QUARTERLY OF APPLIED MATHEMATICS
October 1983

ON EXISTENCE OF SOLUTION OF THE DIRICHLET PROBLEM OF FOURTH ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS*

By

S. BALASUNDARAM AND P. K. BHATTACHARYYA

Indian Institute of Technology, New Delhi

Abstract. Sufficient conditions for the existence and uniqueness of the solution of the Dirichlet problem of fourth order elliptic partial differential equations with variable coefficients have been derived. In a number of examples of practical interest, the easy applicability of these results has been shown.

1. Introduction. A large class of problems of mathematical physics lead to fourth order elliptic partial differential equations with variable coefficients, the equations of the bending problems of elastic—isotropic, orthotropic and anisotropic—plates with variable (or constant) thickness being very particular cases of the general problem considered here. Hence, it is necessary to find sufficient conditions for the existence and uniqueness of the solution of the Dirichlet problem of these equations in weak form. This paper contains new results in this direction.

2. Notations. Let Ω be a domain in $\mathbb{R}^2$ with piecewise smooth boundary Γ such that $\overline{\Omega} = \Omega \cup \Gamma$. Let $H^m(\Omega)$ be the usual Sobolev space [1, 4] of integral order $m \geqslant 0$ equipped with inner product $\langle \cdot, \cdot \rangle_{m,\Omega}$, norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$ such that $H^0(\Omega) \equiv L^2(\Omega)$,

$$H_0^2(\Omega) = \left\{v\colon v \in H^2(\Omega), \gamma_0 v = v\,|_\Gamma = 0, \gamma_1 v = (\partial v/\partial n)\,|_\Gamma = 0\right\} \equiv \overline{D(\Omega)}, \quad (2.1)$$

where $\gamma_k\colon H^2(\Omega) \to H^{2-k-1/2}(\Gamma)$ are trace operators with $k = 0, 1$; $H^{3/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ being the fractional order Sobolev spaces on Γ [1, 4]; $D(\Omega)$ is the space of test functions on Ω [5].

3. The variational problem. To the Dirichlet problem (P) defined by: For given $f \in L^2(\Omega)$, find u such that

$$\Lambda u = f \quad \text{in } \Omega, \qquad u\,|_\Gamma = 0, \qquad (\partial u/\partial n)\,|_\Gamma = 0, \qquad (3.1)$$

* Received September 14, 1982.

where

$$(\Lambda u)(x) \equiv \frac{\partial^2}{\partial x_k \partial x_l}\left(a_{ijkl}\frac{\partial^2 u}{\partial x_i \partial x_j}\right)(x)$$
$$\equiv (a_{ijkl}u,_{ij})_{,kl}(x) \quad \text{for } x \in \Omega \tag{3.2}$$

(*in* (3.2) *and also in the sequel, the Einstein's summation convention has been followed*), we associate the Galerkin Variational Problem (P_G) defined by: Find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = l(v) \quad \forall v \in H_0^2(\Omega), \tag{3.3}$$

where the continuous, symmetric bilinear form $a(\cdot, \cdot)$ and the continuous linear form $l(\cdot)$ are defined by: $\forall v, w \in H_0^2(\Omega)$,

$$a(v, w) = \langle \Lambda v, w \rangle_{0,\Omega} = \int_\Omega a_{ijkl}v,_{ij}w,_{kl}\, d\Omega = a(w, v), \tag{3.4}$$

$$l(v) = \langle f, v \rangle_{0,\Omega} = \int_\Omega fv\, d\Omega \quad \forall v \in H_0^2(\Omega);$$

the coefficients a_{ijkl} satisfy the following conditions: $\forall i, j, k, l = 1, 2,$

$$a_{ijkl} \in C^0(\overline{\Omega}); \qquad a_{ijkl}(x) \geqslant 0, \qquad a_{ijkl}(x) = a_{klij}(x) \quad \forall x \in \overline{\Omega}. \tag{A1}$$

