domain of the (Z_1, Z_2)-plane characterizing the cross section of a cylindrical body in its undeformed configuration. We assume that the cylindrical body is subjected to a plane deformation so that the position of a point (Z_1, Z_2) after deformation is (z_1, z_2). The deformation may be represented by a transformation.

$$z_a = z_a(Z_A) \quad \text{for all } Z_A \in D \tag{2.1}$$

which maps D onto a domain d of the same plane.

Let F_{aA} be the components of the deformation-gradient tensor associated with the deformation (2.1). Then

$$F_{aA} = z_{a,A}, \tag{2.2}$$

and its fundamental scalar invariants may be taken as

$$J = \det \mathbf{F} = \Lambda_1 \Lambda_2, \tag{2.3}$$

$$I = F_{aA} F_{aA} = \Lambda_1^2 + \Lambda_2^2, \tag{2.4}$$

where Λ_1 and Λ_2 are the principal stretch ratios.

The plane-strain elastic potential U for the class of harmonic materials discussed in [2, 4–7] is given by

$$U = 2\mu[H(R) - J], \qquad R = \Lambda_1 + \Lambda_2, \tag{2.5}$$

where μ is a positive constant and H a given function of R. The function $H(R)$ cannot be completely arbitrary. A thorough investigation of the various restrictions may be found in [5]. For our purposes it suffices to list the following properties:

(I) $H(2) = 1, H'(2) = 1.$ (2.6)

(II) $RH''(R) - H'(R) > 0$ for $0 < R < \infty$, (2.7)

(III) There exists an $R_0 \in (1,2)$ such that

$$H'R < 0 \quad \text{for } 0 < R < R_0, \qquad > 0 \quad \text{for } R_0 < R < \infty. \tag{2.8}$$[1]

(IV) $H''(R) > \frac{1}{2}$ for $0 < R < \infty$. (2.9)[2]

In terms of the function H, the components σ_{aA} of the Piola stress tensor $\boldsymbol{\sigma}$ are

$$\sigma_{aA} = \frac{\partial U}{\partial F_{aA}} = 2\mu\left\{\frac{H'(R)}{R}F_{aA} + \left[\frac{H'(R)}{R} - 1\right]\varepsilon_{ab}\varepsilon_{AB}F_{bB}\right\} \tag{2.10}$$

where ε_{ab}, ε_{AB} are the components of the two-dimensional alternator. In the absence of body forces, the equations of equilibrium are just

$$\sigma_{aA,A} = 0 \quad \text{on } D. \tag{2.11}$$

We shall be working primarily with the Piola stresses, but the components τ_{ab} of the Cauchy stress tensor $\boldsymbol{\tau}$ are given by

$$\tau_{ab} = 2\mu\left\{\frac{H'(R)}{RJ}F_{aA}F_{bA} + \left[\frac{H'(R)}{R} - 1\right]\delta_{ab}\right\} \tag{2.12}$$

where δ_{ab} is the Kronecker delta.

Let C be a curve in D defined by

$$Z_A = C_A(L) \tag{2.13}$$

where L measures the arc length along C. The unit tangent and normal vectors $\mathbf{S}$ and $\mathbf{N}$ of C are defined by their respective components

$$S_A = C'_A(L), \qquad N_A = \varepsilon_{AB}C'_B(L) \tag{2.14}$$

The image of C under the mapping (2.1) is a curve c in d defined by

$$z_a = c_a(l) \tag{2.15}$$

where l measures the arc length along c. The components of the unit tangent and normal vectors $\mathbf{s}$ and $\mathbf{n}$ of c are, respectively,

$$s_a = c'_a(l), \qquad n_a = \varepsilon_{ab}c'_b(l), \tag{2.16}$$

where

$$c'_a(l) = F_{aA}C'_A(L)\,dL/dl, \tag{2.17}$$

$$(dl)^2 = F_{aA}F_{aB}C'_A(L)C'_B(L)(dL)^2 \tag{2.18}$$

The traction vector acting on an arc element dl is just

$$\mathbf{t}\,dl = \tau_{ab}n_b\mathbf{i}_a\,dl = \sigma_{aA}N_A\mathbf{i}_a\,dL = \mathbf{T}\,dL \tag{2.19}$$

where $\mathbf{T}$ and $\mathbf{t}$ are, respectively, the Piola and Cauchy traction vectors.

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[1] The Baker-Ericksen inequality requires $H'(R) > 0$.

[2] The stronger condition $H''(R) > 1$ holds for $R > R_0$.

