A PERTURBATION METHOD FOR SOLVING A QUADRATIC EVOLUTION EQUATION*

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Abstract. A quadratic evolution equation of the form

$$\dot{u} = Lu + \varepsilon Qu$$

is considered where L and Q are particular linear and quadratic integral operators respectively. This equation has been proposed to describe the variation with time of $u(x, t)$, the volume density of an ensemble of particles undergoing concurrent coalescence and fracture.

The equation is solved in an important special case by standard perturbation techniques where ε is the sufficiently small parameter. This method, in combination with certain results from the theory of semigroups of linear operators, provides computable approximations as well as an existence proof. An example is also given.

A number of mechanical and physiochemical processes involve ensembles of particles undergoing concurrent coalescence and fracture (see bibliography in [4]). The result is an ensemble volume density that varies with time.

An equation proposed by T. H. Courtney (see [4]) to describe the evolution of the volume density with time is:

$$\frac{\partial u}{\partial t} = -Bx^{\alpha}u(x,t) - Cx^{\gamma}u(x,t)\int_0^{V_0-x} y^{\gamma}u(y,t)\,dy$$
$$+2B\int_x^{V_0} y^{\alpha-1}u(y,t)\,dy + \frac{C}{2}\int_0^x y^{\gamma}(x-y)^{\gamma}u(y,t)u(x-y,t)\,dy$$
$$\text{with } u(x,0) = u_0(x) \text{ for } 0 \leq x \leq V_0. \quad (1)$$

In (1), $u(x, t)\,dx$ is the number of particles at time t with volume between x and $x + dx$, $u_0(x)$ is the initial volume distribution and

$$V_0 = \int_0^{V_0} xu_0(x)\,dx = \text{total volume of all particles in the ensemble.} \quad (2)$$

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Parameters B and α are positive and represent the "tendency to fracture". C is a positive constant representing the "coalescence tendency" and γ gives the dependence of coalescence on particle size. If particles are spherical and coalescence depends on surface area then $\gamma = 2/3$.

In [4] it was shown that a unique, nonnegative, volume conserving solution to (1) exists for all $t \geqslant 0$ in the L_1 space,

$$X = \left\{ f\colon \int_0^{V_0} |f(x)|\, x\, dx < \infty \right\}.$$

In the present paper an alternate representation for this solution is derived which may in some cases be more useful computationally. Since this coincides with the solution constructed in [4] (by means of Picard iteration) no attempt will be made to rederive the above properties.

Equation (1) will be simplified by introducing the substitution

$$w(x,t) = x^{\gamma} u(x,t), \tag{3}$$

in which case (1) becomes

$$\frac{\partial w}{\partial t} = Lw + \varepsilon Qw, \quad w(x,0) = x^{\gamma} u_0(x) = w_0(x) \tag{4}$$

where

$$Lw = -Bx^{\alpha} w(x,t) + 2Bx^{\gamma} \int_x^{V_0} y^{\alpha-\gamma-1} w(y,t)\, dy$$

$$Qw = \frac{B}{2} x^{\gamma} \int_0^x w(y,t) w(x-y,t)\, dy - Bx^{\gamma} w(x,t) \int_0^{V_0 - x} w(y,t)\, dy$$

$$\varepsilon = C/B.$$

L is the linear "fracture" operator (terms with coefficient B) and Q is the quadratic "coalescence" operator.

It is convenient from several standpoints to solve (4) in $L_2[0, V_0]$, that is, seek $w(\cdot, t) \in L_2[0, V_0]$ for each $t \geqslant 0$ which solves (4). It is thus assumed that (i) $\gamma < 3/2$ and (ii) $\gamma < \alpha$. Condition (i) ensures that $u(\cdot, t) \in X$ whenever $w(\cdot, t) \in L_2[0, V_0]$. Condition (ii) guarantees that L is bounded from $L_2[0, V_0]$ into itself (see Proposition 1 below).