But without loss of generality, we can always assume that $\forall i, j, k, l = 1, 2,$

$$a_{ijkl}(x) = a_{klij}(x) = a_{ijlk}(x) = a_{jilk}(x) \quad \forall x \in \overline{\Omega} \tag{A2}$$

since if, for example, $a_{ijkl} \neq a_{jikl}$ or $a_{ijkl} \neq a_{ijlk}$ (see Case (I) in Sec. 4.1) for some $i, j, k, l = 1, 2$, we can always define $\forall i, j, k, l = 1, 2,$

$$\bar{a}_{ijkl} = (a_{ijkl} + a_{jikl} + a_{jilk} + a_{ijlk})/4 \tag{3.5}$$

such that $\forall i, j, k, l = 1, 2,$

$$\bar{a}_{ijkl}(x) = \bar{a}_{klij}(x) = \bar{a}_{jilk}(x) = \bar{a}_{jikl}(x) \quad \forall x \in \overline{\Omega}$$

and $\forall v, w \in H_0^2(\Omega)$,

$$a_{ijkl}v,_{ij}w,_{kl} = \bar{a}_{ijkl}v,_{ij}w,_{kl}, \qquad a(v, w) = a(w, v).$$

Now, we prove the main theorem on the $H_0^2(\Omega)$-ellipticity of a $(\cdot, \cdot)$.

THEOREM (3.1). If the coefficients a_{ijkl} satisfy (A1)–(A2) and

$$\begin{aligned} &\inf_{x\in\overline{\Omega}} (a_{1111} - a_{1112} - a_{1122})(x) > 0,\\ &\inf_{x\in\overline{\Omega}} (a_{1212} - a_{1211} - a_{1222})(x) > 0,\\ &\inf_{x\in\overline{\Omega}} (a_{2222} - a_{2211} - a_{2212})(x) > 0, \end{aligned} \tag{3.6}$$

then $a(\cdot, \cdot)$ defined in (3.4) is $H_0^2(\Omega)$-elliptic.

Proof. From (A2), we have $\forall x \in \overline{\Omega}$, $\forall \xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) \in \mathbb{R}^4$ with $\xi_{21} = \xi_{12}$,

$$\begin{aligned} a_{ijkl}(x)\xi_{ij}\xi_{kl} = {} & \big(a_{1111}(x)\xi_{11}^2 + 2a_{1122}(x)\xi_{11}\xi_{22} + a_{2222}(x)\xi_{22}^2\big)\\ & + 4\big(a_{1211}(x)\xi_{12}\xi_{11} + a_{1212}(x)\xi_{12}^2 + a_{2212}(x)\xi_{22}\xi_{12}\big). \end{aligned} \tag{3.7}$$

$\forall x \in \bar{\Omega}, \forall \xi_{11}, \xi_{22} \in \mathbb{R}$,

$$\begin{aligned}&\left(a_{1111}(x)\xi_{11}^2 + 2a_{1122}(x)\xi_{11}\xi_{22} + a_{2222}(x)\xi_{22}^2\right)\\ &\qquad - \left(a_{1111}(x) + a_{2222}(x) - a_{1122}(x)\right)\left(\xi_{11}^2 + \xi_{22}^2\right)\\ &\qquad + \left(a_{1111}(x)\xi_{22}^2 + a_{2222}(x)\xi_{11}^2\right)\\ &= a_{1122}(x)(\xi_{11} + \xi_{22})^2 \geqslant 0.\end{aligned}$$

Therefore, $\forall x \in \bar{\Omega}, \forall \xi_{11}, \xi_{22} \in \mathbb{R}$,

$$\begin{aligned}&a_{1111}(x)\xi_{11}^2 + 2a_{1122}(x)\xi_{11}\xi_{22} + a_{2222}(x)\xi_{22}^2\\ &\quad \geqslant (a_{1111} - a_{1122})(x)\xi_{11}^2 + (a_{2222} - a_{1122})(x)\xi_{22}^2.\end{aligned} \tag{3.8}$$