3. Perturbation about a state of finite uniform deformation. Deformations which differ only slightly from a state of uniform strain may be represented by a transformation of the form

$$z_a = \lambda_a \delta_{aA} Z_A + u_a(Z_A) \quad \text{(no sum on } a\text{)} \tag{3.1}$$

where λ_a define the primary deformation. The functions u_a, together with their derivatives, are assumed to be small in comparison with the primary deformation. For the purpose of the present paper a set of equations linear in u_a will be derived. Terms that are nonlinear in u_a are henceforth to be neglected. Using (3.1), we obtain from (2.2)–(2.5).

$$\mathbf{F} = \begin{pmatrix} \lambda_1 + u_{1,1} & u_{1,2} \\ u_{2,1} & \lambda_2 + u_{2,2} \end{pmatrix}, \tag{3.2}$$

$$I = (\lambda_1^2 + \lambda_2^2) + 2(\lambda_1 u_{1,1} + \lambda_2 u_{2,2}), \tag{3.3}$$

$$J = \lambda_1 \lambda_2 + (\lambda_1 u_{2,2} + \lambda_2 u_{1,1}), \tag{3.4}$$

$$R = r + (u_{1,1} + u_{2,2}), \qquad r = \lambda_1 + \lambda_2. \tag{3.5}$$

The elastic potential U may now be written as

$$U = U_0 + U_1 + U_2 + \cdots \tag{3.6}$$

where

$$\begin{aligned} U_0 &= 2\mu[H'(r) - \lambda_1\lambda_2], \\ U_1 &= 2\mu[H'(r)(u_{1,1} + u_{2,2}) - (\lambda_1 u_{2,2} + \lambda_2 u_{1,1})], \\ U_2 &= \mu\Big[(H''(r) - \tfrac{1}{2})(u_{1,1} + u_{2,2})^2 + \tfrac{1}{2}(u_{1,1} - u_{2,2})^2 \\ &\quad + (H'(r)/r - \tfrac{1}{2})(u_{1,2} - u_{2,1})^2 + \tfrac{1}{2}(u_{1,2} + u_{2,1})^2\Big] \end{aligned} \tag{3.7}$$

We note in passing that U_2 is positive definite only if

$$H''(r) > \tfrac{1}{2}, \qquad H'(r)/r > \tfrac{1}{2}. \tag{3.8}$$

A reference of (2.6)–(2.9) indicates that the material function H admits the possibility

$$H'(r)/r < \tfrac{1}{2}, \tag{3.9}$$

and hence U_2 is not always positive definite. Indeed, the several buckling solutions obtained in this paper are a direct consequence of this admissible condition. On the other hand, (3.9) does not necessarily lead to a negative second variation of the total energy which is an indication of instability (see e.g. [8]).

The Piola stress components computed from (2.10) are

$$\sigma_{aA} = \mathring{\sigma}_{aA} + \dot{\sigma}_{aA} \tag{3.10}$$

where

$$\mathring{\sigma}_{aA} = \partial U_1/\partial u_{a,A}, \qquad \dot{\sigma}_{aA} = \partial U_2/\partial u_{a,A}. \tag{3.11}$$

The equations of equilibrium governing u_a may now be determined by substituting (3.11) into (2.11), viz.,

$$u_{a,AA} + \left[\frac{rH''(r)}{H'(r)} - 1\right]\delta_{aA}\delta_{bB}u_{b,BA} = 0. \tag{3.12}$$

For the purpose of establishing certain boundary conditions, we shall restrict ourselves to the two special situations:

A. Arbitrary D and $\lambda_1 = \lambda_2$.

B. Rectangular D and arbitrary λ_a.

It is clear that for either case the normal to the boundary ∂D is not altered by the primary deformation. Treating the boundaries ∂D and ∂d, respectively, as the curves C and c defined in Sec. 2, we obtain

$$dl/dL = \lambda_{(S)} + \delta_{aA}u_{a,B}C'_A C'_B, \tag{3.13}$$

$$c'_a = \delta_{aA}C'_A + (u_{a,A} - \delta_{aA}\delta_{cD}u_{c,B}C'_B C'_D)C'_A/\lambda_{(S)} \tag{3.14}$$

where $\lambda_{(S)}$ denotes the primary stretch ratio along C. For convenience, we shall write

$$s_a = \delta_{aA}C'_A + \dot{s}_a, \qquad n_a = \varepsilon_{aA}C'_A + \dot{n}_a, \tag{3.15}$$

where $\dot{n}_a = \varepsilon_{ab}\dot{s}_b$ and $\dot{s}_a$ is just the second term of (3.14). The following components of the Piola and Cauchy traction vectors are computed:

$$\mathbf{T}\cdot\mathbf{N} - \mathring{\sigma}_{(N)} = \dot{T}_N = N_A N_B \delta_{Ba}\dot{\sigma}_{aA}, \tag{3.16}$$

$$\mathbf{T}\cdot\mathbf{S} = \dot{T}_S = N_A S_B \delta_{Ba}\dot{\sigma}_{aA}, \tag{3.17}$$

$$\mathbf{t}\cdot\mathbf{n} - \frac{\mathring{\sigma}(N)}{\lambda_{(S)}} = \dot{t}_n = \frac{1}{\lambda_{(S)}}\left[\dot{T}_N - \frac{\mathring{\sigma}_{(N)}}{\lambda_{(S)}}\delta_{Aa}u_{a,B}S_A S_B\right], \tag{3.18}$$

$$\mathbf{t}\cdot\mathbf{s} = \dot{t}_s = \frac{1}{\lambda_{(S)}}\left[\dot{T}_S + \frac{\mathring{\sigma}_{(N)}}{\lambda_{(S)}}\delta_{Aa}u_{a,B}N_A S_B\right], \tag{3.19}$$

where

$$\mathring{\sigma}_{(N)} = \mathring{\sigma}_{aA}\delta_{aB}N_A N_B. \tag{3.20}$$

