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POTENTIAL RELATED TO A STRIP WITH A PAIR OF *V* OR *U* NOTICES*

By

CHIH-BING LING

Virginia Polytechnic Institute and State University

1. Synopsis and introduction. By a strip is here meant a plane region of infinite length bounded by two parallel straight lines. The strip is further indented symmetrically by a pair of V or U notches, one on each edge. The V notch consists of two equal line segments inclined symmetrically to the normal of the edge of the strip and connected smoothly at the closed end by a circular arc. The open end is divergent. The entire notch possesses an axis of symmetry. The U notch is a particular case of the V notch in which the two line segments are parallel and both intersect the edge normally.

The purpose of this paper is to present a potential solution for such a region with each boundary at a prescribed potential equal in magnitude but opposite in sign.

2. The problem. The geometry of the given strip with the pair of V notches on the edges is shown in Fig. 1. For convenience, the dimension of the strip is measured by a typical length b or the actual depth of either notch. In this manner, the straight portions of the lower and upper edges are denoted by $y = 0$ and $y = 2a$, respectively, where $a > 1$ on physical considerations. The radius of the circular arc DCG of the lower V notch $ADCGA'$ is denoted by λ. The angle of inclination of the line segments AD and GA' to the normal of the edge is denoted by ψ. Besides, the opening of the AA' is denoted by 2γ, where

$$\gamma = \{\lambda + (1 - \lambda)\sin\psi\}/\cos\psi. \tag{1}$$

The upper V notch is of the same size. Furthermore, let the potential on the lower boundary of the strip be v_0 and that on the upper boundary be $-v_0$. No generality is lost if we take $v_0 = 1$.

3. A set of harmonic functions. Define two pairs of polar coordinates by

$$z = x + iy = ir\exp(-i\theta),$$
$$z^* = z - 2ia = ir^*\exp(-i\theta^*), \tag{2}$$

such that

$$x = r\sin\theta, \qquad y = r\cos\theta,$$
$$x = r^*\sin\theta^*, \quad y - 2a = r^*\cos\theta^*. \tag{3}$$

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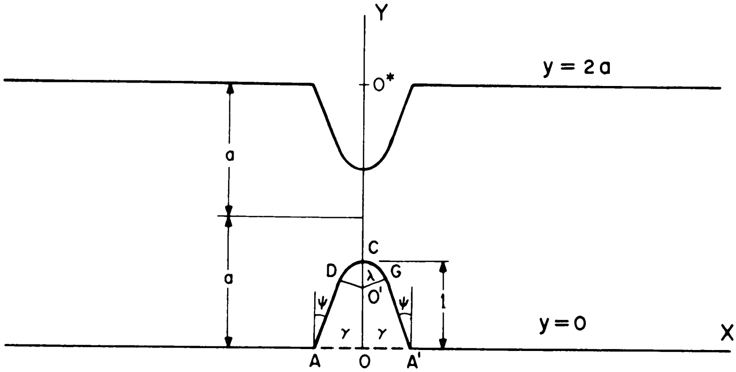

FIG. 1. The strip with a pair of V notches; length beong measured in unit b.

The pair (r, θ) has its pole at the origin 0 and the pair (r^*, θ^*) has its pole on the y axis at the point 0* or $y = 2a$. The angle in either pair is measured clockwisely from the y axis.

Consider a function in the region $0 \leqslant y \leqslant 2a$:

$$H_0(x, y) = \frac{\cos\theta}{r} + \frac{\cos\theta^*}{r^*} + \int_0^\infty f(k)\sinh k(y-a)\cos kx\,dk. \tag{4}$$

The function consists of three parts. The first part has a singularity at 0 and the second a singularity at 0*. It is harmonic, save at the singularities, and is even in x. It gives the following values along the lines $y = 0$, a and $2a$, respectively:

$$\begin{aligned} H_0(x,0) &= -\frac{2a}{x^2+4a^2} - \int_0^\infty f(k)\sinh ka\cos kx\,dk, \\ H_0(x,a) &= 0, \\ H_0(x,2a) &= \frac{2a}{x^2+4a^2} + \int_0^\infty f(k)\sinh ka\cos kx\,dk. \end{aligned} \tag{5}$$