Similarly, the following inequalities can be established: $\forall x \in \bar{\Omega}, \forall \xi_{11}, \xi_{12}, \xi_{22} \in \mathbb{R}$,

$$\begin{aligned}&(a_{1111} - a_{1122})(x)\xi_{11}^2 + 2a_{1112}(x)\xi_{11}(2\xi_{12}) + a_{1212}(x)(2\xi_{12})^2\\ &\quad \geqslant (a_{1111} - a_{1112} - a_{1122})(x)\xi_{11}^2 + (a_{1212} - a_{1112})(x)(2\xi_{12})^2;\end{aligned} \tag{3.9}$$

$$\begin{aligned}&(a_{2222} - a_{2211})(x)\xi_{22}^2 + 2a_{1222}(x)\xi_{22}(2\xi_{12}) + (a_{1212} - a_{1211})(x)(2\xi_{12})^2\\ &\quad \geqslant (a_{2222} - a_{2211} - a_{2212})(x)\xi_{22}^2 + (a_{1212} - a_{1211} - a_{1222})(x)(2\xi_{12})^2.\end{aligned} \tag{3.10}$$

Then, using (3.7)–(3.10), we have: $\forall x \in \bar{\Omega}, \forall \xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) \in \mathbb{R}^4$ with $\xi_{21} = \xi_{12}$,

$$\begin{aligned}a_{ijkl}(x)\xi_{ij}\xi_{kl} &\geqslant (a_{1111} - a_{1112} - a_{1122})(x)\xi_{11}^2\\ &\quad + (a_{1212} - a_{1211} - a_{1222})(x)(2\xi_{12})^2 + (a_{2222} - a_{2211} - a_{2212})(x)\xi_{22}^2\\ &\Rightarrow a(v, v) = \int_\Omega a_{ijkl}(x) v,_{ij} v,_{kl}\, d\Omega\\ &\geqslant \int_\Omega \Big[(a_{1111} - a_{1112} - a_{1122})(x)(v,_{11})^2\\ &\qquad + 4(a_{1212} - a_{1211} - a_{1222})(x)(v,_{12})^2\\ &\qquad\qquad + (a_{2222} - a_{2212} - a_{2211})(x)(v,_{22})^2\Big]\, d\Omega\\ &\geqslant \alpha_0 \int_\Omega v,_{ij} v,_{ij}\, d\Omega = \alpha_0 |v|_{2,\Omega}^2 \quad \forall v \in H_0^2(\Omega),\end{aligned}$$

where

$$\begin{aligned}\alpha_0 = \min\Big\{ &\inf_{x\in\bar{\Omega}} (a_{1111} - a_{1112} - a_{1122})(x),\\ &2 \inf_{x\in\bar{\Omega}} (a_{1212} - a_{1211} - a_{1222})(x), \inf_{x\in\bar{\Omega}} (a_{2222} - a_{2211} - a_{2212})(x)\Big\} > 0.\end{aligned}$$

Then, the result follows from the application of the Poincare-Friedrichs inequality [4].

Now, if

$$\inf_{x\in\bar\Omega}(a_{1111}-a_{1112}-a_{1122})(x)=0,$$
$$\inf_{x\in\bar\Omega}(a_{1212}-a_{1211}-a_{1222})(x)=0,$$
$$\inf_{x\in\bar\Omega}(a_{2222}-a_{2211}-a_{2212})(x)=0.$$

(See Case (II) in the Sec. (4.1)), then the Theorem (3.1) is not applicable. But we have

THEOREM (3.2). If $\forall i, j, k, l=1,2$,

$$a_{ijkl}=A_{ijkl}+\beta_{ijkl}, \tag{3.11}$$

where $\forall i, j, k, l=1,2$, A_{ijkl} satisfy the conditions (A1)–(A2) and (3.6) and β_{ijkl} satisfy

$$\int_\Omega \beta_{ijkl}v,_{ij}v,_{kl}\,d\Omega\geqslant 0\quad\forall v\in H_0^2(\Omega), \tag{3.12}$$

then $a(\cdot,\cdot)$ is $H_0^2(\Omega)$-elliptic.

Proof. Using (3.11), we obtain: $\forall v\in H_0^2(\Omega)$,

$$a(v,v)=\int_\Omega(A_{ijkl}+\beta_{ijkl})v,_{ij}v,_{kl}\,d\Omega$$
$$\geqslant\int_\Omega A_{ijkl}v,_{ij}v,_{kl}\,d\Omega$$

(by virtue of (3.12)), from which the result follows by the Theorem (3.1), since A_{ijkl} satisfy (A1)–(A2) and (3.6).

Remark (3.1). β_{ijkl} in (3.11) do not satisfy (A1)–(A2) and (3.6) in general.

