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A NONSTANDARD NONLINEAR BOUNDARY-VALUE PROBLEM FOR HARMONIC FUNCTIONS*

By

NIMA GEFFEN

Tel-Aviv University, Tel-Aviv

Abstract. Existence and uniqueness are proved for a nonstandard, nonlinear boundary-value problem for 2-dimensional harmonic functions. The problem models an ideal flow-field, and a few cases of applied interest are considered. Slight generalizations are derived in the appendix.

1. Introduction. Incompressible, inviscid fluid flow about given boundaries is potential, and satisfies standard Neumann or mixed linear boundary conditions. The pressure is a single valued function of the square of the velocity, which, in turn, is a gradient of the potential φ:

$$p = p_0 - \rho_0 \frac{q^2}{2} \qquad \mathbf{q} = \nabla\varphi. \tag{$*$}$$

The continuity equation: $\nabla \cdot \mathbf{q} = 0$, implies $\nabla^2\varphi = 0$.

If, instead of no-flow, we wish to prescribe the pressure about a given boundary, we have (by $(*)$) the nonlinear boundary condition: $q = |\nabla\varphi| = f$. The pressure distribution determines the forces and moments acting on the surface, and the question under what conditions it can be prescribed comes up in design problems. The motivation for the present work came from yet another direction. As an alternative to known aerodynamic design procedures involving quasi-linear equations of mixed type with mixed linear boundary conditions of the Dirichlet-Neumann variety (e.g. [1]), we proposed a method involving nonstandard boundary conditions (e.g. [3]). The mathematical model treated here can be viewed as a limiting case of the transonic nonlinear equation, and the conjecture (supported by numerical evidence) is that the difficulties (and ways out) associated with the nonlinear boundary condition are essentially the same in both cases [3].

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The original motivation for the formulation and analysis above is due to Dr. Sara Yaniv of Tel-Aviv University, who long ago suggested a free-sonic-line approach to supercritical airfoil design. The main "complex" idea is due to Professor Stanley Osher of UCLA. Relevant questions, comments, references and classifications have been suggested by Professors Jim Ralston, Gregory Eskin, and Julian Cole at UCLA. Dr. Davie Levine of Tel-Aviv University provided valuable information related to integral representations and behaviour of singularities.

2. First nonstandard boundary-value problem. Consider Laplace's equation:

$$\nabla^2\varphi = \varphi_{xx} + \varphi_{yy} = 0 \tag{1}$$

in a domain D with the magnitude of $\nabla\varphi$ prescribed on the boundary ∂D (Fig. 1):

$$|\nabla\varphi| = \sqrt{\varphi_x^2 + \varphi_y^2} = f \quad \text{on } \partial D, \qquad f > 0. \tag{2}$$

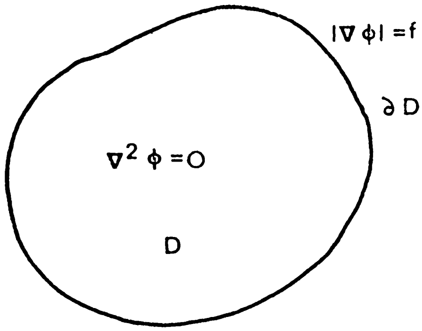

FIG. 1.

THEOREM 1. The gradient field $\mathbf{q} = (u, v) = (\varphi_x, \varphi_y) = \nabla\varphi$ satisfying (1) and (2) is determined up to a rotation. In other words:

(i) $\mathbf{q} = \nabla\varphi$, such that (1), (2) hold, exists.

(ii) If (u, v) is a solution to (1), (2), so is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

for $0 < \alpha < 2\pi$.