The first expression is given by the second and third parts only. The first part is zero along $y = 0$ save at the origin. The third expression is given by the first and third parts only. The second part is zero along $y = 2a$ save at the point 0*. Both the first and third expressions vanish identically provided that by Fourier cosine transform,

$$\begin{aligned} f(k) &= -\frac{2}{\pi\sinh ka}\int_0^\infty \frac{2a}{x^2+4a^2}\cos kx\,dx \\ &= -e^{-2ka}/\sinh ka. \end{aligned} \tag{6}$$

The second expression is given by all the three parts. These relations indicate that the line $y = a$ is a line of antisymmetry. With the value of $f(k)$ in (6), the function becomes

$$H_0(x, y) = \mathrm{Re}\left[\frac{i}{z} + \frac{i}{z-2ia}\right] - \int_0^\infty \frac{e^{-2ka}}{\sinh ka}\sinh k(y-a)\cos kx\,dk. \tag{7}$$

The preceding function is useful in constructing the required potential solution. Successive differentiation of the function with respect to x gives functions also with the desired properties but odd derivatives must be excluded since they are not even in x and cannot enter into the desired symmetrical solution. The set of harmonic functions is therefore

$$\begin{aligned} H_{2s}(x, y) &= \frac{(-1)^s}{(2s)!}\frac{\partial^{2s}}{\partial x^{2s}} H_0(x, y) \\ &= \mathrm{Re}\left[\frac{i(-1)^s}{z^{2s+1}} + \frac{i(-1)^s}{(z-2ia)^{2s+1}}\right] \\ &\quad - \frac{1}{(2s)!}\int_0^\infty \frac{e^{-2ka}}{\sinh ka} k^{2s} \sinh k(y-a)\cos kx \, dk. \end{aligned} \tag{8}$$

Note that here a constant factor is introduced for convenience.

4. The solution. The potential solution for the given strip is constructed as follows:

$$V(x, y) = 1 - \frac{y}{a} + \sum_{s=0}^{\infty} A_{2s} H_{2s}(x, y), \tag{9}$$

where A_{2s} are parametric coefficients. The solution is harmonic, even in x, and antisymmetric with respect to the line $y = a$ as desired. It possesses two singularities with alternate signs on the edges of the strip, one at the origin and the other at the poin 0^*. Both such singularities are excluded from the strip owing to the presence of the notches. The solution gives potentials on the lower and upper edges of values 1 and -1, respectively, save at the singularities.

5. The mapping function. To adjust the remaining boundary conditions on the notches, a mapping function to transform the notch is needed. Such a transformation for the V and U notches has been considered by the author in a recent paper [1]. However, the value of λ here is defined differently. With the present λ, the mapping function of the V notch is given in two stages by

$$z = i(1-\lambda) - \lambda e^{-i\psi}\{1 + iqW_2(c)/W_1(c)\}, \tag{10}$$

$$c = 1 + i(\zeta + 1)/(\zeta - 1), \tag{11}$$

where

$$\begin{aligned} q &= 2^{3-2\delta}\Gamma(2-\delta)\Gamma(\tfrac{1}{2}+\delta)\cos\psi/\pi^{1/2}, \\ \delta &= \psi/\pi \qquad (o \leqslant \delta < \tfrac{1}{2}), \\ W_1(c) &= {}_2F_1(\tfrac{1}{2}-\delta, \tfrac{3}{2}-\delta; 3-2\delta; c), \\ W_2(c) &= c^{-(2-2\delta)}(1-c)\,{}_2F_1(\delta+\tfrac{1}{2}, \delta-\tfrac{1}{2}; 2; 1-c), \end{aligned} \tag{12}$$

