# OPTIMAL TEMPERATURE PATHS FOR THERMORHEOLOGICALLY SIMPLE VISCOELASTIC MATERIALS WITH CONSTANT POISSON'S RATIO ARE CANONICAL*

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**Abstract.** In this note we discuss the thermal stress problem for a thermorheologically-simple linearly-viscoelastic body, subjected to a spatially-uniform temperature field and homogeneous boundary conditions, assuming that Poisson's ratio is constant and inertia negligible. In particular, we consider the following optimization problem: of all temperature paths $\theta(t)$, $0 \leqslant t \leqslant t_f$, which belong to a given function class, is there one which renders a given stress measure a minimum at time $t_f$. We show that a resulting optimal path $\theta(t)$ (if it exists) is canonical: $\theta(t)$ is independent of the shape of the body and of the particular homogeneous boundary conditions.

**The viscoelastic problem.** We consider a thermorheologically-simple viscoelastic material subject to a spatially-uniform temperature field. We assume that Poisson's ratio $\nu$ is constant. The stress $\mathbf{S}(\mathbf{x}, t)$, strain $\mathbf{E}(\mathbf{x}, t)$, and temperature[1] $\theta(t)$ are then related through the constitutive equations[2]

$$\mathbf{S}(\mathbf{x}, t) = \int_0^t G(\xi(t) - \xi(s))\dot{\mathbf{Q}}(\mathbf{x}, s)\, ds,$$

$$\mathbf{Q} = \mathbf{E} + \frac{1}{1 - 2\nu}(\operatorname{tr}\mathbf{E})\mathbf{I} - \frac{(1 + \nu)\alpha}{1 - 2\nu}\eta\mathbf{I}, \qquad (1)^3$$

with $G(t)$ the scalar relaxation function, $\alpha$ the coefficient of termal expansion,

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[1] Actually, $\theta$ is the temperature *increment* relative to a fixed reference temperature.

[2] Cf., e.g., Muki and Sternberg [1].

[3] We use the following notation (cf., e.g., Gurtin [2]): upper-case boldface letters are second-order tensors; $\mathbf{A}^T$ and $\operatorname{tr}\mathbf{A}$ are the transpose and trace of $\mathbf{A}$; $|\mathbf{A}| = (A_{ij}A_{ij})^{1/2}$ (summation over repeated subscripts implied); $\mathbf{I}$ is the identity; $\operatorname{div}\mathbf{S}$ is the vector field with components $\partial S_{ij}/\partial x_j$; $\nabla\mathbf{u}$ is the tensor field with components $\partial u_i/\partial x_j$.

$$\xi(t) = \int_0^t \varphi(\theta(\beta))\, d\beta \tag{2}$$

the reduced time, and $\varphi(\theta)$ the shift factor.

As a customary, we assume quasi-static conditions; the remaining field equations then take the form

$$\operatorname{div} \mathbf{S} = \mathbf{0}, \qquad \mathbf{E} = \tfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \tag{3}$$

with $\mathbf{u}(\mathbf{x}, t)$ the displacement. Here and in what follows the time is confined to the interval $0 \leqslant t < \infty$, it being tacit that $\mathbf{u}$, $\mathbf{E}$, $\mathbf{S}$, and $\theta$ all vanish for $t < 0$.

We limit our attention to the homogeneous boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \mathcal{S}_1, \qquad \mathbf{Sn} = \mathbf{0} \quad \text{on } \mathcal{S}_2, \tag{4}$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are complementary subsets of — and $\mathbf{n}$ the outward unit normal to — the boundary of the body $\mathcal{B}$.

The boundary-value problem (1)–(4) can be solved, at least formally, as follows. Write $(1)_1$ in the form

$$\mathbf{S} = G \,\#\, \mathbf{Q}. \tag{5}$$

Since $\theta$ is independent of position, the operation # commutes with spatial differentiation. Hence $(3)_2$ becomes

$$G \,\#\, (\operatorname{div} \mathbf{Q}) = \mathbf{0},$$

which is satisfied by taking

$$\operatorname{div} \mathbf{Q} = \mathbf{0}. \tag{6}$$

Similarly, the boundary condition $(4)_2$ is satisfied provided

$$\mathbf{Qn} = \mathbf{0} \quad \text{on } \mathcal{S}_2. \tag{7}$$

Our problem thus reduces to solving the linear thermoelastic problem defined by the field equations $(1)_2$, $(3)_2$, and (6) and the boundary conditions $(4)_1$ and (7). Let $\mathbf{Q} = \mathbf{P}(\mathbf{x})$ denote the solution to the boundary-value problem corresponding to $\theta(t) = 1$, for all $t$. Since the boundary data are null and the field equations linear, the solution of the thermoelastic problem is[4]

$$\mathbf{Q}(\mathbf{x}, t) = \theta(t)\mathbf{P}(\mathbf{x});$$

hence (5) yields

$$\begin{aligned} \mathbf{S}(\mathbf{x}, t) &= \tau(t)\mathbf{P}(\mathbf{x}), \\ \tau &= G \,\#\, \theta. \end{aligned} \tag{8}$$