In (4) the εQw term is to be viewed as a perturbation on the linear problem, $\partial w/\partial t = Lw$. When this approach is taken one often seeks a solution valid for ε sufficiently small. Since $\varepsilon = C/B$ this defines the condition $C \ll B$, or a situation in which the effect of coalescence is small compared to fracture. This case is of interest to researchers in the field. It turns out, however, that the solution developed below is good for arbitrary ε. Inequality (15) below implies a trade-off between the size of ε and the time interval $[0, T]$ on which the series solution is known to be valid. Regardless how large ε becomes, a sufficiently small T can always be found so that (3) and (5) give a unique, nonnegative solution to (1) valid on $[0, T]$. It is an open question whether stronger results can be obtained, for example, if the series solution remains valid outside $[0, T]$ and possibly for all $t \geqslant 0$.

Following standard perturbation procedures (see [3] and [5]), a solution in the form

$$w(x,t) = \sum_{k=0}^{\infty} \varepsilon^k p_k(x,t) \tag{5}$$

is substituted into (4). Equating coefficients of ε^k the following equations obtain:

$$\frac{\partial p_0}{\partial t} = Lp_0 \quad \text{for } k = 0,$$

$$\frac{\partial p_k}{\partial t} = Lp_k + f_k \quad \text{for } k \geqslant 1$$

where

$$\begin{aligned} f_k(x,t) = {} & \frac{B}{2} x^\gamma \sum_{n=0}^{k-1} \int_0^x p_n(y,t) p_{k-n-1}(x-y,t)\, dy \\ & - Bx^\gamma \sum_{n=0}^{k-1} p_n(x,t) \int_0^{V_0 - x} p_{k-n-1}(y,t)\, dy. \end{aligned} \tag{6}$$

Once initial conditions are specified these equations can be solved recursively since f_k involves previously computed p_j's. Now,

$$w(x,0) = \sum_{k=0}^{\infty} \varepsilon^k p_k(x,0) = w_0(x),$$

therefore choose

$$p_0(x,0) = w_0(x), \quad p_k(x,0) \equiv 0, \qquad k \geqslant 1,$$

leading to the fully specified initial value problems

$$\begin{aligned} \frac{\partial p_0}{\partial t} &= Lp_0, \quad p_0(x,0) = w_0(x) \\ \frac{\partial p_k}{\partial t} &= Lp_k + f_k, \quad p_k(x,0) \equiv 0, \qquad k \geqslant 1. \end{aligned} \tag{7}$$

To solve (7) the theory of semigroups of linear operators is utilized (see [6, Chap. IX], [1, Chap. I] and [2, Chap. XVII, Sec. 1.7]). The correspondences:

$$\begin{aligned} p_k(\cdot, t) &\leftrightarrow \Psi_k(t), \\ f_k(\cdot, t) &\leftrightarrow F_k(t), \\ w(\cdot, t) &\leftrightarrow \Phi(t) \end{aligned}$$

are made where $\{\Psi_k\}$, $\{F_k\}$ and Φ are functions from $[0, \infty)$ into $L_2[0, V_0]$. The system (7) then becomes

$$\begin{aligned} \dot{\Psi}_0 &= L\Psi_0, \quad \Psi_0(0) = w_0, \\ \dot{\Psi}_k &= L\Psi_k + F_k, \quad \Psi_k(0) = 0, \qquad k \geqslant 1. \end{aligned} \tag{8}$$

Hereafter the norm $\|\cdot\|$ will denote the usual $L_2[0, V_0]$ norm. The same notation will also denote the operator norm for operators in $B(L_2, L_2)$.

PROPOSITION 1. If $\alpha > \gamma$ then L is a bounded, linear operator from $L_2[0, V_0]$ into itself.

