FREE BOUNDARIES IN ONE DIMENSIONAL FLOW*

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Abstract. Several problems are discussed regarding flow in a horizontal channel. The channel bed may be impervious to infiltration, or may allow a constant rate of infiltration, or may be impervious to the left of some point in the channel bed and allow a constant rate of infiltration to the right. The free boundary is the time history of the motion of a piston at which the water height $\mu(x)$ has been specified. The case $\mu(x) \equiv 0$ is the dam breaking problem. In the dam breaking problem in which the channel bed allows a constant rate of infiltration a more general form of a centered rarefaction wave is required, i.e., the characteristics are not straight lines. Two problems are formulated for channel beds that are partly impervious and partly porous. Shock formation may arise here. This possibility is exhibited in a single nonconservation equation.

1. Introduction. In this section we formulate a number of problems in channel flow involving fixed and free boundaries. In these examples the channel is horizontal, the flow is frictionless, the channel cross section is uniform and rectangular, there is no lateral inflow, but the channel bed may be porous. For the convenience of the reader we give a derivation, in Sec. 2, of the equations of channel flow under more general conditions, i.e., positive slope, friction, lateral inflow, and infiltration. The problems we formulate can also be stated in terms of one-dimensional gas flow in a cylinder of uniform cross section with porous walls if the adiabatic equation is $p = \frac{1}{2}g\rho^2$.

We formulate first a fixed boundary problem. The channel extends to infinity in both directions and has an impervious bed, i.e., the infiltration is 0. There is a piston at $x = 0$, and stationary water to the left of the piston at height h_0. The piston moves to the right according to the specified motion $x = s(t)$, where

$$0 \leqslant s'(t) \leqslant 2c_0, \quad s''(t) \geqslant 0, \quad c_0 = (gh_0)^{1/2}. \tag{1}$$

If u and h are velocity and height then

$$h_t + (uh)_x = 0, \qquad u_t + \left(\tfrac{1}{2}u^2 + gh\right)_x = 0. \tag{2}$$

There is an interface $x = r(t)$ separating moving from stationary water which moves to the left with speed c_0, i.e., $r(t) = -c_0 t$. Thus

$$h(-c_0 t, t) = h_0, \quad u(-c_0 t, t) = 0, \qquad t \geqslant 0. \tag{3}$$

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At the piston face the water particles have the velocity of the piston, so

$$u(s(t), t) = s'(t), \quad s(0) = 0, \qquad t \geqslant 0. \tag{4}$$

There are two cases:

$$\text{(a)} \quad s'(0) = 0, \qquad \text{(b)} \quad s'(0) > 0. \tag{5}$$

In the fixed boundary problem (2), (3), (4), the equations (2) are satisfied in $-c_0 t \leqslant x \leqslant s(t)$. This problem is discussed in Sec. 3.

The problem above becomes a free boundary problem if $s(t)$ is not specified but the height $\mu(x)$ at the piston face is specified, $\mu'(x) \leqslant 0$. Then (2) and (3) still apply but (4) is replaced by

$$h(s(t), t) = \mu(s(t)), \quad u(s(t), t) = s'(t), \quad s(0) = 0. \tag{6}$$

There are two cases:

$$\text{(a)} \quad \mu(0) = h_0, \qquad \text{(b)} \quad \mu(0) < h_0. \tag{7}$$

Thus (2), (3), (6) is a free boundary problem, with $x = s(t)$ the free boundary. The case $\mu(x) \equiv 0$ is the dam breaking problem discussed by Stoker [2, p. 313]. This problem is also discussed in Sec. 3. The solution in the case (7a) does not exist for all $\mu(x)$, $\mu'(x) \leqslant 0$, but there is a solution if

$$\int_0^{\varepsilon} \frac{dx}{h_0 - \mu(x)} < \infty. \tag{8}$$

The solution always exists in the case (7b).

We conside now the free boundary problem above, in the case $\mu(x) \equiv 0$, when the entire channel is porous with constant infiltration rate α(volume/area time). This is the dam breaking problem with a porous channel bed. Again there is an interface $x = r(t)$ separating moving from stationary but subsiding water, the level of the water to the left of $x = r(t)$ declining according to $h(x, t) = h_0 - \alpha t$. Thus the interface does not move with constant speed, but $r(t)$ is defined by

$$r'(t) = -[g(h_0 - \alpha t)]^{1/2}, \qquad r(0) = 0,$$

so

$$r(t) = -\frac{2}{3\lambda}\left[c_0^3 - \left(c_0^2 - \lambda t\right)^{3/2}\right], \qquad \lambda = g\alpha. \tag{9}$$