THEOREM (3.3). If the coefficients a_{ijkl} satisfy (A1)–(A2) and (3.6) (resp. (3.11) and (3.12)), the problem (P_G) has a unique solution.

Proof. The result follows from the Theorem (3.1) (resp. Theorem (3.2)) and the Lax-Milgram lemma [4].

Remark (3.2). Since $a(\cdot,\cdot)$ is symmetric, the Ritz variational problem (P_R) corresponding to (P) can be defined as follows: For given $f\in L^2(\Omega)$, find $u\in H_0^2(\Omega)$ such that

$$J(u)=\inf_{v\in H_0^2(\Omega)}J(v), \tag{3.13}$$

where

$$J(v)=\tfrac{1}{2}a(v,v)-l(v)\quad\forall v\in H_0^2(\Omega). \tag{3.14}$$

PROPOSITION (3.1). If a_{ijkl} satisfy (A1)–(A2) and (3.6) (resp. (3.11)–(3.12)) and $u\in H_0^2(\Omega)$ is the solution of (P_G), then $u\in H_0^2(\Omega)$ is also the unique solution of (P_R).

4. Examples. First of all, we shall consider the biharmonic problem whose results are well known and then the bending problems of elastic—isotropic, orthotropic and anisotropic—plates with constant and variable thickness in order to illustrate the generality of the results obtained. In all the examples given below, only the $H_0^2(\Omega)$-ellipticity of the corresponding bilinear form $a(\cdot,\cdot)$ has been proved by Theorem (3.1) (resp. Theorem (3.2)), since the existence and uniqueness of the solution $u \in H_0^2(\Omega)$ of the corresponding problem (P_G) follows immediately from the Theorem (3.3).

4.1 *The biharmonic problem.*

Case (I). For $a_{ijkl} = \delta_{ik}\delta_{jl}$, $\Lambda u = \Delta\Delta u$. But $a_{ijkl} \neq a_{jikl}$, $a_{ijkl} \neq a_{ijlk}$ in general, i.e. (A2) is not satisfied, although (A1) holds, since $a_{ijkl} = a_{klij}$. Define $\bar{a}_{ijkl}$ by (3.5) such that $\bar{a}_{iiii} = 1$, $\bar{a}_{1212} = \bar{a}_{1221} = \bar{a}_{2112} = \bar{a}_{2121} = \frac{1}{2}$, $\bar{a}_{ijkl} = 0$ otherwise. Then, $\bar{a}_{ijkl}$ satisfy (A1)–(A2) and also (3.6), since $\bar{a}_{1111} - \bar{a}_{1112} - \bar{a}_{1122} = 1$, $\bar{a}_{1212} - \bar{a}_{1211} - \bar{a}_{1222} = \frac{1}{2}$, $\bar{a}_{2222} - \bar{a}_{2211} - \bar{a}_{2212} = 1$, and the $H_0^2(\Omega)$-ellipticity of $a(\cdot,\cdot)$ defined by

$$a(v,w) = \int_\Omega \delta_{ik}\delta_{jl} v,_{ij} w,_{kl}\, d\Omega = \int_\Omega v,_{ij} w,_{ij}\, d\Omega$$

follows from the Theorem (3.1).

Case (II). $a_{ijkl} = \delta_{ij}\delta_{kl}$ yield $\Lambda u = \Delta\Delta u$ and $a(v,w) = \int_\Omega \Delta u \Delta w\, d\Omega$. Then a_{ijkl} satisfy (A1)–(A2), but not (3.6), since $a_{1111} - a_{1112} - a_{1122} = a_{1212} - a_{1211} - a_{1222} = a_{2222} - a_{2211} - a_{2212} = 0$. Hence, Theorem (3.1) is not applicable. In order to apply the Theorem (3.2), define $\delta_{ij}\delta_{kl} = \bar{a}_{ijkl} + \beta_{ijkl}$, where $\bar{a}_{ijkl}$ are those given above in Case (I) which satisfy (A1)–(A2) and (3.6), and β_{ijkl} are given by: $\beta_{1122} = \beta_{2211} = 1$, $\beta_{1212} = \beta_{2112} = \beta_{2121} = \beta_{1221} = -\frac{1}{2}$, $\beta_{ijkl} = 0$ otherwise (see Remark (3.1)). $\forall v \in D(\Omega)$,

$$\int_\Omega \beta_{ijkl} v,_{ij} v,_{kl}\, d\Omega = 2\int_\Omega \left(v,_{11} v,_{22} - (v,_{12})^2\right) d\Omega = 0.$$

Since $D(\Omega)$ is dense in $H_0^2(\Omega)$, β_{ijkl} satisfy (3.12). Now, the $H_0^2(\Omega)$-ellipticity of $a(\cdot,\cdot)$ follows from Theorem (3.2).