in which ${}_2F_1$ is a Gauss' hypergeometric function. The resulting mapping function transforms the curve of the V notch $ADCGA'$ in the z plane through the real axis in an intermediate c plane into a portion of the circumference of a unit circle $\rho = 1$ in the ζ plane from $\phi = -\beta$ to $\phi = \beta$. Here (ρ, ϕ) are a pair of polar coordinates defined by

$$\zeta = i\rho e^{-i\phi}. \tag{13}$$

The value of β is given by

$$\beta = \pi/2 + 2\tan^{-1}(1-\alpha), \tag{14}$$

where α is the positive real root between 0 and 1 of the following equation of c:

$$\Gamma(2-\delta)\Gamma(\tfrac{1}{2}+\delta)\frac{2^{3-2\delta}}{\pi^{1/2}}\frac{W_2(c)}{W_1(c)}=\frac{1}{\lambda\cos^2\psi}-\frac{1}{1+\sin\psi}. \tag{15}$$

The transformations of the points on the cirve of V notch $ADCGA'$ are shown in Table 1.

TABLE 1. Transformations of points on curve of V notch.

Point	z	c	ζ	$\phi\ (\rho=1)$
A	$-\{\lambda+(1-\lambda)\sin\psi\}/\cos\psi$	α	$ie^{i\beta}$	$-\beta$
D	$i(1-\lambda)-\lambda e^{-i\psi}$	1	-1	$-\pi/2$
C	i	2	i	0
G	$i(1-\lambda)+\lambda e^{i\psi}$	$\pm\infty$	1	$\pi/2$
A'	$\{\lambda+(1-\lambda)\sin\psi\}/\cos\psi$	$-\alpha/(1-\alpha)$	$ie^{-i\beta}$	β

In the case of the U notch, the corresponding mapping function is obtained by simply putting $\psi=0$ in (10) whereas (11) remains unchanged. It was further found that the hypergeometric functions involved in this case can be expressed in terms of complete elliptic integrals. The mapping function (10) becomes alternately

$$z=-\lambda+i(1-\lambda)-2i\lambda\frac{(2-c)E'-cK'}{(2-c)E-2(1-c)K}. \tag{16}$$

Accordingly, the equation (15) is to be replaced by

$$\frac{(2-c)E'-cK'}{(2-c)E-2(1-c)K}=\frac{1-\lambda}{2\lambda}. \tag{17}$$

The complex variable c is also the square of the modulus of the complete elliptic integrals. Note that here the customary notations for the complete elliptic integrals are used.

6. Boundary conditions on notches. Let any point on the curve of the lower notch in the z plane be denoted by z_0. It is seen that by the mapping functions, z_0 is first transformed into c_0 on the real axis of the c plane and then c_0 is transformed into a portion of the circumference of a unit circle in the ζ plane as a function of ϕ. Thus, the resulting mapping function expresses z_0 as a function of ϕ. By (11), c_0 and ϕ are connected by

$$c_0=1+\tan(\pi/4+\phi/2). \tag{18}$$

Write

$$z_0=x_0+iy_0. \tag{19}$$

Then the function V on the curve of the lower notch is

$$V(x_0,y_0)=1-\frac{y_0}{a}+\sum_{s=0}^{\infty}A_{2s}H_{2s}(x_0,y_0). \tag{20}$$

Expand this function into a Fourier cosine series in ϕ over the interval $(-\beta, \beta)$ in the form:

$$V(x_0, y_0) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi\phi}{\beta}, \tag{21}$$

where by symmetry, for $m \geqslant 0$,

$$a_m = \frac{2}{\beta}\int_{-\beta}^{0} V(x_0, y_0)\cos\frac{m\pi\phi}{\beta}\,d\phi. \tag{22}$$