**Optimal temperature paths.** Assume we are given a (spatial) *stress measure* $\Sigma(\mathbf{A})$ defined for all sufficiently nice (symmetric, second-order) tensor functions $\mathbf{A}(\mathbf{x})$ on $\mathcal{B}$. Regarding the properties of $\Sigma$, we need only suppose that $\Sigma \geqslant 0$ and that

$$\Sigma(\beta \mathbf{A}) = |\beta| \Sigma(\mathbf{A}) \tag{9}$$

[4] We assume the data such that both elastic and viscoelastic problems are well posed (in the sense of existence and uniqueness).

for any scalar constant $\beta$. Examples of stress measures are the $L^2$-norm

$$\Sigma(\mathbf{A}) = \left[\int_{\mathfrak{B}} |\mathbf{A}|^2 \, dV\right]^{1/2};$$

the sup-norm

$$\Sigma(\mathbf{A}) = \sup_{x \in \mathfrak{B}} |\mathbf{A}(\mathbf{x})|;$$

the $\mathbf{x}_0$-evaluation of $|\mathbf{A}|$ ($\mathbf{x}_0$ a particular point of $\mathfrak{B}$)

$$\Sigma(\mathbf{A}) = |\mathbf{A}(\mathbf{x}_0)|;$$

the $L^2$-norm, sup-norm, or $\mathbf{x}_0$-evaluation of either the octahedral shear stress or the principal-stress magnitude corresponding to $\mathbf{A}$.

The optimization problem under consideration consists in finding temperature paths that minimize the associated stress measure.[5] To state this problem succinctly, let us agree to use the term *temperature path* for a function $\theta(t)$, $0 \leqslant t \leqslant t_f$, which belongs to some prescribed function class $\mathfrak{F}$. For example, $\mathfrak{F}$ may be the set of all sufficiently regular functions that have $\theta(0)$ and $\theta(t_f)$ equal to given initial and final temperatures. Given a temperature path, we write

$$\mathbf{S}_f(\mathbf{x}) = \mathbf{S}(\mathbf{x}, t_f), \tau_f = \tau(t_f)$$

for the corresponding functions computed using (8). (Here and in what follows $t_f > 0$ and $\mathfrak{F}$ are fixed.)

OPTIMIZATION PROBLEM. Find a temperature path $\theta(t)$ that minimizes the stress measure $\Sigma(\mathbf{S}_f)$.

By $(8)_1$ and (9),

$$\Sigma(\mathbf{S}_f) = |\tau_f| \Sigma(\mathbf{P}). \tag{10}$$

If

$$\Sigma(\mathbf{P}) = 0, \tag{11}$$

then all temperature paths are optimal; we henceforth exclude from our discussion situations in which (11) is satisfied. We then conclude from (10) that the optimization problem is equivalent to the

REDUCED OPTIMIZATION PROBLEM. Find a temperature path $\theta(t)$ that minimizes $|\tau_f|$; i.e., that minimizes the functional

$$\left|\int_0^{t_f} G\left(\int_s^{t_f} \varphi(\theta(\beta)) \, d\beta\right) \dot{\theta}(s) \, ds\right|. \qquad (12)^6$$

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[5] This problem — within an essentially one-dimensional framework — was studied by Weitsman and Ford [3], Weitsman [4], and Gurtin and Murphy [5]. Problems of this type arise, for example, in the curing of polymeric materials (cf. [3–5]).

[6] Generally, solutions of this problem will not be everywhere differentiable [3–5], so that (12) must be interpreted in some generalized sense (cf. [5]).

A direct consequence of this equivalence is the following interesting result: *the optimal path, if it exists, is independent of the shape of the body, of the particular homogeneous boundary conditions, and of the particular stress measure.*

The foregoing equivalence is important as it allows one to determine the optimal temperature path without solving the underlying boundary-value problem.

Actually, a slightly stronger result is possible. Let us agree to call a temperature path $\hat{\theta}$ better than a temperature path $\breve{\theta}$ provided

$$\Sigma(\hat{\mathbf{S}}_f) \leqslant \Sigma(\tilde{\mathbf{S}}_f)$$

(using obvious notation). It then follows from (10) that: *if $\hat{\theta}$ is better than $\tilde{\theta}$ for a given body under given homogeneous boundary conditions, $\hat{\theta}$ is better than $\tilde{\theta}$ for any body under all such boundary conditions, and for any choice of stress measure.*

This result is important experimentally as it shows that a "good" temperature path determined experimentally using a configuration amenable to testing is "good" for bodies of all shapes.

## References

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