The complete formulation of the problem is

$$\begin{aligned} &h_t + (uh)_x = -\alpha, \qquad u_t + \left(\tfrac{1}{2}u^2 + gh\right)_x = 0, \\ &h(r(t), t) = h_0 - \alpha t, \quad u(r(t), t) = 0 \qquad 0 \leqslant t \leqslant t_0, \\ &h(s(t), t) = 0, \quad s(0) = 0, \qquad 0 \leqslant t \leqslant t_0, \end{aligned} \tag{10}$$

where the differential equations in (10) are satisfied between $r(t)$ and $s(t)$ and below $t_0 = h_0/\alpha$. There is no water in the channel after $t = t_0$. This problem is discussed in Sec. 4. We note that there is only one condition at the free boundary, in contrast to the two conditions exhibited in (6). This is plausible physically because $\mu(x) \equiv 0$ means that the piston motion has no effect on the flow. This is also true in the case $\alpha = 0$, so the second condition in (6) can be omitted. But there is a difference: the equation $u(s(t), t) = s'(t)$ is true when $\alpha = 0$, but not true when $\alpha > 0$. In both cases, i.e., $\mu(x) \equiv 0$, $\alpha = 0$ and $\mu(x) \equiv 0$, $\alpha > 0$, the solution is a centered rarefaction wave with center at the origin, but, while the CRW is easy to obtain in the case $\alpha = 0$ because the characteristics composing it are straight lines, this is no longer true in the case $\alpha > 0$.

In Sec. 5 we formulate two problems in which the channel bed is partly porous and partly not. In the subsidence problem we have the infiltration rate $f(x) = 0$, $x < 0$, $f(x) = \alpha$, $x > 0$, and $h(x, 0) = h_0 =$ constant, $u(x, 0) = 0$. There is a left moving interface $x = -c_0 t$, and a right moving interface $x = s(t)$, which separates moving from stationary but subsiding water. The formulation is (43), (44). The main question here is the possibility of shock formation, which may occur before time $t_0 = h_0/\alpha$, when there is no water to the right of $s(t_0)$. After t_0 the formulation is (43), (45), (46). Shock formation may occur after t_0. The second problem is the dam breaking problem, i.e., $\mu(x) \equiv 0$, with $f(x) = 0$, $x < x_0$, $f(x) = \alpha$, $x > x_0$; the formulation is (47). The free boundary is the time history of the water edge $x = s(t)$. Shock formation may also occur in this problem.

In Sec. 6 we consider the initial problem value problem

$$u_t + uu_x = \phi(x), \qquad u(x, 0) = u_0(x), \tag{11}$$

for the purpose of determining the effect of $\phi(x)$ and $u_0(x)$ on shock formation. If $\phi(x) \not\equiv 0$ the differential equation in (11) is a nonconservation law, so there is an analogy to the problems of Sec. 5. If $u_0(x) =$ constant and $\phi(x)$ has a point of decrease then shock formation does occur in (11). This lends some support to the possibility of shocks in the problems of Sec. 5.

2. The equations of channel flow. We derive the equations of flow in a wide channel of uniform rectangular cross section. This is done in Stoker [2, pp. 452–455], except for infiltration, which we include in this discussion. Let ρ be the density of water, b the channel width, θ the constant angle which the channel makes with the horizontal (the channel slopes down to the right), $h(x, t)$ and $u(x, t)$ the depth and velocity, $S_f(u, h)$ the friction slope ($\rho g b S_f h \Delta x$ is the resisting force to a slab of the thickness Δx), and $q(x, t)$ and $f(x, t)$ the lateral inflow (rainfall) and infiltration rates in volume per unit area per unit time. Let $s_1(t)$ and $s_2(t)$, $s_2(t) > s_1(t)$, be plane sections moving with the water. The mass between the sections is

$$m(t) = \rho b \int_{s_1}^{s_2} h\, dx.$$

Then, writing $h_k = h(s_k(t), t)$ and $u_k = u(s_k(t), t)$, $k = 1, 2$,

$$m'(t) = \rho b \left[\int_{s_1}^{s_2} h_t\, dx + (uh)_2 - (uh)_1 \right]. \tag{12}$$

We have also

$$m'(t) = \rho b \int_{s_1}^{s_2} (q - f)\, dx. \tag{13}$$

Equating (12) and (13), dividing by $s_2 - s_1$, and letting $s_1 \uparrow x$ and $s_2 \downarrow x$ we get the continuity equation

$$h_t + (uh)_x = q - f. \tag{14}$$

The momentum $M(t)$ between the sections is

$$M(t) = \rho b \int_{s_1}^{s_2} uh\, dx.$$

Then

$$M'(t) = \rho b \left[\int_{s_1}^{s_2} (uh)_t\, dx + (u^2 h)_2 - (u^2 h)_1 \right]. \tag{15}$$

$M'(t)$ is the sum of three terms: the rate of momentum loss through infiltration (there is no gain or loss of momentum in the x direction through lateral inflow), the body force, and the net pressure force at the two sections. The first of these terms is

$$\rho b \int_{s_1}^{s_2} fu\, dx. \tag{16}$$

The body force is

$$\rho g b \sin\theta \int_{s_1}^{s_2} h\, dx - \rho g b \int_{s_1}^{s_2} h S_f\, dx. \tag{17}$$