Hence, the boundary condition on the curve of the lower notch is satisfied if for $m = 0, 1, 2, \ldots,$

$$a_m = 2\delta_{0,m}, \tag{23}$$

where $\delta_{n,m}$ is a Kronecker delta. This leads to a set of equations for determining A_{2s} as follows: For $m = 0, 1, 2, \ldots,$

$$\sum_{s=0}^{\infty} {}^{m}G_{2s}A_{2s} = 2\delta_{0,m} - g_m, \tag{24}$$

where

$$g_m = \frac{2}{\beta}\int_{-\beta}^{0}\left(1 - \frac{y_0}{a}\right)\cos\frac{m\pi\phi}{\beta}\,d\phi,$$

$${}^{m}G_{2s} = \frac{2}{\beta}\int_{-\beta}^{0} H_{2s}(x_0, y_0)\cos\frac{m\pi\phi}{\beta}\,d\phi. \tag{25}$$

By antisymmetry, when the boundary condition on the curve of the lower notch is satisfied, the relevant boundary condition on the curve of the upper notch is automatically satisfied. The solution is therefore complete when the coefficients A_{2s} are solved from (24).

7. Numerical examples. The Gauss' hypergeometric function when $|c| < 1$ is represented by the series:

$${}_2F_1(\alpha_1, \beta_1; \gamma_1; c) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\beta_1)_n}{(\gamma_1)_n}\frac{c^n}{n!}, \tag{26}$$

where

$$(h)_n = \Gamma(n+h)/\Gamma(h). \tag{27}$$

As mentioned before, the root α of the equation (15) or (17) lies between 0 and 1. The range of c in the integrals (25) is from α to 2 as shown in Table 1. Hence, the total range of c to be considered in computation is at most from 0 to 2. For the sake of better convergence of the series, the following scheme is adapted in computation:

$$
\begin{aligned}
W_1(c) &= {}_2F_1(\tfrac{1}{2}-\delta, \tfrac{3}{2}-\delta; 3-2\delta; c) \quad \text{for } 0 < c \leqslant 0.7, \\
&= \frac{\Gamma(3-2\delta)}{(\frac{3}{2}-\delta)\Gamma^2(\frac{3}{2}-\delta)} \left\{ 1 + \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-\delta)_{n+1}(\frac{3}{2}-\delta)_{n+1}}{n!(n+1)!} (1-c)^{n+1} K_n(c) \right\} \\
&\qquad \text{for } 0.3 \leqslant c \leqslant 1.7, \\
&= \frac{ie^{-i\psi}\Gamma(3-2\delta)}{(\frac{3}{2}-\delta)\Gamma^2(\frac{3}{2}-\delta)c^{(1-2\delta)/2}} \left\{ 1 - \sum_{n=0}^{\infty} \frac{(\frac{1}{2}-\delta)_{n+1}(\delta-\frac{3}{2})_{n+1}}{n!(n+1)} \frac{L_n(c)}{c^{n+1}} \right\} \\
&\qquad \text{for } 1.4 \leqslant c \leqslant 2, \qquad (28) \\
W_2(c) &= \frac{\Gamma(2\delta-2)(1-c)}{(\delta-\frac{1}{2})\Gamma^2(\delta-\frac{1}{2})} {}_2F_1(\tfrac{3}{2}-\delta, \tfrac{5}{2}-\delta; 3-2\delta; c) \\
&\quad + \frac{\Gamma(2-2\delta)(1-c)}{(\frac{3}{2}-\delta)\Gamma^2(\frac{3}{2}-\delta)c^{2-2\delta}} {}_2F_1(\delta+\tfrac{1}{2}, \delta-\tfrac{1}{2}; 2\delta-1; c) \\
&\qquad \text{for } 0 < c \leqslant 0.7, \\
&= c^{-(2-2\delta)}(1-c)\,{}_2F_1(\delta+\tfrac{1}{2}, \delta-\tfrac{1}{2}; 2; 1-c) \quad \text{for } 0.3 \leqslant c \leqslant 1.7, \\
&= c^{-(3-2\delta)/2}(1-c)\,{}_2F_1\left(\delta-\tfrac{1}{2}, \tfrac{3}{2}-\delta; 2; \frac{c-1}{c}\right) \quad \text{for } 1.4 \leqslant c \leqslant 2,
\end{aligned}
$$