Proof. Consider the complex potential, assumed analytic:

$$z = x + iy,$$
$$F(z) = \varphi(x, y) + i\psi(x, y)$$

where

$$u = \varphi_x = \psi_y, \qquad v = \varphi_y = -\psi_x.$$

Let the complex velocity be

$$w(z) = F'(z) = \varphi_x + i\psi_x = u - iv$$

and set

$$w(z) = qe^{i\theta}, \tag{3}$$
$$|w(z)| = |F'(z)| = q = |\nabla\varphi| = |\nabla\psi|$$

assuming

$$F'(z) = w \neq 0.$$

The function:[1]

$$\ln w(z) = \ln q + i\theta \tag{4}$$

is also analytic, i.e.

$$\nabla^2(\ln q) = 0 \quad \text{in } D \tag{5}$$

and

$$\ln|w(z)| = \ln q = \ln f, \tag{6}$$

is given on ∂D by (2).

Equations (5) and (6) specify a Dirichlet problem for $\sigma = \ln q = \ln|\nabla\varphi|$ which is well posed. It has a unique solution, which can be written down at least for 'reasonable' problems, e.g., via conformal mapping or Green's function (e.g. [4]).

The angle θ is the harmonic conjugate of $\sigma = \ln q$ (Eq. (4)), hence, using Cauchy-Riemann conditions:

$$\sigma_x = (\ln q)_\alpha = \theta_y, \qquad \sigma_y = (\ln q)_y = -\theta_x$$

it can be determined from $\sigma = \ln q$ up to a constant θ_0, and:

$$w(z) = q^{i(\theta' + \theta_0)} = u - iv.$$

The vector field $(u, v)(x, y)$ forms a one-parameter family with the same speed (i.e. magnitude of $q = \sqrt{u^2 + v^2}$). The velocity vector $\mathbf{q}$ can be rotated by the same angle θ_0 at all points of the field.

Example. Let

$$\nabla^2\varphi = 0 \quad \text{on } D,$$
$$|\nabla\varphi| = C \quad \text{on } \partial D.$$

The solution is

$$\varphi = Ux + Vy$$

where U and V are constants, such that

$$U^2 + V^2 = C^2,$$

i.e. a uniform vector field of magnitude C and an arbitrary ('free') direction.

The same result holds for the 3-dimensional case:

$$\varphi = Ux + Vy + Wz, \qquad U^2 + V^2 + W^2 = C^2.$$

3. Second nonlinear boundary-value problem. In this case we look for a harmonic function satisfying the equation:

$$\nabla^2\varphi = \varphi_{xx} + \varphi_{yy} = 0 \quad \text{in } D \tag{1}$$

and the mixed boundary conditions

$$|\nabla\varphi| = \sqrt{\varphi_x^2 + \varphi_y^2} = f \quad \text{on } \partial D_1, \tag{2a}$$

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[1] The following representation is due to S. Osher of UCLA.

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial D_2 \tag{2b}$$

where $\partial D = \partial D_1 + \partial D_2$ (Fig. 2).

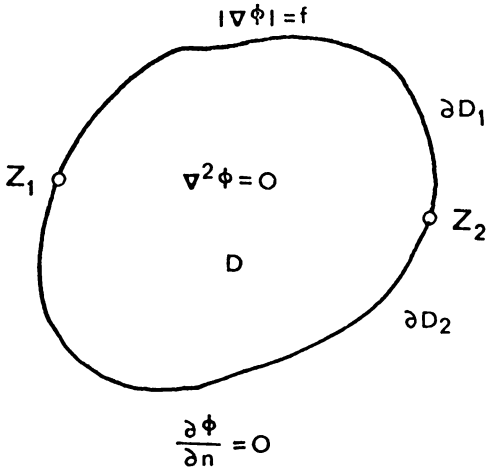

FIG. 2.

THEOREM 2. The gradient field $\mathbf{q} = \nabla\varphi = (\varphi_x, \varphi_y)$ satisfying (1), (2a), (2b) exists and is unique up to a sign, i.e.: if $\mathbf{q}_1$ is a solution, so is $-q_1$.