4.2 *Bending problems of elastic plates.* The bending problem of a clamped thin elastic plate is defined by (P) of the corresponding plate operator Λ given by (3.1), where $u = u(x_1, x_2)$ denotes the normal deflection at any point (x_1, x_2) of the middle plane $\bar{\Omega}$ of the elastic plate, Γ being its boundary along which the plate is clamped, the coefficients a_{ijkl} denote elastic properties and thickness of the plate, $f \in L^2(\Omega)$ denotes the load function. We shall consider elastic plates first with constant thickness and then with variable thickness.

a) *Plates with thickness h = constant.*

(I) *For anisotropic case* [3], $a_{iiii} = D_{ii}$, $a_{1212} = a_{1221} = a_{2121} = a_{2112} = D_{66}$, $a_{1112} = a_{1121} = a_{1211} = a_{2111} = D_{16}$, $a_{1222} = a_{2122} = a_{2212} = a_{2221} = D_{26}$ and $a_{2211} = a_{1122} = D_{12}$, where D_{ij} denote rigidities [3] having the properties: $D_{11}, D_{22}, D_{66} > 0$; $D_{12} = \nu_1 D_{22} = \nu_2 D_{11}$, $0 \leqslant \nu_i < \frac{1}{2}$ $(i = 1,2)$; $0 \leqslant D_{16} < (1-\nu_2)D_{11}$; $0 \leqslant D_{26} < (1-\nu_1)D_{22}$ and $D_{16} + D_{26} < D_{66}$; and the Anisotropic plate operator Λ is given by: $\Lambda u = D_{11}u,_{1111} + 4D_{16}u,_{1112} + 2(D_{12} + 2D_{66})u,_{1122} + 4D_{26}u,_{1222} + D_{22}u,_{2222}$. The coefficients a_{ijkl} satisfy (A1)–(A2) and also (3.6), since $a_{1111} - a_{1112} - a_{1122} = D_{11}(1-\nu_2) - D_{16} > 0$, $a_{1212} - a_{1211} - a_{1222} = D_{66} - (D_{16} + D_{26}) > 0$, $a_{2222} - a_{2212} - a_{2211} = D_{22}(1-\nu_1) - D_{26} > 0$. Hence, the $H_0^2(\Omega)$-ellipticity of the bilinear form $a(\cdot,\cdot)$ defined by

$$a(v,w)=\int_\Omega[(D_{11}v,_{11}+2D_{16}v,_{12}+D_{12}v,_{22})w,_{11}$$
$$+2(D_{16}v,_{11}+2D_{66}v,_{12}+D_{26}v,_{22})w,_{12}$$
$$+(D_{12}v,_{11}+2D_{26}v,_{12}+D_{22}v,_{22})w,_{22}]\,d\Omega$$

follows from the Theorem (3.1).

(II) *For orthotropic case* [2, 3, 6, 7], $a_{iiii}=D_i$,

$$\begin{aligned} &a_{1122}=a_{2211}=D_{12}=\nu_1D_2=\nu_2D_1,\\ &a_{1212}=a_{1221}=a_{2112}=a_{2121}=(H-\nu_2D_1)/2,\\ &a_{1211}=a_{1112}=a_{1121}=a_{2111}=0,\\ &a_{2212}=a_{1222}=a_{2122}=a_{2221}=0, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} &D_i=E_ih^3/(12(1-\nu_1\nu_2)),\\ &H=D_{12}+2D_t,\ D_t=Gh^3/12,\\ &G=E_1E_2/(E_1+(1+2\nu_1)E_2)=E_1E_2\ \ /(E_2+(1+2\nu_2)E_1),\\ &E_1\nu_2=E_2\nu_1, \end{aligned} \tag{4.2}$$