Proof. Using the triangle inequality,

$$\|Lh\| \leqslant B\|x^\alpha h(x)\| + 2B\left\|x^\gamma \int_0^{V_0} y^{\alpha-\gamma-1} h(y)\,dy\right\|.$$

For the first term,

$$\|x^\alpha h(x)\|^2 = \int_0^{V_0} x^{2\alpha} h(x)^2\,dx \leqslant V_0^{2\alpha}\|h\|^2.$$

Using the Schwarz inequality in the second term, it is clear that

$$\left\|x^\gamma \int_x^{V_0} y^{\alpha-\gamma-1} h(y)\,dy\right\|^2 = \int_0^{V_0} dx\, x^{2\gamma}\left(\int_x^{V_0} y^{\alpha-\gamma-1} h(y)\,dy\right)^2$$
$$\leqslant \begin{cases} \dfrac{\|h\|^2 V_0^{2\alpha}}{2\alpha(2\gamma+1)} & \text{if } \alpha \neq \gamma + \dfrac{1}{2}, \\ \|h\|^2 V_0^{2\gamma+1} & \text{if } \alpha = \gamma + \dfrac{1}{2}. \end{cases}$$

From this it follows

$$\|L\| \leqslant \begin{cases} BV_0^\alpha\left(1 + \sqrt{\dfrac{2}{\alpha(2\gamma+1)}}\right) & \text{if } \alpha \neq \gamma + \dfrac{1}{2}, \\ 3BV_0^\alpha & \text{if } \alpha = \gamma + \dfrac{1}{2}. \end{cases} \qquad \text{Q.E.D.}$$

Using the usual variation of parameters formula (see [6, Chap. XIV, Sec. 5] and [1, Chap. I, Sec. 3.3]), the formal solutions to (8) are

$$\Psi_0(t) = e^{tL} w_0,$$
$$\Psi_k(t) = \int_0^t e^{(t-s)L} F_k(s)\,ds, \qquad k \geqslant 1, \tag{9}$$

and so

$$\Phi(t) = e^{tL} w_0 + \sum_{k=1}^{\infty} \varepsilon^k \int_0^t e^{(t-s)L} F_k(s)\,ds. \tag{10}$$

PROPOSITION 2. If F_k is continuous from $[0, \infty)$ to $L_2[0, V_0]$, so is LF_k.

Remark. Hereafter the term continuity will mean strong continuity.

Proof. This follows directly from the boundedness of L. Q.E.D.

THEOREM 1. The functions $\{\Psi_k\colon k \geqslant 0\}$ given in (8) and $\{F_k\colon k \geqslant 1\}$ given by (6) are continuous functions from $[0, \infty)$ to $L_2[0, V_0]$.

Proof. The proof is inductive and in three steps. Step 1. Show Ψ_0 is continuous. Step 2. Show F_k is continuous if $\Psi_0, \Psi_1, \ldots, \Psi_{k-1}$ are. Step 3. Show if F_k is continuous, so is Ψ_k.

Step 1: The properties of the exponential operator (see [6, Chap. IX, Sec. 1]) give

$$\Psi_0(t + \Delta t) - \Psi_0(t) = [e^{(t+\Delta t)L} - e^{tL}]w_0 = e^{tL}[e^{\Delta t L} - I]w_0.$$

Therefore,

$$\|\Psi_0(t+\Delta t) - \Psi_0(t)\| \leqslant \|e^{tL}\| \, \|e^{\Delta t L} - I\| \, \|w_0\|$$
$$\leqslant e^{t\|L\|}(e^{|\Delta t| \, \|L\|} - 1)\|w_0\| \to 0 \quad \text{as } \Delta t \to 0.$$

Step 2: Repeated use of the triangle and Schwarz inequalities yields (omitting the details):

$$\|F_k(t+\Delta t) - F_k(t)\|$$
$$\leqslant \frac{3B}{2} V_0^{\gamma+1/2} \sum_{n=0}^{k-1} \{\|\Psi_n(t+\Delta t)\| \, \|\Psi_{k-n-1}(t+\Delta t) - \Psi_{k-n-1}(t)\|$$
$$+ \|\Psi_{k-n-1}(t)\| \, \|\Psi_n(t+\Delta t) - \Psi_n(t)\|\}$$

This proves step 2.