The pressure force at the section $x = s_k(t)$ is

$$\pm \rho g b \int_0^{h_k} (h_k - \eta)\, d\eta,$$

where $+$ goes with $k = 1$ and $-$ with $k = 2$. Thus the net pressure force is

$$-\tfrac{1}{2} \rho g b \left(h_2^2 - h_1^2 \right). \tag{18}$$

From (15)–(18) we get, on dividing by $s_2 - s_1$, letting $s_1 \uparrow x$ and $s_2 \downarrow x$, and writing $S = \sin\theta$,

$$(uh)_t + (u^2 h)_x = -fu + gh(S - S_f) - \left(\tfrac{1}{2} g h^2\right)_x. \tag{19}$$

If $v = uh$ then (19) becomes

$$v_t + \left(\frac{v^2}{h} + \frac{1}{2} g h^2 \right)_x = -\frac{fv}{h} + gh(S - S_f). \tag{20}$$

Thus (14), with $v = uh$, and (20) can be taken as the continuity and momentum equations for channel flow. But, using (14) in (19), we get

$$u_t + \left(\frac{1}{2} u^2 + gh \right)_x = g(S - S_f) - \frac{qu}{h}, \tag{21}$$

so that we may also use (14) and (21) as the equations of channel flow. In the discussion of shock formation we need to use (14), with $v = uh$, and (20) because (20), rather than (21), is the differential expression of momentum conservation.

3. The fixed and free boundary problems with zero infiltration. If we introduce $c = (gh)^{1/2}$ in (2) we get

$$(u + 2c)_t + (u + c)(u + 2c)_x = 0, \qquad (22)$$
$$(u - 2c)_t + (u - c)(u - 2c)_x = 0.$$

Thus the Riemann invariants $u + 2c$ and $u - 2c$ are constant, respectively, along the characteristics C_1 and C_2, where

$$C_1\colon \ \frac{dx}{dt} = u + c, \qquad C_2\colon \ \frac{dx}{dt} = u - c. \qquad (23)$$

The characteristics originating on $x = -c_0 t$ carry the constant value $u + 2c = 2c_0$ since $u = 0$ and $c = c_0$ on $x = -c_0 t$. Therefore if the C_1 characteristics cover the entire domain D between $x = -c_0 t$ and $x = s(t)$ then $u + 2c = 2c_0$ in D. Since $u - 2c$ is constant on a C_2 characteristic, both u and c are constant on a C_2 characteristic, so each C_2 characteristic is a straight line. The value of dx/dt on the C_2 characteristic originating at the point $(s(t), t)$ is

$$u(s(t), t) - c(s(t), t) = \tfrac{3}{2}s'(t) - c_0.$$

We consider now case (5a), $s'(0) = 0$. In this case the C_2 characteristic at the origin coincides with $x = -c_0 t$. Let (x, t) be any point in D, and let the C_2 characteristic through (x, t) intersect $x = s(t)$ at $t = T$. Then

$$\frac{x - s(T)}{t - T} = \frac{3}{2}s'(T) - c_0. \qquad (24)$$

The two functions of T, x and t fixed, $0 \leqslant T \leqslant t$,

$$y = \frac{x - s(T)}{t - T}, \qquad y = \frac{3}{2}s'(T) - c_0, \qquad (25)$$

have the following properties: the second is a nondecreasing function (because $s''(t) \geqslant 0$) which has the value $-c_0$ at $T = 0$, and the first, which is dx/dt on the segment joining (x, t) to $(s(T), T)$, is a decreasing function which has the value x/t at $T = 0$ and goes to $-\infty$ as $T \to t$. Thus, since $x/t > -c_0$, (25) has a unique intersection $T(x, t)$ so (24) has a unique solution. But u is constant along the C_2 characteristic through (x, t) so

$$u(x, t) = u(s(T), T) = s'(T(x, t)). \qquad (26)$$

From $u + 2c = 2c_0$ in D we get

$$c(x, t) = c_0 - \tfrac{1}{2}s'(T(x, t)). \qquad (27)$$

The right side of (27) is $\geqslant 0$ because $s'(t) \leqslant 2c_0$. The solution of (2), (3), (4) in the case $s'(0) = 0$ is given by (26) and (27).

In the case (5b), $s'(0) > 0$, the value of dx/dt along C_2 characteristics originating at $(s(t), t)$ goes to $b = \tfrac{3}{2}s'(0) - c_0 > -c_0$ as $t \to 0$. In the region between $x = bt$ and $x = s(t)$ the discussion above applies, so u and c are defined by (26) and (27). In the region between $x = -c_0 t$ and $x = bt$ there is the CRW

$$u = \frac{2c_0}{3} + \frac{2x}{3t}, \qquad c = \frac{2c_0}{3} - \frac{x}{3t}, \qquad (28)$$

which coincides with (26) and (27) on the common boundary $x = bt$.