The following three types of homogeneous boundary conditions will be considered:

Constant Dead-Load Traction

$$\dot{T}_N = \dot{T}_S = 0; \tag{3.21}$$

Constant Hydrostatic Traction

$$\dot{t}_n = \dot{t}_s = 0; \tag{3.22}$$

Lubricated Contact Surface

$$\dot{t}_s = 0 \quad \text{and} \quad u_a n_a = \delta_{aA}u_a N_A = 0. \tag{3.23}$$

For the solutions of (3.12) we follow the complex formulation established in [2]. Thus

$$Z = Z_1 + iZ_2, \tag{3.24}$$

$$u(Z) = u_1 + iu_2 = \kappa W(Z) - Z\overline{W'(Z)} - \overline{w(Z)}, \tag{3.25}$$

where

$$\kappa(r) = \frac{rH''(r) + H'(r)}{rH''(r) - H'(r)}, \tag{3.26}$$

where W', w' are holomorphic functions of the complex variable Z in D. The Piola stress components may be combined to yield

$$\dot{\sigma}_{22} - i\dot{\sigma}_{12} = 2\mu\left[\chi W' + \overline{W'} + Z\overline{W''} + \overline{w'}\right], \tag{3.27}$$

$$\dot{\sigma}_{11} + i\dot{\sigma}_{21} = 2\mu\left[\chi W' + \overline{W'} - Z\overline{W''} - \overline{w'}\right], \tag{3.28}$$

where

$$\chi(r) = \frac{4H'(r)H''(r) - rH''(r) - H'(r)}{rH''(r) - H'(r)}. \tag{3.29}$$

Using these relations, we may combine the various terms introduced in (3.16)–(3.23) to obtain

$$\frac{1}{2\mu}\left(\dot{T}_N + i\dot{T}_S\right) = \overline{C'}\frac{d}{dL}\left[\chi W + Z\overline{W'} + \overline{w}\right], \tag{3.30}$$

$$\frac{1}{2\mu}\left(\dot{t}_n + i\dot{t}_s\right) = \overline{C'}\frac{d}{dL}\left[\left(\chi W + Z\overline{W'} + \overline{w}\right) - \frac{\mathring{\sigma}_{(N)}}{2\mu\lambda_{(S)}}\left(\kappa W - Z\overline{W'} - \overline{w}\right)\right] \tag{3.31}$$

$$= \left[\chi W' + \overline{W'} + \left(Z\overline{W''} + \overline{w}'\right)\overline{C'}^{2}\right] - \frac{\mathring{\sigma}_{(N)}}{2\mu\lambda_{(S)}}\left[\kappa W' - \overline{W'} - \left(Z\overline{W''} + \overline{w}'\right)\overline{C'}^{2}\right], \tag{3.32}$$

$$u_n + iu_s = i\overline{C'}\left(\kappa W - Z\overline{W'} - \overline{w}\right), \tag{3.33}$$

where $C(L) = C_1(L) + iC_2(L)$. Equations (3.30) and (3.31) are immediately applicable for the conditions (3.21) and (3.22).

To derive the conditions needed in (3.23), we note from (3.33) that the vanishing of u_n is simply

$$\operatorname{Im}\overline{C'}\left(\kappa W - Z\overline{W'} - \overline{w}\right) = 0. \tag{3.34}$$

Differentiating the above with respect to L, and then applying the result to the condition of vanishing $\dot{t}_s$, we obtain

$$\operatorname{Im}\left\{(\chi + \kappa)W' + \left(1 + \mathring{\sigma}_{(N)}/2\mu\lambda_{(S)}\right)\overline{C''}\left(\kappa W - Z\overline{W'} - \overline{w}\right)\right\} = 0. \tag{3.35}$$

4. Circular regions. Let D be the annular region defined by $A < |Z| < B$. The primary deformation is defined by $z_a = \lambda\delta_{aA}Z_A$ so that

$$r = 2\lambda, \qquad \mathring{\sigma}_{11} = \mathring{\sigma}_{22} = \mathring{\sigma} \equiv 2\mu\left[H'(r) - \lambda\right]. \tag{4.1}$$

The objective is to determine whether nontrivial solutions of the form (3.1) exist for certain values of λ when D is subjected to certain types of boundary conditions.