Proof. Recapitulating the definitions and reasoning above we have

$$\begin{aligned} F(z) &= \varphi + i\psi, \\ w(z) &= F'(z) = u - iv, \\ (u, v) &= (\varphi_x, \varphi_y) = (\psi_y, -\psi_x), \\ w(z) &= qe^{i\theta}, \\ \ln w(z) &= \ln(q) + i\theta = \sigma + i\theta, \\ q^2 &= u^2 + v^2; \qquad \tan\theta = -v/u. \end{aligned}$$

Assuming again that $F(z)$ is analytic and $F'(z) \neq 0$, $w(z)$ is analytic, and on the boundary ∂D_2:

$$\frac{\partial \varphi}{\partial n} = 0 = q_n \quad \text{and} \quad \frac{\partial \varphi}{\partial n} = \frac{\partial \psi}{\partial l} = 0.$$

Thus, along the boundary ∂D_2:

$$\frac{\partial \psi}{\partial l} = 0 \Rightarrow \psi = C$$

which simply expresses the fact that ∂D_2 is a streamline (i.e., no-through-flow surface).

On such a surface we have

$$\psi = C\colon d\psi = \psi_x dx + \psi_y dy = 0$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\psi=C} = -\left.\frac{\psi_x}{\psi_y}\right|_{\psi=C} = \frac{v}{u} = \tan(-\theta) \in \partial D_2$$

let ∂D_2 be given as

$$y = Y(X)$$

i.e.,

$$\tan(-\theta)\,\big|_{y=Y} = Y'(x).$$

Using the relation:

$$\frac{\partial(\ln q)}{\partial n} = \frac{\partial \sigma}{\partial n} = \frac{\partial \theta}{\partial l}$$

we get

$$\frac{\partial \sigma}{\partial n} = g \in \partial D_2$$

where

$$g = \frac{\partial}{\partial l}\left[-\tan^{-1} Y'(x)\right]$$

and the problem to solve is:

$$\begin{aligned} \nabla^2(\ln q) &= 0 \in D, \\ \ln q &= \ln f \in \partial D_1, \\ \frac{\partial}{\partial n}(\ln q) &= g \in \partial D_2 \end{aligned}$$

which is a standard linear mixed boundary-value problem for the harmonic function $\sigma = \ln q$ in D (Fig. 3).

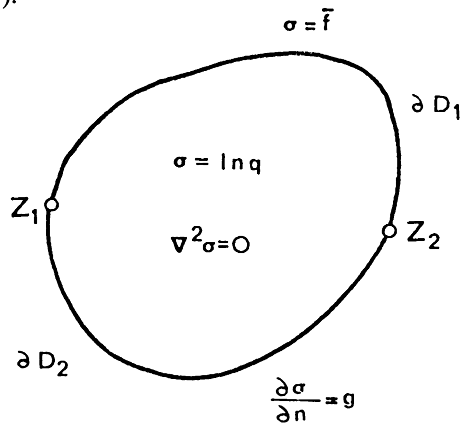

FIG. 3.

The problem is well posed, and has a unique solution in D, which can be written explicitly in various forms and has been investigated extensively (e.g. [4]).

To determine θ we can either use the Cauchy Riemann equations, or solve for θ directly. The angle function θ is harmonic (i.e. $\nabla^2\theta = 0$) and satisfies the conjugate boundary conditions:

$$\frac{\partial \theta}{\partial n} = \frac{\partial(\ln q)}{\partial l} = \frac{\partial}{\partial l}(\ln f) = f_1 \quad \text{on } \partial D_1$$

and:

$$\theta = \tan^{-1}(-Y'(x)) = g_1 \quad \text{on } \partial D_2.$$

The problem is again a standard linear mixed boundary-value problem, which has a unique solution. Thus q and θ are uniquely defined for this problem, and since: $w(z) = qe^{i\theta} = u - iv$, the vector $\mathbf{q} = (u, v)$ is defined up to a sign, i.e., if $\mathbf{q}_1 = (u_1, v_1)$ is a solution, so is $q_2 = -q_1 = (-u_1, -v_1)$ and these are the only two solutions.

COROLLARY. The 'external' problem (Fig. 4):

$$\begin{aligned} \nabla^2\varphi &= 0 \text{ outside } S, \\ \nabla\varphi &= f \in S_1, \qquad f > 0, \\ \frac{\partial\varphi}{\partial n} &= 0 \in S_2, \\ (\varphi_x, \varphi_y) &= U(1,0), \\ |x| + |y| &\to \infty \end{aligned}$$

has a unique solution.