Step 3. From (9) we have

$$\Psi_k(t+\Delta t) - \Psi_k(t) = \int_0^{t+\Delta t} e^{(t+\Delta t-s)L} F_k(s)\, ds$$
$$- \int_0^t e^{(t-s)L} F_k(s)\, ds.$$

If $\int_0^{t+\Delta t} e^{(t-s)L} F_k(s)\, ds$ is added and subtracted then taking norms, (see [2, Chap. XVII, Sec. 1.7]) the following inequality obtains:

$$\|\Psi_k(t+\Delta t) - \Psi_k(t)\| \leqslant (e^{|\Delta t| \, \|L\|} - 1) \int_0^{t+\Delta t} e^{|t-s| \, \|L\|} \|F_k(s)\|\, ds$$
$$+ \int_t^{t+\Delta t} e^{|t-s| \, \|L\|} \|F_k(s)\|\, ds.$$

Since F_k is continuous it follows Ψ_k is also. This proves the theorem. Q.E.D.

THEOREM 2. The functions $\{\Psi_k\}$ given in (9) are strong solutions to (8).

Proof. Theorem 1 and Proposition 2 imply the hypotheses of Proposition 3.2, Chap. I of [1] are satisfied. This proves Theorem 2. Q.E.D.

COROLLARY 1. $\dot{\Psi}_k$ exists and is continuous for $k \geqslant 0$.

Proof. This follows from (9) and Theorems 1 or 2. Q.E.D.

Although (9) gives solutions to (8), it remains to be shown that the series in (10) converges and solves (4).

Assume $0 \leqslant t \leqslant T$ for some $T > 0$. From (9) it follows that

$$\|\Psi_k(t)\| \leqslant e^{T\|L\|} \int_0^T \|F_k(s)\|\, ds. \tag{11}$$

To get an estimate for $\|F_k(s)\|$ an argument similar to the proof of Theorem 1, step 2 yields

$$\|F_k(t)\| \leqslant \frac{3B}{2} V_0^{\gamma+1/2} \sum_{n=0}^{k-1} \|\Psi_n(t)\| \, \|\Psi_{k-n-1}(t)\|. \tag{12}$$

Substituting (12) into (11) gives

$$\|\Psi_k(t)\| < \frac{3B}{2} V_0^{\gamma+1/2} e^{T\|L\|} \sum_{n=0}^{k-1} \int_0^T \|\Psi_n(s)\| \, \|\Psi_{k-n-1}(s)\| \, ds. \tag{13}$$

PROPOSITION 3. For $k \geqslant 0$ and $0 \leqslant t \leqslant T$ it follows

$$\|\Psi_k(t)\| \leqslant bC_0^k a_k \tag{14}$$

where

$$b = \|w_0\| e^{T\|L\|}, \qquad C_0 = \frac{3B}{2} bTV_0^{\gamma+1/2} e^{T\|L\|}$$

and

$$a_{k+1} = \sum_{n=0}^{k} a_n a_{k-n}, \qquad a_0 = 1.$$

Proof. Inequality (14) is clear for $k = 0$. Suppose it is true for $k = 0, 1, \ldots, N$. We show (14) then follows if $k = N + 1$. Using (13) we have

$$\|\Psi_{N+1}(t)\| \leqslant \frac{3B}{2} V_0^{\gamma+1/2} e^{T\|L\|} \sum_{n=0}^{N} \int_0^T \|\Psi_n(s)\| \, \|\Psi_{N-n}(s)\| \, ds.$$

Applying (14) under the integral yields

$$\begin{aligned} \|\Psi_{N+1}(t)\| &\leqslant \frac{3B}{2} V_0^{\gamma+1/2} e^{T\|L\|} Tb^2 C_0^N \sum_{n=0}^{N} a_n a_{N-n} \\ &= bC_0^{N+1} a_{N+1}. \quad \text{Q.E.D.} \end{aligned}$$

THEOREM 3. Let $T > 0$ and $\varepsilon > 0$ satisfy

$$\varepsilon < 1/4C_0 \tag{15}$$

where C_0 is given in Proposition 3. Then the series $\sum_{k=0}^{\infty} \varepsilon^k \Psi_k$ and $\sum_{k=0}^{\infty} \varepsilon^k \dot{\Psi}_k$ converge in $L_2[0, V_0]$ uniformly in t for $0 \leqslant t \leqslant T$.