In the special case $s(t) = at$, $a < 2c_0$, we have $b = \frac{3}{2}a - c_0$. In the region between $x = bt$ and $x = at$ there is the constant regime $u = a$, $c = c_0 - \frac{1}{2}a$, and, in the region between $x = -c_0 t$ and $x = bt$, the CRW (28). To the left of $x = -c_0 t$ there is the constant regime $u = 0$, $c = c_0$ so the CRW separates two constant regimes. When $a = 2c_0$ then $b = 2c_0$, and there is only the CRW between $x = -c_0 t$ and $x = 2c_0 t$. On $x = 2c_0 t$, $c = 0$, so $h = 0$. This is the solution to the dam breaking problem.

In the free boundary problem (2), (3), (6) we assume that $\mu'(x) < 0$ for small x. We consider first the case (7a), $\mu(0) = h_0$. Since $u + 2c = 2c_0$, $u(s(t), t) = s'(t)$, and $c(s(t), t) = [g\mu(s(t))]^{1/2}$ we get

$$s'(t) + 2[g\mu(s(t))]^{1/2} = 2(gh_0)^{1/2}, \qquad s(0) = 0. \tag{29}$$

The initial value problem (29) has only the solution $s(t) \equiv 0$ if $[\mu(x)]^{1/2}$ has a continuous derivative in the neighborhood of $x = 0$, but if (8) applies then

$$t = \zeta(x) = \frac{1}{2g^{1/2}} \int_0^x \frac{d\zeta}{h_0^{1/2} - [\mu(\zeta)]^{1/2}} \tag{30}$$

is the solution of (29); here $t = \zeta(x)$ is the function inverse to $x = s(t)$. The integral in (30) is convergent since it is less than

$$2h_0^{1/2} \int_0^x \frac{d\zeta}{h_0 - \mu(\zeta)} < \infty.$$

Since $\zeta'(0) = +\infty$, $s'(0) = 0$. Also

$$\zeta'(x) \geqslant \tfrac{1}{2}(gh_0)^{-1/2} = (2c_0)^{-1}$$

so $0 \leqslant s'(t) \leqslant 2c_0$. When $\zeta'(x) = (2c_0)^{-1}$ (so $s'(t) = 2c_0$), $\mu(x) = 0$. From

$$s''(t) = -g^{1/2}[\mu(s(t))]^{-1/2}\mu'(s(t))s'(t)$$

we get $s''(t) \geqslant 0$. With this $s(t)$ we then determine u and c by (26) and (27).

In the case (7b), $s'(0) > 0$. We define $s(t)$ by (30) (condition (8) is not needed) and determine u and c as in the fixed boundary problem.

When $\mu(x) \equiv 0$ the second condition in (6) can be omitted, i.e., the free boundary problem can be formulated as (2), (3), and $h(s(t), t) = 0$. This problem has the solution $s(t) = 2c_0 t$, with u and $h = g^{-1}c^2$ given by the CRW (28), so $s'(t) = 2c_0$ and $u(s(t), t) = 2c_0$. Thus the second condition in (6) is true anyway, even though it can be omitted. The solution of (2), (3), and $h(s(t), t) = 0$ is unique if we impose the requirement that $h(x, t) > 0$ when $x < s(t)$. For if $s(t)$, u, h is a solution and P is a point of $x = s(t)$ not on $x = 2c_0 t$ then, on the C_2 characteristic issuing from P into $-c_0 t < x < s(t)$, $h = 0$. Thus there is no such P, so $s(t) = 2c_0 t$ and u and c are given by (28).

4. The dam breaking problem with constant infiltration. This is problem (10). The problem has the equivalent formulation ($r(t)$ is given by (9))

$$\begin{gathered}(u + 2c)_t + (u + c)(u + 2c)_x = -\lambda/c,\\ (u - 2c)_t + (u - c)(u - 2c)_x = \lambda/c,\\ c(r(t), t) = (c_0^2 - \lambda t)^{1/2}, \quad u(r(t), t) = 0, \qquad 0 \leqslant t \leqslant t_0,\\ c(s(t), t) = 0, \quad s(0) = 0, \qquad 0 \leqslant t \leqslant t_0.\end{gathered} \tag{31}$$