The constant dead-load traction condition along a circular boundary $|Z| = \rho$ may be derived from (3.21) and (3.30). We shall use $D[\rho]$ to denote this condition and

$$D[\rho] \equiv \left[\chi W + Z\overline{W'} + \overline{w}\right]_{|Z|=\rho} = 0. \tag{4.2}$$

The constant hydrostatic traction condition along a circular boundary $|Z| = \rho$ may be derived from (3.22) and (3.31). In view of (4.1) and the fact $H' \neq 0$, this condition, denoted by $H[\rho]$, is simply

$$H[\rho] \equiv \left[W + Z\overline{W'} + \overline{w}\right]_{|Z|=\rho} = 0. \tag{4.3}$$

We note in passing that $H[\rho]$ is independent of the primary deformation. Finally, the lubricated contact condition along a circular boundary $|Z| = \rho$, denoted by $L[\rho]$, may be obtained by combining (3.34), (3.35) and using (4.1). It is

$$L[\rho] \equiv \left[W' - \left(1 + \frac{H'}{rH''}\right)\frac{W}{Z} - \frac{H'}{rH''}\overline{W'} + \left(1 - \frac{H'}{rH''}\right)\frac{\overline{w}}{\overline{Z}}\right]_{|Z|=\rho} = 0. \tag{4.4}$$

In terms of the annular region $D[A < |Z| < B]$, it can be shown that of the three boundary-value problems

$$H[A] = H[B] = 0, \tag{4.5}$$

$$L[A] = L[B] = 0, \tag{4.6}$$

$$D[A] = D[B] = 0, \tag{4.7}$$

only the last one allows nontrivial solutions. We shall first consider this case in detail.

The mathematical problem is that of the determination of W and w holomorphic in D and satisfying the boundary conditions (4.7). It follows from these considerations that W and w must be of the form

$$W = C_n\frac{Z^{n+1}}{B^n} + C_{-n}\frac{A^n}{Z^{n-1}}, \tag{4.8}$$

$$w = c_n\frac{Z^{n-1}}{B^{n-2}} + c_{-n}\frac{A^{n+2}}{Z^{n+1}}, \tag{4.9}$$

where n $(n > 1)$ is an integer and $C_{\pm n}$, $c_{\pm n}$ are arbitrary complex constants. These constants are zero unless the characteristic condition

$$\chi^2(r) = \chi_n^2(\alpha) \equiv \frac{(n^2-1)(1-\alpha)^2\alpha^{n-1}}{(1-\alpha^{n-1})(1-\alpha^{n+1})}, \qquad \alpha = \left(\frac{A}{B}\right)^2, \tag{4.10}$$

is satisfied. Using the inequality

$$\sum_{k=0}^{n-1} \alpha^k > n\alpha^{(n-1)/2}, \tag{4.11}$$

one can show that

$$\chi_n^2(\alpha) < 1 \quad \text{for all } n \text{ and } 0 \leqslant \alpha < 1. \tag{4.12}$$

Moreover,

$$\chi_n^2(\alpha) \to (n^2-1)\alpha^{n-1} \quad \text{for } n > 1 \text{ as } \alpha \to 0, \tag{4.13}$$

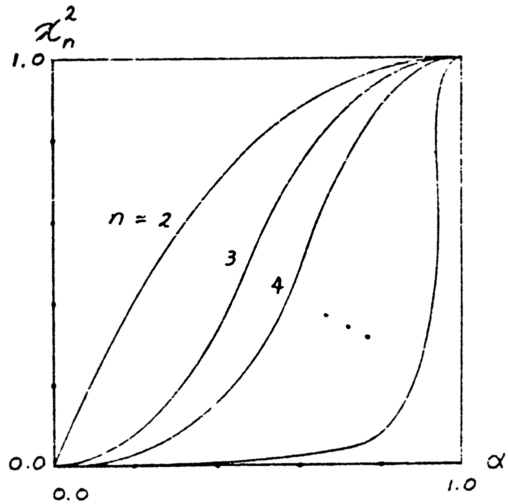

FIG. 1.

$$\chi_n^2(\alpha) \to 1 - n^2(1-\alpha)^2/12 \quad \text{for } n > 1 \text{ as } \alpha \to 1, \tag{4.14}$$

$$\lim_{n\to\infty} \chi_n^2(\alpha) = 0 \quad \text{for } 0 \leqslant \alpha < 1. \tag{4.15}$$

The qualitative behavior of $\chi_n^2(\alpha)$ is provided in Fig. 1.