where, for $n \geqslant 0$,

$$
\begin{aligned}
K_n(c) &= \ln(1-c) + \frac{1}{n+1} + \frac{1}{n+\frac{3}{2}-\delta} \\
&\quad + 2\sum_{m=0}^{n} \left(\frac{1}{m+\frac{1}{2}-\delta} - \frac{1}{m+1} \right) - 2\sum_{m=2}^{\infty} \omega_m (\tfrac{1}{2}+\delta)^{m-1}, \\
L_n(c) &= \ln(-c) - \frac{1}{n+1} - \frac{1}{\frac{3}{2}-\delta} - \frac{1}{\frac{1}{2}-\delta} \\
&\quad + \sum_{m=0}^{n} \left(\frac{2}{m+1} - \frac{1}{m+\frac{1}{2}-\delta} - \frac{1}{m-\frac{3}{2}+\delta} \right) + 2\sum_{m=2}^{\infty} \omega_m (\tfrac{1}{2}+\delta)^{m-1}, \qquad (29)
\end{aligned}
$$

in which, for $m \geqslant 2$,

$$\omega_m = \sum_{n=1}^{\infty} \frac{1}{n^m}, \tag{30}$$

and for the logarithmic terms involved,

$$
\begin{aligned}
\ln(1-c) &= \ln|1-c| - \begin{cases} 0, & \text{if } c < 1, \\ i\pi, & \text{if } c > 1, \end{cases} \\
(1-c)\ln(1-c) &= 0, \quad \text{if } c = 1, \\
\ln(-c) &= \ln c - i\pi, \quad \text{if } c > 0.
\end{aligned} \tag{31}
$$

Note that two of the preceding functions are the original functions in (12) while the rest are transformed functions [2]. The overlapping values of c may be used for checking purpose. The mapping function in (10) is also valid in the limiting case $\delta = 0$. Nevertheless, the second of the first two transformed functions for W_2 in (28) does not hold for $\delta = 0$. The next function may be used for the combined range $0 < c \leqslant 1.7$. On the other hand, if the mapping function in (16) is used, the complete elliptic integrals may be evaluated by Gauss method of arithmetic-geometric mean. This method is useful even when the modulus is complex [3].

The root α of the equation (15) or (17) is solved by using a method devised by the author. By this method, the root can be computed one digit at a time until the desired accuracy is reached. A description of the method will be given in a separate paper to appear elsewhere. The subsequent computation of β is straightforward. The results are shown in Tables 2 and 3. The values corresponding to $\delta = 0$ or $\psi = 0$ are for the U notch.

TABLE 2. Values of α.

δ	ψ	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$
0	0°	0.5569562	0.6873197	0.7658807	0.8216731
0.05	9	0.5122765	0.6453784	0.7267243	0.7847184
0.10	18	0.4586555	0.5929492	0.6768990	0.7373738
0.15	27	0.3958618	0.5283969	0.6139599	0.6766972
0.20	36	0.3245698	0.4505061	0.5353760	0.5992477
0.25	45	0.2471183	0.3594965	0.4394602	0.5018117

TABLE 3. Values of β.

δ	ψ	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$
0	0°	2.404905	2.176894	2.030751	1.923740
0.05	9	2.478353	2.252369	2.104321	1.994887
0.10	18	2.563144	2.343936	2.195823	2.084449
0.15	27	2.657710	2.452143	2.307625	2.196188
0.20	36	2.758887	2.575705	2.440693	2.333106
0.25	45	2.861482	2.710137	2.592595	2.495191

The following two cases, namely, $a = 2$, $\lambda = \frac{1}{5}$ and $\psi = 0$ and 9°, are computed as illustrative examples. To facilitate evaluation, the function H_{2s} in (7) and (8) is developed by expansion and integration into the following series for $s \geqslant 0$:

$$H_{2s}(x, y) = \sum_{n=0}^{\infty} \operatorname{Re}\left[\frac{i(-1)^s}{(z + 2nai)^{2s+1}} + \frac{i(-1)^s}{(z - 2nai - 2ai)^{2s+1}}\right]$$
$$= \sum_{n=0}^{\infty}\left[\frac{\cos(2s+1)\theta_n}{\{x^2 + (y + 2na)^2\}^{(2s+1)/2}} + \frac{\cos(2s+1)\theta_n^*}{\{x^2 + (y - 2na - 2a)^2\}^{(2s+1)/2}}\right], \tag{32}$$

where θ_n and θ_n^* are

$$\theta_n = \tan^{-1}\frac{x}{y + 2na}, \qquad \theta_n^* = \tan^{-1}\frac{x}{y - 2na - 2a}. \tag{33}$$