The characteristics are given by (23). The curve $x = r(t)$ is the C_2 characteristic passing through the origin. Analogous to the case $\lambda = 0$ we look for a CRW with center at the origin; this is a solution of (31) such that all C_2 characteristics pass through the origin. To this end we introduce characteristic coordinates. Let $\alpha(x, t) = \alpha$ and $\beta(x, t) = \beta$ be, respectively, the family of C_1 and C_2 characteristics. Along a C_1 characteristic α is constant and along a C_2 characteristic β is constant. We choose the parameters α and β as follows: α is the t coordinate of the intersection of the C_1 characteristic with $x = r(t)$, and β is the limit of $u(x, t)$ on a C_2 characteristic as $(x, t) \to (0,0)$. Thus $\beta(x, t) = 0$ is the C_2 characteristic $x = r(t)$. We assume that through each point to the right of $x = r(t)$ there is exactly one C_1 and one C_2 characteristic. Thus the mapping of the domain to the right of $x = r(t)$ onto the first quadrant of the (α, β) plane is one to one, except for $x = 0$, $t = 0$, which maps onto the β axis. The C_2 characteristic $x = r(t)$ maps onto the α axis, the other C_2 characteristics map onto $\beta =$ constant, and the C_1 characteristics map onto $\alpha =$ constant. In the (α, β) plane we get, after some calculation,

$$\begin{aligned} &\text{(a)}\quad u_\beta + 2c_\beta = -\lambda t_\beta / c, \qquad &&\text{(c)}\quad x_\beta = (u + c)t_\beta, \\ &\text{(b)}\quad u_\alpha - 2c_\alpha = \lambda t_\alpha / c, &&\text{(d)}\quad x_\alpha = (u - c)t_\alpha. \end{aligned} \tag{32}$$

The boundary conditions on the α axis are given by (33) below; these conditions are a consequence of (1) the α axis is the map of $x = r(t)$, and (2) the choice of the parameter α as the t coordinate of the intersection of the C_1 characteristic with $x = r(t)$:

$$\begin{aligned} &u(\alpha,0) = 0, \qquad &&c(\alpha,0) = \left(c_0^2 - \lambda\alpha\right)^{1/2}, \\ &x(\alpha,0) = r(\alpha), &&t(\alpha,0) = \alpha. \end{aligned} \tag{33}$$

On the β axis the boundary conditions are given by (34) below. The conditions $x(0, \beta) = 0$ and $t(0, \beta) = 0$ are clear (because the β axis maps onto the origin), and the condition $u(0, \beta)$ follows from the choice of the parameter β as the limit of $u(x, t)$ as (x, t) goes to the origin along a C_2 characteristic. If we set $\alpha = 0$ in (32a) we get, since $t_\beta(0, \beta) = 0$,

$$u_\beta(0, \beta) + 2c_\beta(0, \beta) = 0,$$

so

$$u(0, \beta) + 2c(0, \beta) = u(0,0) + 2c(0,0) = 2c_0.$$

This implies the condition on $c(0, \beta)$ in (34):

$$u(0, \beta) = \beta, \quad c(0, \beta) = c_0 - \tfrac{1}{2}\beta, \quad x(0, \beta) = 0, \quad t(0, \beta) = 0. \tag{34}$$

We note that $c(0, 2c_0) = 0$ and $c(\lambda^{-1}c_0^2, 0) = 0$. The free boundary in the (α, β) plane is therefore the curve $c(\alpha, \beta) = 0$ joining the two points $(0, 2c_0)$ and $(\lambda^{-1}c_0^2, 0)$. Thus the solution of (31) is obtained by solving (32), (33), (34) in the domain bounded by $\alpha = 0$, $\beta = 0$, and $c(\alpha, \beta) = 0$. The free boundary $x = s(t)$ in the (x, t) plane is defined implicitly by $c(\alpha(x, t), \beta(x, t)) = 0$.

In the case $\lambda = 0$ the problem (32), (33), (34) is easily solved. Indeed we get, from (32a) and (32b) and the boundary conditions (33) and (34), $u + 2c = 2c_0$ and $u - 2c = 2\beta - 2c_0$, so

$$u(\alpha, \beta) = \beta, \qquad c(\alpha, \beta) = c_0 - \tfrac{1}{2}\beta. \tag{35}$$

From (32d) and (35) we get $x = (\frac{3}{2}\beta - c_0)t$ so, from (32c),

$$2(2c_0 - \beta)t_\beta - 3t = 0. \tag{36}$$

Thus, from (36) and (33), we get

$$t(\alpha, \beta) = \alpha\left(\frac{2c_0}{2c_0 - \beta}\right)^{3/2}, \qquad x(\alpha, \beta) = (\tfrac{3}{2}\beta - c_0)t(\alpha, \beta). \tag{37}$$

The solution is valid in the strip $\{\alpha \geqslant 0, 0 \leqslant \beta \leqslant 2c_0\}$. From (37) we get $\beta = (2/3)(c_0 + x/t)$ which, inserted in (35) gives the CRW (28). The free boundary, which is the locus $c(\alpha, \beta) = 0$, is $\beta = 2c_0$, or, in the (x, t) plane, $x = 2c_0 t$.