The characteristic equation (4.10) now becomes

$$\chi(r) = \pm \chi_n(\alpha) \tag{4.16}$$

or

$$\frac{H'(r)}{r} = (1 \pm \chi_n(\alpha)) \Big/ \left(4 - \frac{1 \mp \chi_n(\alpha)}{H''(r)}\right). \tag{4.17}$$

The properties of the two sides of (4.17) may be established by using (2.6)–(2.9):

$$\begin{aligned} &[H'(r)/r]' > 0, \\ &\quad H'(r)/r < 0\ (0 < r < R_0), \quad H'(R_0)/R_0 = 0, \quad H'(2)/2 = \tfrac{1}{2}, \\ &\quad H'(r)/r > 0\ (R_0 < r < \infty), \quad H'(r)/r \to 1 \text{ as } r \to \infty, \end{aligned} \tag{4.18}$$

$$\frac{1 \pm \chi_n}{4} < (1 \pm \chi_n) \Big/ \left(4 - \frac{1 \mp \chi_n}{H''(r)}\right) < \frac{1 \pm \chi_n}{3 \pm \chi_n} < \frac{1}{2} \quad \text{for } 1 < R_0 < r < 2. \tag{4.19}$$

It follows that each of the two equations (4.16) has at least one root. Let $r_c^+(\alpha, n)$ and $r_c^-(\alpha, n)$ be, respectively, the roots associated with the equations with $+\chi_n$ and $-\chi_n$. Then (4.18), (4.19) together with the condition

$$(1 + \chi_n) \Big/ \left(4 - \frac{1 - \chi_n}{H''}\right) > (1 - \chi_n) \Big/ \left(4 - \frac{1 + \chi_n}{H''}\right), \tag{4.20}$$

lead to the conclusion that $r_c^-(\alpha, n) < r_c^+(\alpha, n)$. Moreover,

$$1 < R_0 < r_c^-(\alpha, 2) \ldots < r_c^-(\alpha, \infty) = r_c^+(\alpha, \infty) < \ldots < r_c^+(\alpha, 2) < 2 \tag{4.21}$$

which indicates that the buckling load is a hydrostatic compression, i.e.,

$$\mathring{\sigma} = \mathring{\sigma}_c^{\pm}(\alpha,n) = 2\mu\left[H'\left(r_c^{\pm}(\alpha,n)\right) - \tfrac{1}{2}r_c^{\pm}(\alpha,n)\right] < 0, \tag{4.22}$$

$$\mathring{\sigma}_c^{-}(\alpha,2) < \ldots < \mathring{\sigma}_c^{-}(\alpha,\infty) = \mathring{\sigma}_c^{+}(\alpha,\infty) < \ldots < \mathring{\sigma}_c^{+}(\alpha,2) < 0. \tag{4.23}$$

Finally, (4.13)–(4.16) imply that if r_c is a root of the equation

$$\chi(r_c) = 0, \tag{4.24}$$

then

$$\lim_{\alpha\to 0} r_c^{\pm}(\alpha,n) = \lim_{n\to\infty} r_c^{\pm}(\alpha,n) = r_c. \tag{4.25}$$

Let $(C_n, C_{-n}, c_n, c_{-n}) = (C_n^{\pm}, C_{-n}^{\pm}, c_n^{\pm}, c_{-n}^{\pm})$ be the constants associated with the roots $r = r_c^{\pm}(\alpha,n)$ of the characteristic equation (4.10). These constants can be expressed in terms of a single constant, and the relations are

$$C_{-n}^{\pm} = \pm\left(\frac{n+1}{n-1}\right)^{1/2}\left(\frac{1-\alpha^{n+1}}{1-\alpha^{n-1}}\right)^{1/2}\frac{1}{\alpha^{1/2}}\overline{C_n^{\pm}}, \tag{4.26}$$

$$c_n^{\pm} = -(n+1)\frac{1-\alpha^n}{1-\alpha^{n-1}}C_n^{\pm}, \tag{4.27}$$

$$c_{-n}^{\pm} = (n-1)\frac{1-\alpha^n}{1-\alpha^{n+1}}C_{-n}^{\pm}. \tag{4.28}$$

This formally concludes the class of solutions governed by the Dead-Load Traction conditions. This class of problems does not seem to have been discussed in the literature.

As an illustration, we consider a cylinder made of Standard Harmonic Material for which the material function $H(R)$ has the form

$$H(R) = \frac{1}{2}\frac{(1-\nu)}{1-2\nu}R^2 - \frac{1}{1-2\nu}R + \frac{1}{1-2\nu} \tag{4.29}$$

where ν is a material constant that may be identified with Poisson's ratio for $R \approx 2$. The roots of the characteristic equation are found to be

$$r_c^{\pm}(\alpha,n) = \frac{3-2\nu}{2(1-\nu)} \pm \frac{1-2\nu}{2(1-\nu)}\chi_n(\alpha). \tag{4.30}$$

The associated buckling loads are

$$\mathring{\sigma}_c^{\pm}(\alpha,n) = \left[-1 \pm \chi_n(\alpha)\right]\frac{\mu}{2(1-\nu)} \tag{4.31}$$

and

$$\mathring{\sigma}_c^{-}(\alpha,2) < \ldots < \mathring{\sigma}_c^{-}(\alpha,\infty) = -\frac{\mu}{2(1-\nu)} = \mathring{\sigma}_c^{+}(\alpha,\infty)$$
$$< \ldots < \mathring{\sigma}_c^{+}(\alpha,2) < 0 \quad \text{for } \alpha \neq 0. \tag{4.32}$$