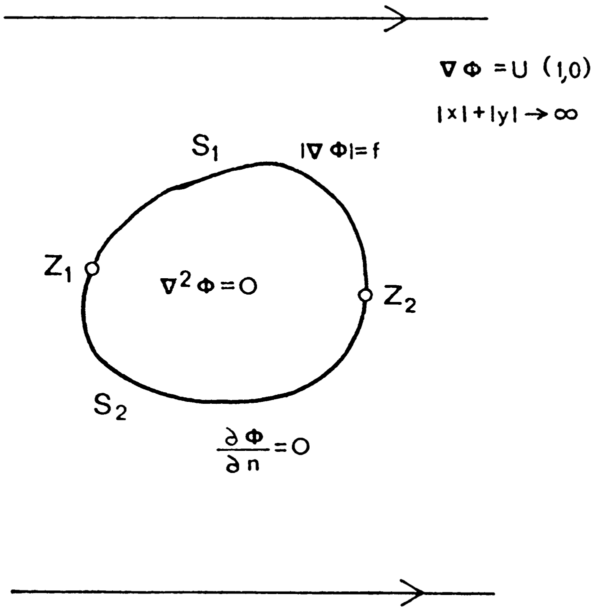

FIG. 4.

The proof follows the lines above; the external problem for $\sigma = \ln q$ is well posed and has a unique solution. The conjugate harmonic function can be determined directly or via the Cauchy-Riemann equations, as before. Here it is uniquely determined, with the direction fixed by the given uniform far-field.

Singular points. The assumption

$$F'(z) = w \neq (0, \infty)$$

excluding 'stagnation' and singular points, can be relaxed if we know the location and a type of singularities, which is often the case.

We then remove them by considering the function:

$$P = \prod_{j=1}^{k} f(z)(z - z_j)^{\beta_j}$$

where z_j are the 'special' points and β_j are chosen to remove the singularities or troublesome zeros at $z = z_j$.

The analysis above can be repeated for the corresponding noon-standard boundary-value problems for $P(z)$ and the same conclusions drawn.

Remarks on non-homogeneous Neumann conditions. For the homogeneous Neumann Condition $\partial\varphi/\partial n = 0$ the nonlinear boundary-value problem for φ or $\mathbf{q} = \nabla\varphi$ becomes linear for the auxiliary function:

$$\sigma = \ln q = \ln |\nabla\varphi| .$$

In addition, the boundary-value problems for σ and θ decouple and result in standard linear mixed boundary-value problems for σ and θ separately, of the types frequently encountered in potential theory and its physical applications (i.e., elasticity, hydrodynamics [4]).

This decoupling fails for a nonhomogeneous condition on ∂D_2, where

$$q_n = \partial\varphi/\partial n = Q \neq 0.$$

For this case let ∂D_2 be $y = Y(x)$ and $\tan\alpha = Y'(x)$. We again have (Fig. 5):

$$\tan\theta = -\frac{v}{u},$$

$$q_n = q\sin(-\theta - \alpha) = -\beta\sin(\theta + \theta).^2$$

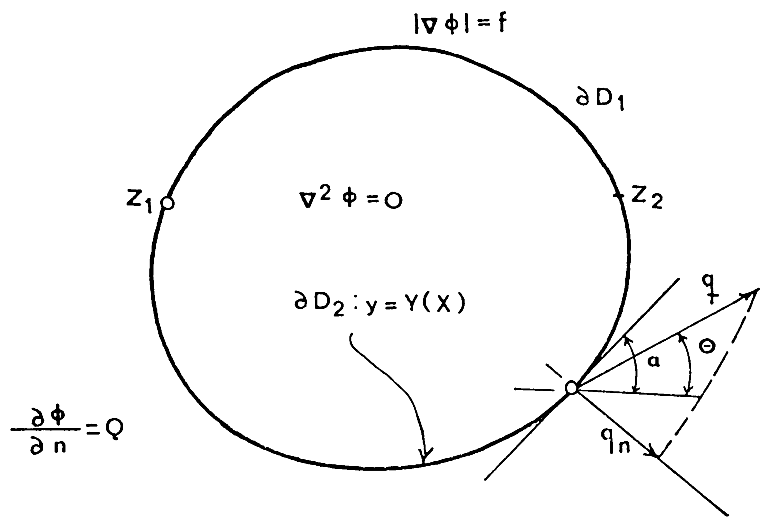

FIG. 5.