Proof. Using (14),

$$\left\| \sum_{k=N+1}^{\infty} \varepsilon^k \Psi_k(t) \right\| \leqslant \sum_{k=N+1}^{\infty} \varepsilon^k \|\Psi_k(t)\| \leqslant \sum_{k=N+1}^{\infty} b\varepsilon^k C_0^k a_k.$$

Now,

$$a_k = \frac{(2k)!}{k!\,(k+1)!}$$

so that $a_k \leqslant 4^k$ for $k \geqslant 0$. Thus,

$$\left\| \sum_{k=N+1}^{\infty} \varepsilon^k \Psi_k(t) \right\| \leqslant b \sum_{k=N+1}^{\infty} (4C_0\varepsilon)^k.$$

If $4C_0\varepsilon < 1$ this quantity is small if N is large. The bound is independent of t. From (8) it follows

$$\|\dot{\Psi}_k\| \leqslant \|L\|\,\|\Psi_k\| + \|F_k\| \quad \text{for } k \geqslant 1.$$

Then using (12) and (14) it follows

$$\|\dot{\Psi}_k\| \leqslant bC_0^k a_k\left(\|L\| + Te^{-T\|L\|}\right)$$

and so the same argument gives the uniform convergence of $\sum_{k=0}^{\infty} \varepsilon^k \dot{\Psi}_k$. Q.E.D.

THEOREM 4. If ε and T satisfy (15) then the solution given in (10) solves (4) for $0 \leqslant t \leqslant T$.

Proof. Using (8) and the linearity of L:

$$\sum_{k=0}^{N} \varepsilon^k \dot{\Psi}_k = L \sum_{k=0}^{N} \varepsilon^k \Psi_k + \sum_{k=0}^{N} \varepsilon^k F_k \qquad (F_0 = 0).$$

Letting $N \to \infty$, and now using the continuity of L:

$$\sum_{k=0}^{\infty} \varepsilon^k \dot{\Psi}_k = L\left(\sum_{k=0}^{\infty} \varepsilon^k \Psi_k\right) + \sum_{k=0}^{\infty} \varepsilon^k F_k.$$

The proof is complete if it can be shown that

$$\frac{d}{dt}\sum_{k=0}^{\infty} \varepsilon^k \Psi_k = \sum_{k=0}^{\infty} \varepsilon^k \dot{\Psi}_k \quad \text{and} \tag{i}$$

$$\varepsilon Q\left(\sum_{k=0}^{\infty} \varepsilon^k \Psi_k\right) = \sum_{k=0}^{\infty} \varepsilon^k F_k. \tag{ii}$$

Equality (i) follows just as in advanced calculus since $\sum_{k=0}^{\infty} \varepsilon^k \dot{\Psi}_k$ converges uniformly and $\dot{\Psi}_k$ is continuous. Equality (ii) follows from the continuity of Q, the way $\{F_k\}$ is defined and the convergence of $\sum_{k=0}^{\infty} \varepsilon^k \|F_k\|$. This proves the theorem. Q.E.D.