For the case $\alpha = 0$, we have

$$r = r_c = \frac{3 - 2\nu}{2(1 - \nu)}, \qquad \mathring{\sigma} = \mathring{\sigma}_c = -\frac{\mu}{2(1 - \nu)},$$
$$\kappa(r_c) = 2(1 - \nu) \tag{4.33}$$

The associated explicit results are:

Circular Cylinder of Radius B

$$W(Z) = C_n B\left(\frac{Z}{B}\right)^{n+1}, \qquad w(Z) = -(n + 1)C_n B\left(\frac{Z}{B}\right)^{n-1}, \tag{4.34}$$

$$u(Z) = B\left[2(1 - \nu)C_n\left(\frac{Z}{B}\right)^{n+1} + (n + 1)\bar{C}_n\left(1 - \frac{Z\bar{Z}}{B^2}\right)\left(\frac{\bar{Z}}{B}\right)^{n-1}\right], \tag{4.35}$$

$$(u_r + iu_\theta)|_{Z=Be^{i\theta}} = 2(1 - \nu)BC_n e^{in\theta}. \tag{4.36}$$

Cylindrical Hole of Radius A

$$W(Z) = C_{-n}A\left(\frac{Z}{A}\right)^{-(n-1)}, \qquad w(Z) = (n - 1)C_{-n}A\left(\frac{Z}{A}\right)^{-(n+1)}, \tag{4.37}$$

$$u(Z) = A\left[2(1 - \nu)C_{-n}\left(\frac{Z}{A}\right)^{-(n-1)} - (n - 1)\bar{C}_{-n}\left(1 - \frac{Z\bar{Z}}{A^2}\right)\left(\frac{\bar{Z}}{A}\right)^{-(n+1)}\right], \tag{4.38}$$

$$(u_r + iu_\theta)|_{Z=Ae^{i\theta}} = 2(1 - \nu)AC_{-n}e^{-in\theta}. \tag{4.39}$$

It is seen that the buckling conditions for the last two cases are the same Eq. (4.24) which, as we shall see in the sequel, also governs the buckling of a number of other cases. It should also be mentioned that the specific results (4.34)–(4.36) agree with those described by Sensenig [9].

We now return to the statement that neither (4.5) not (4.6) allows nontrivial solutions. The statement is obvious for (4.5) since the primary deformation characterized by r is not involved in the boundary conditions. For the case defined by (4.6), nontrivial solutions of the form defined by (4.8) and (4.9) are possible only if the condition

$$\left(\frac{H'}{rH''}\right)^2 = \frac{n^2}{1 - \chi_n^2(\alpha)} \qquad (n \geqslant 2) \tag{4.40}$$

is satisfied. This is, however, impossible since $\chi_n^2 < 1$ by (4.12) and hence the condition (2.7) would be violated.

5. Regions with curvilinear boundaries. The class of problems defined by (4.7) can obviously be generalized to include regions that may be conformally mapped onto an annular region. We take the case of a solid cylinder with an arbitrary cross section D as an example. We suppose that D may be mapped onto the unit circle $|\zeta| \leqslant 1$ in the complex ζ $(= \zeta_1 + i\zeta_2)$-place via the transformation

$$Z = m(\zeta) \quad \text{for all } \zeta \in D_\zeta\{|\zeta| < 1\}. \tag{5.1}$$

Moreover, define

$$\Omega(\zeta) = W(m(\zeta)), \qquad \omega(\zeta) = w(m(\zeta)) \tag{5.2}$$

which are holomorphic in D_ζ. The dead-load condition becomes (cf. (3.21) and (3.30))

$$\chi\Omega(\zeta) + \frac{m(\zeta)}{m'(\zeta)}\overline{\Omega'(\zeta)} + \overline{\omega(\zeta)} = 0, \qquad |\zeta| = 1. \tag{5.3}$$

The solution is clearly zero unless $\chi(r) = 0$. For that case the solution may be expressed in terms of a single arbitrary function, viz.,

$$\Omega(\zeta) = f(\zeta), \qquad \omega(\zeta) = -\frac{m(1/\bar{\zeta})}{m'(\zeta)}f'(\zeta) \tag{5.4}$$

where f is holomorphic in D_ζ. More general cases may be treated by a similar approach. The case of an elliptic cylinder with a confocal crack is perhaps an interesting problem.