[2] For the no-flow condition: $q_n = 0$, i.e., $-\theta = \alpha$.

The boundary condition on ∂D_2 is

$$\bar{Q} = -Q = q \sin(\theta + \alpha)$$

and

$$Q = \ln \bar{Q} = \ln q + \ln \sin(\theta + \alpha)$$

or

$$Q = \sigma + \ln \sin(\theta + \alpha)$$

on ∂D_2: $y = Y(X)$ and $\tan \alpha = Y'(x)$.

The conjecture here is that this problem also has a unique solution (up to a sign) at least for the case where:

$$\Phi_{12} = \int_{z_1}^{z_2} q_n \cdot dl = 0$$

i.e., where there is no net flux through the boundary.

4. An alternative 'real' analysis. Circumventing the complex representation, with a view towards possible generalizations (see Appendix) the following holds:

THEOREM 3. Consider the function

$$\mathsf{q} = \mathbf{q} \cdot \mathbf{q}$$

where $\mathbf{q}$ is a vector field. There exist functions f twice differentiable and monotone (i.e., $f' \neq 0$) such that $f(\mathsf{q})$ is harmonic, i.e.,

$$\nabla^2(f(\mathsf{q})) = 0 \tag{7}$$

if and only if:

$$\frac{\nabla^2 \mathsf{q}}{\nabla \mathsf{q} \cdot \nabla \mathsf{q}} = -F(\mathsf{q}) \tag{8}$$

and in this case:

$$f(\mathsf{q}) = \int^{\mathsf{q}} \left[e^{\int^{\mathsf{q}'} F(q'')\, dq''} \right] d\mathsf{q}. \tag{9}$$

Proof. The proof is trivially straightforward, via a direct calculation:

$$\nabla \cdot \nabla(f(\mathsf{q})) = f'' \nabla \mathsf{q} \cdot \nabla \mathsf{q} + f' \nabla^2 \mathsf{q} = 0$$

or

$$f''/f' = \nabla^2 \mathsf{q} / (\nabla \mathsf{q})^2.$$

Since

$$f''/f' = (\ln f')' = +F(\mathsf{q}) \tag{10}$$

we must have

$$\nabla^2 \mathsf{q} / (\nabla \mathsf{q})^2 = -F(\mathsf{q}) \tag{8}$$

and integrating (10) twice gives Eq. (6). Since Eq. (10) is a second-order differential equation for f, it has a two-parameter family of solutions.

For 2-dimensional 'harmonic' potential fields:

$$\mathbf{q} = (\varphi_x, \varphi_y), \qquad \nabla^2\varphi = 0$$

direct calculation gives:

$$\nabla^2 q = \left(\varphi_x^2 + \varphi_y^2\right)_{xx} + \left(\varphi_x^2 + \varphi_y^2\right)_{yy} = 4\left(\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy}\right),$$

$$(\nabla q)^2 = 4\left[(\varphi_x\varphi_{xx} + \varphi_y\varphi_{yx})^2 + (\varphi_x\varphi_{xy} + \varphi y\varphi_{yy})^2\right]$$

$$= 4\left(\varphi_x^2 + \varphi_y^2\right)\left(\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy}\right).$$

Hence:

$$F(q) = -1/q \quad \text{and} \quad f(q) = \ln c q^{\alpha}$$

choosing $c = 1$, $\theta = \frac{1}{2}$ we recover (5) and the rest of the proof follows.

Appendix: A note on a wrong question with some simple answers.