When $\lambda > 0$ an explicit solution is not available; an existence theorem for (32), (33), (34) is required. We provide, instead, a plausibility argument which is, at the same time, a (crude) computational procedure. Let k be small and let $\alpha = ik$, $\beta = jk$, $i, j = 0, 1, 2, \cdots$ be two sets of equidistant parallel lines in the first quadrant of the (α, β) plane. Let $u_{ij}, c_{ij}, x_{ij}, t_{ij}$ be the values of u, c, x, and t at $\alpha = ik$, $\beta = jk$. Then a discretized version of (32) is

$$\begin{aligned}
&\text{(a)}\quad c_{i(j-1)}u_{ij} + 2c_{i(j-1)}c_{ij} + \lambda t_{ij} = c_{i(j-1)}u_{i(j-1)} + 2c_{i(j-1)}^2 + \lambda t_{i(j-1)},\\
&\text{(b)}\quad c_{(i-1)j}u_{ij} - 2c_{(i-1)j}c_{ij} - \lambda t_{ij} = c_{(i-1)j}u_{(i-1)j} - 2c_{(i-1)j}^2 - \lambda t_{(i-1)j},\\
&\text{(c)}\quad (u_{i(j-1)} + c_{i(j-1)})t_{ij} - x_{ij} = (u_{i(j-1)} + c_{i(j-1)})t_{i(j-1)} - x_{i(j-1)},\\
&\text{(d)}\quad (u_{(i-1)j} - c_{(i-1)j})t_{ij} - x_{ij} = (u_{(i-1)j} - c_{(i-1)j})t_{(i-1)j} - x_{(i-1)j}.
\end{aligned} \tag{38}$$

In (38) we have replaced the non-differential terms in (32) by their values at the lattice point to the left or below the point $\alpha = ik$, $\beta = jk$. The system (38) is linear in the four unknowns $u_{ij}, c_{ij}, t_{ij}, x_{ij}$ and has the determinant

$$4c_{i(j-1)}c_{(i-1)j}[u_{i(j-1)} - u_{(i-1)j} + c_{(i-1)j} + c_{i(j-1)}]. \tag{39}$$

Thus if $c_{i(j-1)} \neq 0$, $c_{(i-1)j} \neq 0$, and the bracketed term in (39) is not 0, equations (38) determine u_{ij}, c_{ij}, t_{ij}, and x_{ij} in terms of the values of u, c, t and x at $((i-1)k, jk)$ and $(ik, (j-1)k)$. The linear system (38) is supplemented by the initial specifications, derived from (33) and (34),

$$\begin{aligned}
&u_{i0} = 0, \quad c_{i0} = (c_0^2 - \lambda ik)^{1/2}, \quad x_{i0} = r(ik), \quad t_{i0} = ik,\\
&u_{0j} = jk, \quad c_{0j} = c_0 - \tfrac{1}{2}jk, \quad x_{0j} = 0, \quad t_{0j} = 0,
\end{aligned} \tag{40}$$

where $i, j = 0, 1, 2, \ldots$. Although the linear system (38) does not depend on k, k appears in the solution through the initial specifications (40). We note that, in the bracketed term in (39), $u_{i(j-1)} - u_{(i-1)j}$ is small (i.e. of order k) while $c_{(i-1)j} + c_{i(j-1)}$ is not small until c gets close to 0. Since $u_{0j}, c_{0j}, r_{0j}, x_{0j}$ and $u_{i0}, c_{i0}, t_{i0}, x_{i0}$ are known from (40) we can use (38) to calculate u_{ij}, c_{ij}, t_{ij} and x_{ij} up to, very nearly, the locus $c(\alpha, \beta) = 0$.

The free boundary $x = s(t)$, which is the time history of the water edge, begins at the origin and terminates at the point $x_0 = -2c_0^3/3\lambda$, $t_0 = c_0^2/\lambda = h_0/\alpha$. At time t_0 there is no water in the channel. The graph of $x = s(t)$ is the locus of points on the C_2 characteristics on which $c = 0$. Since u has the least upper bound $2c_0$ this implies that the curve $x = s(t)$ enters the first quadrant with slope $2c_0$, advances to some maximum value of x, and then retreats to x_0 at time t_0. Since $h = 0$ on $x = s(t)$ we get