6. Rectangular regions. Let D^+ be the rectangular region defined by

$$D^+: -L < Z_1 < +L, \qquad 0 < Z_2 < 2B. \tag{6.1}$$

The rectangle is loaded on the sides $Z_1 = \pm L$ so that the prebuckling deformation is defined by the pair of values (λ_1, λ_2) leading to the solution:

$$\mathring{\sigma}_{11} = 2\mu[H'(r) - \lambda_2], \qquad \mathring{\sigma}_{22} = 2\mu[H'(r) - \lambda_1] = 0,$$
$$\mathring{\sigma}_{12} = \mathring{\sigma}_{21} = 0. \tag{6.2}$$

The objective is to determine whether nontrivial solutions exist under the following boundary conditions:

$$\dot{T}_N = \dot{T}_S = 0 \quad \text{on } Z_2 = 0, 2B; \tag{6.3}$$

$$\dot{t}_s = u_n = 0 \quad \text{on } Z_1 = \pm L. \tag{6.4}$$

Bucklings of rectangles have been studied by many researchers (see, e.g., [1, 8, 10–14]).

In terms of the complex formulation and (3.30)–(3.33), these conditions are

$$\chi W + Z\overline{W'} + \bar{w} = 0 \quad \text{for } Z_2 = 0, 2B, |Z_1| < L. \tag{6.5}$$

$$\begin{cases} \operatorname{Re}[\kappa W - Z\overline{W'} - \bar{w}] = 0 \\ \operatorname{Re}[\chi W + Z\overline{W'} + \bar{w}] = 0 \end{cases} \quad \text{for } Z_1 = \pm L, 0 < Z_2 < 2B. \tag{6.6}$$

Since both H' and H'' are positive, there is no possibility for the sum $\kappa + \chi$ to vanish. It follows from (6.6) that

$$\operatorname{Re} W = \operatorname{Re}(Z\overline{W'} + \bar{w}) = 0 \quad \text{for } Z_1 = \pm L, 0 < Z_2 < 2B. \tag{6.7}$$

Define a new function $W_H(Z)$ such that

$$W_H(Z) = \begin{cases} W(Z) & \text{for } Z \in D^+, \\ -\dfrac{1}{\chi}\left[Z\overline{W'(\bar{Z})} + \overline{w(\bar{Z})}\right] & \text{for } Z \in D^-[|Z_1| < L, -2B < Z_2 < 0]. \end{cases} \tag{6.8}$$

It follows from the first of (6.5) that $W_H(Z)$ is holomorphic in $D = D^+ \cup D^-$. Moreover,

$$w(Z) = -\chi\overline{W_H(\bar{Z})} - ZW_H'(Z) \quad \text{for } Z \in D^+. \tag{6.9}$$

The two conditions (6.7) can now be reduced to the single condition

$$W_H(Z) + \overline{W_H(Z)} = 0 \quad \text{for } Z_1 = \pm L, |Z_2| < 2B. \tag{6.10}$$

Also, the second of (6.5) now becomes

$$\chi[W_H(Z) - W_H(\bar{Z})] + (Z - \bar{Z})\overline{W_H'(Z)} = 0 \quad \text{for } |Z_1| < L, Z_2 = 2B. \tag{6.11}$$

The solutions and the associated buckling condition are:

$$W_H(Z) = \left.\begin{matrix} A_n \\ iA_n \end{matrix}\right\}\left\{ e^{in\pi(Z-iB)/2L} \mp e^{-in\pi(Z-iB)/2L} \right\} \quad \begin{matrix} (n \text{ odd}), \\ (n \text{ even}), \end{matrix} \tag{6.12}$$

$$\chi(r)\sinh\frac{n\pi B}{L} = \frac{n\pi B}{L}, \tag{6.13}$$

where A_n are real constants. These are just the solutions obtained in [10]. We note that there are infinitely many eigenmodes and the associated eigenparameters are distinct.

By formally letting $B \to \infty$ in the above, the problem reduces to that of the buckling of a semi-infinite strip of width $2L$ subjected to lateral load in the sense of (6.3) and (6.4). The explicit results are simply

$$W_H(Z) = \left.\begin{matrix} A_n \\ iA_n \end{matrix}\right\} e^{in\pi Z/2L} \quad \begin{matrix} (n \text{ odd}), \\ (n \text{ even}), \end{matrix} \tag{6.14}$$

$$\chi(r) = 0. \tag{6.15}$$

The properties of the characteristic equation (6.15) have been studied in [2]. It has at least one and at most a finite number of roots. Associated with each one of the roots, however, there are infinitely many modes of the form (6.14). In fact, any linear combination of the form

$$W_H(Z) = \sum_{n \text{ odd}} A_n e^{in\pi Z/2L} + i \sum_{n \text{ even}} A_n e^{in\pi Z/2L} \tag{6.16}$$

is also a possible mode. Thus the physical meaning of an eigenmode becomes ambiguous. We note again that (6.15) is just (4.24).