1. *The question*. Consider a potential vector field

$$\mathbf{u} = \nabla\varphi \tag{1}$$

where φ satisfies a second-order differential equation:

$$L\varphi = a_{ij}\frac{\partial^2\varphi}{\partial x_i \partial x_j} + b_i\frac{\partial\varphi}{\partial x_i} \tag{2}$$

and summation is implied for indices appearing twice in the same term.

Can one find a function f satisfying:

$$Lf(q) = 0 \tag{3}$$

where q is the square of the magnitude of the vector field $\mathbf{u}$, i.e.,

$$q = \mathbf{u}\cdot\mathbf{u} = (\nabla\varphi)^2?$$

This may provide information on relevant properties of the field directly (e.g. the kinetic energy if $\mathbf{u}$ of the velocity vector), and enable one to treat non-standard boundary values, where q is given on the boundary (e.g. a prescribed pressure in fluid dynamics).

The question is "wrong" because, in general, one cannot expect such a simple relationship between φ and q, except, possibly, in fortuitous cases. The condition for such luck, however, can be easily obtained, and readily checked. The check is constructive, explicitly yielding the function f if it exists; hence the following discussion.

2. *The answer*. Consider a twice, continuously differentiable function $\varphi(x_1, \ldots, x_n)$ satisfying the linear equation

$$L\varphi = \sum_{i,j} a_{ij}\frac{\partial^2\varphi}{\partial x_i \partial x_j} + \sum_i b_i\frac{\partial\varphi}{\partial x_i} = 0. \tag{2}$$

Let

$$\mathbf{u} = \nabla\varphi, \tag{1}$$

$$q = \mathbf{u}\cdot\mathbf{u} = (\nabla\varphi)^2. \tag{3}$$

THEOREM. There exists a twice differential function $f(q)$ such that

$$Lf(q) = 0 \tag{4}$$

if and only if

$$\left(a_{ij}\frac{\partial^2 q}{\partial x_i \partial x_j} + b_i\frac{\partial q}{\partial x_i}\right) \Big/ a_{ij}\frac{\partial q}{\partial x_i}\cdot\frac{\partial q}{\partial x_j} = F(q) \tag{5}$$

and then

$$f(q) = c_1\int_0^q \exp\left(-\int_{q_0}^{q'} F(q'')\,dq''\right) dq' + c_2. \tag{6}$$

In terms of φ the condition is

$$\frac{a_{ij}\left[\left(\dfrac{\partial^2\varphi}{\partial x_l \partial x_j}\right)\left(\dfrac{\partial^2\varphi}{\partial x_l \partial x_i}\right) + \dfrac{\partial\varphi}{\partial x_l}\,\dfrac{\partial^2\varphi}{\partial x_l \partial x_i \partial x_j}\right]}{2a_{ij}\left(\dfrac{\partial\varphi}{\varphi x_l}\,\dfrac{\partial^2\varphi}{\partial x_l \partial x_i}\right)\left(\dfrac{\partial\varphi}{\partial x_l}\,\dfrac{\partial^2\varphi}{\partial x_l \partial x_j}\right)} = \left[F\left(\frac{\partial\varphi}{\partial x_l}\right)\cdot\left(\frac{\partial\varphi}{\varphi x_l}\right)\right].$$

If instead of (3) we set:

$$q = \alpha_{ij}u_i u_j \tag{3'}$$

condition (5) becomes:

$$a_{lm}\left[a_{ij}\left(\frac{\partial u_l}{\partial x_i}\,\frac{\partial u_m}{\partial x_j} + \frac{\partial u_m}{\partial x_i}\,\frac{\partial u_l}{\partial x_j} + u_l\frac{\partial^2 u_m}{\partial x_i \partial x_j} + u_m\frac{\partial^2 u_l}{\partial x_i \partial x_j}\right) + b_i\left(u_l\frac{\partial u_m}{\partial x_i} + u_m\frac{\partial u_l}{\partial x_i}\right)\right]$$
$$\cdot a_{ij}\left[\sum_{l,m}\left(u_l\frac{\partial u_m}{\partial x_i} + u_m\frac{\partial u_l}{\partial x_i}\right)\right]^2 = F(\alpha_{lm}u_l u_m) \tag{5'}$$

with a corresponding change (and complication) of (7). Equation (3′) reduces to (3) when $\alpha_{ij} = \delta_{ij}$ and other choices may be useful (e.g., $\alpha_{ij}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ in an example below).