The series converges rapidly when s is large. The first series for $s = 0$ can be summed into a function in closed form. The sums of the succeeding series can be found from this function by differentiation. The first few such sums are

$$\begin{aligned}
H_0(x, y) &= \frac{\pi}{2a}\operatorname{Re}\left[i\coth\frac{\pi z}{2a}\right],\\
H_2(x, y) &= -\left(\frac{\pi}{2a}\right)^3\operatorname{Re}\left[i\operatorname{csch}^2\frac{\pi z}{2a}\coth\frac{\pi z}{2a}\right],\\
H_4(x, y) &= \left(\frac{\pi}{2a}\right)^5\operatorname{Re}\left[i\left(\frac{1}{3}+\operatorname{csch}^2\frac{\pi z}{2a}\right)\operatorname{csch}^2\frac{\pi z}{2a}\coth\frac{\pi z}{2a}\right],\\
H_6(x, y) &= -\left(\frac{\pi}{2a}\right)^7\operatorname{Re}\left[i\left(\frac{2}{45}+\frac{2}{3}\operatorname{csch}^2\frac{\pi z}{2a}+\operatorname{csch}^4\frac{\pi z}{2a}\right)\operatorname{csch}^2\frac{\pi z}{2a}\coth\frac{\pi z}{2a}\right].
\end{aligned} \tag{34}$$

They are useful in evaluation when s is small. The integrals in (25) are evaluated here by using Simpson rule. The computation is carried out in two parts, one part from $-\beta$ to $-\pi/2$ and the other from $-\pi/2$ to 0. The first part is further divided into 80 double divisions and the second into 100 double divisions. To test the accuracy, the computation is repeated by changing the width of the double divisions by 10%. The values are found stable to five significant figures.

To solve (24), the set of equations is truncated from $m = 0$ to $m = M - 1$, in which the first M coefficients of A_{2s} are retained. When the truncated set of equations is solved by matrix inversion or otherwise, the potential in the strip can be computed. In particular, the value along the line $x = 0$ across the narrowest section of the strip is given by

$$V(0, y) = 1 - \frac{y}{a} + \sum_{s=0}^{M-1} A_{2s}H_{2s}(0, y). \tag{35}$$

It is inferred from existing solutions of notch problems that the resulting values in the present problem are anticipated to exhibit Mitchell-Ling effect of truncation [4], [5], [6]. This is to say that the value of the potential at a given point forms an oscillatory sequence with the value of M, whose elements are alternately below and above, and as M increases

TABLE 4. Values of $V(0, y)$ across narrowest section for $a = 2$ and $\lambda = \frac{1}{5}$.

y	$\psi = 0$	$\psi = 9°$
1	1	1
1.1	0.7326	0.7492
1.2	0.5533	0.5767
1.3	0.4242	0.4487
1.4	0.3264	0.3492
1.5	0.2493	0.2688
1.6	0.1862	0.2018
1.7	0.1325	0.1442
1.8	0.0852	0.0929
1.9	0.0417	0.0455
2	0	0

converge very slowly to, the true value. Owing to this effect, accurate value of the potential can be obtained by merely taking the average value corresponding to two appropriate consecutive integers of M. The values of the potential across the narrowest section of the strip shown in Table 4 are the average values corresponding to $M = 15$ and 16. Some typical values are shown in the table below:

M	$\psi = 0$	$\psi = 9°$
15	0.7330	0.7485
16	0.7323	0.7500
average	0.7326	0.7492

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