The functions defined by all the possible solutions, (6.14) and (6.16), are periodical in Z_1. They can, therefore, be formally extended to $Z_1 = \pm\infty$. Thus the buckling of the (upper) half space may be formally described by (6.15) and

$$W_H(Z) = A(k)e^{ikZ} \qquad (k > 0) \tag{6.17}$$

$$W_H(Z) = \int_0^\infty A(k)e^{ikZ}\,dk. \tag{6.18}$$

The earliest work on the buckling of a half space was carried out by Biot [1]. His work, together with many other later results [15–18] was based on the assumption of periodicity in Z_1. Our Eq. (6.15) and (6.17) are also based upon the same assumption. We shall see, however, that the assumption of periodicity is not essential to the establishment of the buckling condition (6.15). It comes out of the present complex formulation naturally.

The term mode or eigenfunction is customarily used to indicate a specific function dictated by an eigenvalue which very often also determines the "wave length" of the function. For the several classes of buckling problems governed by the buckling condition $\chi(r) = 0$, the buckling solution is completely arbitrary and hence the term mode is no longer appropriate in the sense described. Moreover, the "wave length" of a buckling solution is totally unrelated to the eigenvalue. Still, some of the buckling solutions appear to be more natural than the other in that they come out spontaneously from an innocuous mathematical analysis. All of the buckling solutions obtained in this paper and the cited references are of this nature, and we shall term them "natural modes" with the understanding that the term does not really have a specific definition.

For the case of a half space, there appears to be another class of natural modes which, in physical terms, may be even more appropriate than (6.17). We shall derive them directly from the general complex formulation and, in the process, shall show that the assumption of periodicity is not essential to the establishment of the buckling condition. The mathematical problem is that of the determination of holomorphic functions W and w in $D^+ [Z_2 > 0]$ such that (cf. (6.5))

$$\chi W + Z\overline{W'} + \bar{w} = 0 \quad \text{for } Z_2 = 0. \tag{6.19}$$

For the conditions at infinity we require that $\dot{\sigma}_{aA} \to 0$ so that (cf. (3.27) and (3.28))

$$W'(Z), w'(Z) \to 0 \quad \text{as } |Z| \to \infty \text{ for } Z \in D^+. \tag{6.20}$$

The analytic continuation (6.8), in which D^- is now the lower half space, and (6.19) immediately lead to the conclusion that W_H is holomorphic in the whole plane. This conclusion, together with (6.20), shows that $W_H \equiv 0$ unless $\chi(r) = 0$. Thus the buckling condition is totally unrelated to the buckling solution. Conversely, when the condition $\chi(r) = 0$ is satisfied the condition (6.19) may be easily satisfied by the choice

$$W = f(Z), \qquad w = -Zf'(Z) \tag{6.21}$$

where $f(Z)$ is an arbitrary function holomorphic in D^+ and satisfying the conditions (6.20). One set of "natural modes" would be

$$f(Z) = f_n(Z) = A_n/(Z + i)^n \qquad (n \geqslant 1) \tag{6.22}$$

where A_n are real so that the solution is either symmetric or antisymmetric with respect to the Z_2-axis depending on whether n is odd or even. The traction-free surface displacement associated with (6.22) is

$$u(Z_1) = u_1(Z_1,0) + iu_2(Z_1,0) = \kappa(r_c)A_n(Z_1 + i)^{-n}, \tag{6.23}$$

which shows the bulging of the surface near $Z = 0$.

Practically every paper on the subject of buckling of a half space begins with a statement on possible geological application. With such a statement in mind, it is perhaps not too over-zealous to claim that the localized buckling may be even more applicable than the periodical buckling. Mathematically, though, the set of localized natural modes is neither better nor worse than the set of periodical natural modes. In fact, they should be related. The solution (6.21) may be obtained from (6.18) by setting

$$A(k) = (-i)^n A_n \frac{k^{n-1}}{(n-1)!} e^{-k}. \tag{6.24}$$

7. Post-Buckling. Post-buckling analysis requires the inclusion of higher order terms and the result is in general an amplitude-eigenvalue curve branching off a critical eigenvalue (see, e.g., [19]). To the best of our knowledge, analysis of this nature has not been applied to cases where infinitely many solutions are associated with a single critical eigenvalue. Our attempts have been motivated by this observation and so far we have not been able to overcome certain difficulties involved.

Consider, for example, the buckling of the upper half space. If we try to trace the branch generated by (6.17), the higher order equations will involve a "forcing" term of the form e^{-ikZ}. Similarly, tracing a branch generated by (6.21) will lead to a forcing term of the form $A_n/(Z - i)^n$. This phenomenon is typical of all the cases we have considered. In an ordinary case, the "frequency" of the homogeneous part of the higher order equations is fixed by the choice of mode and hence all one has to do is to remove one secular term. But here the homogeneous part of the higher order equations is not affected by the choice of natural mode and, worse than that, the "forcing" function contains many secular terms. Either there is a way to remove all the scalar terms or the answer has to be that there are no nontrivial solutions. In the latter case, what would be the meaning of a buckling load obtained in this paper and many other cited references?

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