$$h_x(s(t), t)s'(t) + h_t(s(t), t) = 0,$$

and, from the first equation of (10),

$$u(s(t), t)h_x(s(t), t) + h_t(s(t), t) = -\alpha,$$

so

$$h_x(s(t), t)[s'(t) - u(s(t), t)] = \alpha. \tag{41}$$

From (41) we get, since $h = 0$ on $x = s(t)$,

$$h_x(s(t), t) < 0$$

and

$$u(s(t), t) > s'(t). \tag{42}$$

Thus, in the case $\alpha > 0$, the water at the edge makes a positive angle with the horizontal; in the case $\alpha = 0$ that angle is 0. It is clear from (42) that u is positive on that part of $x = s(t)$ for which $s'(t) \geqslant 0$ and remains positive on $x = s(t)$ somewhat beyond the turning point. Indeed it is plausible that $u(x, t) > 0$ for all x and t except for points lying on $x = r(t)$. This is easily proved if we make the equally plausible hypothesis that $h_x(x, t) < 0$ for all x and t. For suppose (x_1, t_1) is in $D = \{r(t) < x < s(t)\}$ and is a point with maximum x for which $u = 0$. If (x_1, t_1) is an isolated point then both u_t and u_x are 0 at that point and, from the second equation of (10), we get $h_x(x_1, t_1) = 0$. If (x_1, t_1) is not an isolated point and $x = \sigma(t)$ is the curve through that point such that $u(\sigma(t), t) = 0$, then, on differentiating this equation with respect to t and noting that $\sigma'(t_1) = 0$, we get $u_t(x_1, t_1) = 0$, so again, from the second equation of (10), we get $h_x(x_1, t_1) = 0$. There remains the possibility that (x_1, t_1) lies on the decreasing part of $x = s(t)$, but this is ruled out by the second equation of (10), which shows that $u_t(x_1, t_1) > 0$. This implies that $u < 0$ on $x = x_1$, $t_2 < t < t_1$ for some t_2, so there are points with larger x at which $u = 0$.

5. Problems involving channel beds that are partly impervious and partly porous. We formulate first a subsidence problem. The channel extends to infinity in both directions, with the infiltration $f(x)$, in volume per unit area per unit time, having the value 0 for $x < 0$ and the positive constant α for $x > 0$. We assume that u and h are initially 0 and h_0. When $0 \leqslant t < t_0 = h_0/\alpha$ there are two interfaces, $x = r(t)$ and $x = s(t)$, separating moving from stationary water. The interface $r(t)$ is given by $r(t) = -c_0 t$, while it is reasonable to suppose the interface $s(t)$ is defined by

$$s'(t) = [g(h_0 - \alpha t)]^{1/2}, \qquad s(0) = 0.$$

The equations are

$$h_t + (uh)_x = -f, \qquad u_t + \left(\tfrac{1}{2}u^2 + gh\right)_x = 0, \tag{43}$$

with boundary conditions

$$h(r(t), t) = h_0, \quad u(r(t), t) = 0, \quad h(s(t), t) = h_0 - \alpha t, \quad u(s(t), t) = 0. \tag{44}$$

The main point of interest in this problem is the possibility of shock formation. There are several possibilities:

(a) there is no shock formation up to time t_0,

(b) a shock $\xi(t)$ forms at time t_1, $0 < t_1 \leqslant t_0$, and $\xi(t) < s(t)$, $t_1 \leqslant t \leqslant t_0$,

(c) a shock forms at time t_1, $\xi(t_1) \leqslant s(t_1)$, but, at some time t^*, $t_1 \leqslant t^* \leqslant t_0$, $\xi(t^*) = s(t^*)$.

After time t_0 the time history of the water edge $x = s(t)$ becomes a free boundary with (44), in case (a) above, replaced by

$$h(r(t),t) = h_0, \quad u(r(t),t) = 0, \quad h(s(t),t) = 0, \quad t \geqslant t_0, \tag{45}$$

and the initial conditions

$$h(x,t_0) = h_0(x), \quad u(x,t_0) = u_0(x), \quad -c_0 t_0 \leqslant x \leqslant 2c_0^3/3\lambda, \tag{46}$$

where $h_0(x)$ and $u_0(x)$ are known from the solution of (43) and (44) at $t = t_0$. It is possible that shock formation occurs after t_0.

The dam breaking problem, which we have discussed in the case of zero infiltration or constant infiltration, can also be formulated for the case $f(x) = 0$, $x < x_0$, $f(x) = \alpha$, $x > x_0$, where x_0 may be positive, negative, or 0. In the case $x_0 \geqslant 0$ the formulation is

$$\begin{aligned} &h_t + (uh)_x = -f, \qquad u_t + \left(\tfrac{1}{2}u^2 + gh\right)_x = 0, \\ &h(-c_0 t, t) = h_0, \quad u(-c_0 t, t) = 0 \qquad t \geqslant 0, \\ &h(s(t),t) = 0, \quad s(0) = 0, \qquad t \geqslant 0. \end{aligned} \tag{47}$$

The case $x_0 < 0$ is more complicated. If $x_0 < 0$ and $|x_0|$ is large then the problem has two distinct parts: the first part is the problem discussed in section 4 and the second part is the subsidence problem formulated above. But if $x_0 < 0$ and $|x_0|$ is small then, in addition to the left moving interface $x = r(t)$ and right moving interface $x = s(t)$, there is an interface $x = \sigma(t)$ between $x = x_0$ and $x = r(t)$ which is moving to the right and which coincides with $x = r(t)$ at $x = x_0/2$ at time $t = t_0 - \lambda^{-1}(\frac{3}{4}\lambda x_0 + c_0^3)^{2/3}$. The first case occurs when $x_0 < -4c_0^3/3\lambda$ and the second when $0 > x_0 > -4c_0^3/3\lambda$. Shock formation may occur in all three of the cases $x_0 < 0$, $x_0 > 0$, $x_0 = 0$.