Before concluding with an example consider

PROPOSITION 4. If L is given in (4) then for $h \in L_2[0, V_0]$

$$e^{tL}h = \exp(-tBx^{\alpha}) \times\left[h(x) + \alpha x^{\gamma}\sum_{k=1}^{\infty}\left(\frac{2tB}{\alpha}\right)^k \frac{1}{k!\,(k-1)!}\int_x^{V_0} y^{\alpha-\gamma-1}(y^{\alpha} - x^{\alpha})^{k-1} h(y)\,dy\right]. \tag{16}$$

Proof. Write $e^{tL} = e^{-tBx^{\alpha}I} \cdot e^{2tBx^{\gamma}U}$ where I is the identity and

$$Uh = \int_x^{V_0} y^{\alpha-\gamma-1} h(y)\,dy$$

is a Volterra integral operator. Clearly

$$e^{-tBx^{\alpha}I}(h) = e^{-tBx^{\alpha}}h(x).$$

Further, a simple induction argument shows

$$(x^{\gamma}U)^k h = \frac{x^{\gamma}}{(k-1)!\alpha^{k-1}}\int_x^{V_0} y^{\alpha-\gamma-1}(y^{\alpha} - x^{\alpha})^{k-1} h(y)\,dy. \tag{17}$$

Using

$$e^{2tBx^\gamma U} = \sum_{k=0}^{\infty} \frac{(2tB)^k}{k!}(x^\gamma U)^k$$

and (17), (16) follows. Q.E.D.

EXAMPLE. Assume initially all particles have the same volume, x_0. Approximate this initial distribution by

$$\phi_0(x) = \frac{V_0}{x_0} x^\gamma \delta(x - x_0) \tag{18}$$

where δ is the Dirac δ-function and the constant V_0/x_0 is introduced to satisfy (2). Substituting (18) into (16) gives

$$p_0(x, t) = e^{tL} w_0 = e^{-tBx^\alpha} \frac{V_0}{x_0} x^\gamma \delta(x - x_0)$$
$$+ \alpha V_0 x_0^{\alpha-2} e^{-tBx^\alpha} x^\gamma H(x_0 - x) \sum_{k=1}^{\infty} \left(\frac{2tB}{\alpha}\right)^k \frac{(x_0^\alpha - x^\alpha)^{k-1}}{k!(k-1)!} \tag{19}$$

where H is the Heavyside unit step function.

Equation (19) gives the solution if $\varepsilon = 0$, that is if there is no coalescence. The pulse at x_0 is decreasing exponentially with time in favor of a continuum of smaller particle volumes.

To observe the effect of coalescence the higher order terms must be included.

$$p_1(x, t) = \int_0^t e^{(t-s)L} F_1(s)\, ds$$

where

$$f_1(x, t) = \frac{B}{2} x^\gamma \int_0^x p_0(y, t) p_0(x - y, t)\, dy - Bx^\gamma p_0(x, t) \int_0^{V_0 - x} p_0(y, t)\, dy.$$

To carry out the computation analytically is, even in this simple example, quite involved. Multiplying the terms in (19) together and integrating yields for f_1 seven very complicated terms. In the interest of brevity the terms in ψ_1 will be described qualitatively. New terms introduced by ψ_1 will be of the form

$$g_1(x, t)\delta(x - 2x_0) + g_2(x, t) H(x - x_0) H(2x_0 - x),$$

where g_1 and g_2 are continuous in both variables. The pulse at $x = 2x_0$ results from the coalescence of particles of volume x_0. The continuum has been extended out to $x = 2x_0$ by the second term. The $O(\varepsilon)$ approximation will never introduce peaks beyond $x = 2x_0$. If the $\varepsilon^2 p_2$ term is included, however, terms like $\delta(x - 3x_0)$ and $\delta(x - 4x_0)$ will be introduced and the continuum of values will be extended out to $x = 4x_0$.

The computational difficulties encountered in this relatively simple example suggest that the equations be integrated numerically. This will be left for future consideration.

Summary. A nonlinear integrodifferential equation describing the evolution with time of the volume density of an ensemble of particles is solved using perturbation theory. While most appropriate in the case $C \ll B$, or fracture dominating coalescence, the solution is

shown to be valid for all ε. Indeed, (15) implies a trade-off between the size of $\varepsilon = C/B$ and the time interval, $[0, T]$, on which the series solution is known to be valid. As ε increases, T must decrease. The question of the convergence of the solution outside $[0, T]$ for fixed ε remains open.

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