E_i and ν_i $(i=1,2)$ are Young's moduli and Poisson's coefficients respectively. Then, the corresponding Orthotropic plate operator Λ is defined by $\Lambda u=D_1u,_{1111}+2Hu,_{1122}+D_2u,_{2222}$. The coefficients a_{ijkl} satisfy (A1)–(A2) and also (3.6), since

$$\begin{aligned} &a_{1111}-a_{1112}-a_{1122}=D_1(1-\nu_2)>0;\\ &a_{1212}-a_{1211}-a_{1222}=D_t>0;\\ &a_{2222}-a_{2211}-a_{2212}=(1-\nu_1)D_2>0. \end{aligned} \tag{4.3}$$

Hence, the $H_0^2(\Omega)$-ellipticity of the corresponding bilinear form $a(\cdot,\cdot)$ defined by

$$a(v,w)=\int_\Omega[(D_1v,_{11}+\nu_2D_1v,_{22})w,_{11}+2(H-\nu_2D_1)v,_{12}w,_{12}$$
$$+(\nu_2D_1v,_{11}+D_2v,_{22})w,_{22}]\,d\Omega \tag{4.4}$$

follows from the Theorem (3.1).

(III) *For isotropic case* [3, 6, 7], which is obtained from the orthotropic case in (II) by putting $E_1=E_2=E$ and $\nu_1=\nu_2=\nu$ in (4.1) and (4.2) such that $D_1=D_2=H=D$, the Isotropic plate operator Λ is defined by: $\Lambda u\equiv D\Delta\Delta u$, and the $H_0^2(\Omega)$-ellipticity of the corresponding bilinear form $a(\cdot,\cdot)$ defined by

$$a(v,w)=\int_\Omega D[(v,_{11}+\nu v,_{22})w,_{11}+2(1-\nu)v,_{12}w,_{12}+(\nu v,_{11}+v,_{22})w,_{22}]\,d\Omega \tag{4.5}$$

follows from (4.1)–(4.3) and Theorem (3.1).

b) *Plates with variable thickness* $h=h(x_1,x_2)>0$: The thickness function h satisfies the following condition:

$$h \varepsilon C^0(\bar{\Omega}), \qquad h_0 = \min_{(x_1,x_2)\in\bar{\Omega}} h(x_1, x_2) > 0. \tag{A3}$$

For the sake of brevity, we shall consider only the isotropic plates with variable thickness, since the proofs for the orthotropic and anisotropic plates with variable thickness h satisfying (A3) are similar. For the isotropic case, $a_{iiii} = D$, $a_{1122} = a_{2211} = \nu D$, $a_{1212} = a_{1221} = a_{2121} = a_{2112} = D(1-\nu)/2$, and $a_{ijkl} = 0$ otherwise, where

$$D = D(x_1, x_2) = \left(Eh^3/\left(12(1-\nu^2)\right)\right)(x_1, x_2) \geqslant Eh_0^3\left(12(1-\nu^2)\right) > 0. \tag{4.6}$$

Then a_{ijkl} satisfy (A1)–(A2) and (3.6), since $a_{1111} - a_{1112} - a_{1122} = (1-\nu)D > 0$, $a_{1212} - a_{1211} - a_{1222} = D(1-\nu)/2 > 0$, $a_{2222} - a_{2211} - a_{2212} = (1-\nu)D > 0$. Hence, the corresponding bilinear form $a(\cdot, \cdot)$ defined by (4.5), in which D is given by (4.6), is $H_0^2(\Omega)$-elliptic by Theorem (3.1).

Bibliography

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

[2] M. T. Huber, *Probleme der Statik technische wichtiger orthotroper Platten*, Warsaw, 1929

[3] S. G. Lekhnitskii, *Anisotropic plates*, Gordon and Breach, New York, 1968

[4] J. L. Lions, *Problèmes aux limites dans les équations dérivées partielles*, Les presses de L'Université de Montréal, Montréal, 1965

[5] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966

[6] S. Timoshenko and S. Woinowsky-Krieger, *Theory of plates and shells*, McGraw-Hill, New York, 1955

[7] M. S. Troitsky, *Stiffened plates*, Elsevier, New York, 1976