Proof. Substituting $f(q)$ in $Lf(q) = 0$, we get

$$f''a_{ij}\frac{\partial^2 q}{\partial x_i \partial x_j} + f'a_{ij}\frac{\partial q}{\partial x_i}\,\frac{\partial q}{\partial x_j} + f'b_i\frac{\partial q}{\partial x_i} = 0,$$

hence:

$$-\frac{f'}{f''} = a_{ij}\frac{\partial^2 q}{\partial x_i \partial x_j}\Big/\left(b_i\frac{\partial q}{\partial x_i} + a_{ij}\frac{\partial q}{\partial x_i}\,\frac{\partial q}{\partial x_j}\right).$$

Since f is a function of q only, so is the LHS of (9) and condition (5) follows. The explicit solution for f, equation (7), is obtained by integrating twice the equation:

$$f'/f'' = -F(q). \tag{10}$$

3. *Examples*. (a) *Laplace's equation*. Let

$$L\varphi = \nabla^2\varphi = 0 \qquad q = (\nabla\varphi)^2. \tag{11}$$

There exists a harmonic function $f(q)$ if and only if

$$\frac{\nabla^2 q}{(\nabla q)^2} = F(q). \tag{12}$$

In two dimensions:

$$q = \varphi_x^2 + \varphi_y^2,$$
$$\nabla \cdot \nabla q = 4\left(\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy}\right),$$
$$(\nabla q)^2 = 4\left(\varphi_{xy}^2 - \varphi_{xx}\varphi_{yy}\right)\left(\varphi_x^2 + \varphi_y^2\right).$$

Hence

$$F(q) = \frac{1}{\varphi_x^2 + \varphi_y^2} = \frac{1}{q}, \qquad \frac{f''}{f'} = -\frac{1}{q}$$

and

$$f = c \ln q^{\alpha}.$$

(b) *Wave equation*. Let

$$L\varphi = \varphi_{xx} - \varphi_{yy} = 0, \qquad q = \varphi_x^2 + \varphi_y^2.$$

For this case

$$Lf(q) = 0$$

implies

$$-\frac{f''}{f'} = \frac{q_{xx} - q_{yy}}{q_x^2 - q_y^2} = F(q)$$

but here

$$q_{xx} - q_{yy} = (\varphi_{xx} - \varphi_{yy})(\varphi_{xx} + \varphi_{yy}) + \varphi_x(\varphi_{xx} - \varphi_{yy})_x + \varphi_y(\varphi_{xx} - \varphi_{yy})_y = 0$$

and q itself satisfies the wave equation!

(c) *A different choice of* α_{ij}. Choosing:

$$\alpha_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

i.e.,

$$q = \varphi_x^2 - \varphi_y^2$$

one gets

$$\varphi_{xx} + \varphi_{yy} = 0 \Rightarrow q_{xx} + q_{yy} = 0$$

and

$$\varphi_{xx} - \varphi_{yy} = 0 \Rightarrow (\ln q)_{xx} - (\ln q)_{yy} = 0$$

(which can be obtained via the change $y \to iy$ in (a) and (b) above).

Additional relations can be derived by further changes in α_{ij} and reiterating the same procedure on the equations for q. This results in higher order equations for φ, which may contain useful information, e.g., via using the maximum principle for Laplace's equation, and conservation laws for the wave equation.

4. *A final apology*. Although a wrong question is asked, the answer is so simple and elementary that it may be worth a try on special occasions. When it works, it "linearizes" a nonlinear problem for q and φ from which complete information can be drawn. Other interesting equations amenable and examples in higher dimensions are yet to be shown, as well as possible connections among them.

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