6. The equation $u_t + uu_x = \phi(x)$. The continuity equations in (43) and (47) are non-conservation equations because of the right sides, $-f(x)$, which are step functions with a single decreasing step. To determine the posssible effect of such a term we consider the initial value problem (11) for a single equation.

In the case $\phi(x) \equiv 0$ the equation is a conservation law. Then if $u_0(x)$ is nondecreasing and is continuous and differentiable, except possibly at isolated points, (11) has a continuous solution in the half plane $t \geqslant 0$. If $u_0(x)$ has a point of decrease then a continuous solution in the half plane does not exist, but there is a generalized solution [1].

Suppose $\phi(x) \not\equiv 0$. If $\phi(x)$ is nondecreasing and if $u_0(x)$ is continuous and $u_0'(x) \geqslant 0$, except at isolated points, then, if u_0 and ϕ are bounded, (11) has a continuous solution in the half plane $t \geqslant 0$. But if, for some x_0, $u_0'(x_0) < 0$, and $\phi'(x) \leqslant 0$, then u_x becomes infinite for finite t on the characteristic originating at $x = x_0$. This can be proved, using the argument in [1], as follows: let $q(t) = u_x$ on that characteristic. Then

$$q' = u_{xt} + uu_{xx}.$$

Differentiating (11) with respect to x we get

$$u_{xt} + uu_{xx} + (u_x)^2 = \phi'(x),$$

so on the characteristic we have

$$q' + q^2 = \phi'(x). \tag{48}$$

Dividing by q^2, integrating, and using $\phi'(x) \leqslant 0$, we get

$$-q(t)^{-1} + q(0)^{-1} + t \leqslant 0,$$

or

$$-q(t) \geqslant \left[(-u_0'(x_0))^{-1} - t\right]^{-1}. \tag{49}$$

Since $u_0'(x_0) < 0$, (49) implies that $q = u_x$ goes to $-\infty$ for finite t on the characteristic originating at $x = x_0$. The assertion remains true if $u_0'(x_0) = 0$, $\phi'(x_0) < 0$, and $\phi'(x) \leqslant 0$, which is the situation if $u_0(x) =$ constant and $\phi(x)$ has the point of decrease $x = x_0$. To prove this we get, from (48), on dividing by $q^2 + 1$ and integrating,

$$\tan^{-1} q = \int_0^t \frac{-q^2 + \phi'(x)}{q^2 + 1}\,dt < -\int_0^t \frac{q^2}{q^2 + 1}\,dt,$$

so

$$-q > \tan \int_0^t \frac{q^2}{q^2 + 1}\,dt. \tag{50}$$

Since $-q$ is an increasing function of t there are the following possibilities:

1. $-q(t) \to a$, $t \to \infty$, $0 < a \leqslant \infty$,
2. $-q(t) \to \infty$, $t \to t_0$, $t_0 < \infty$.

The first possibility is ruled out because the right side of (50) is $+\infty$ for some finite t while the left side is finite for that t. Thus we have the second possibility, which implies shock formation. The following example exhibits this possibility: Let $\alpha > 0$, $\beta > 0$, $u_0(x) = \beta$, $\phi(x) = 0$, $x < 0$, $\phi(x) = -\alpha$, $x > 0$. The shock $\xi(t)$ is, in $\beta/\alpha \leqslant t \leqslant 3\beta/\alpha$,

$$(8\alpha)^{-1}(3\beta - \alpha t)(\beta + \alpha t), \tag{51}$$

and in $t \geqslant 3\beta/\alpha$,

$$(16\alpha)^{-1}(\alpha t - 3\beta)(-3\alpha t + \beta). \tag{52}$$

The shock originates at $x = \beta^2/2\alpha$, $t = \beta/\alpha$, so u is continuous below $t = \beta/\alpha$. In the first quadrant $u = -\alpha t + \beta$, except in the region between the t axis, the shock (51), and the parabola $x = -\frac{1}{2}\alpha t^2 + \beta t$ joining the origin to $(\beta^2/2\alpha, \beta/\alpha)$, where $u = (\beta^2 - 2\alpha x)^{1/2}$. In the second quadrant $u = \beta$ below (52), and

$$u = \tfrac{1}{2}(\beta - \alpha t) - \tfrac{1}{2}\left[(\beta - \alpha t)^2 + 4\alpha x\right]^{1/2}$$

above (52).

The discussion above supports the possibility of shock formation in (43), (44) and in (47). It is possible that an extension of the argument above regarding u_x might be obtained, but that argument is insufficient to specify the shock, if it exists.

References

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[2] J. J. Stoker, *Water waves*, Interscience, New York, 1957