QUARTERLY OF APPLIED MATHEMATICS

VOLUME XLI JULY · 1983 NUMBER 2

BUCKLING OF COLUMNS AND REARRANGEMENTS OF FUNCTIONS*

By

DAVID C. BARNES

Washington State University

I. Introduction. A slender column is subject to an axial compressive load P which may cause it to buckle. We assume the column is not twisted so that in the buckled state it will lie in a plane. The governing differential equations for the displacement $w(x)$ from its straight equilibrium position is [13, 16]

$$+[EI(x)w''(x)]'' + Pw''(x) = 0. \tag{1}$$

Here s is distance measrued from one end of the column, E is the modulus of elasticity and $I(x)$ is the moment of inertia of the cross section of the column about a line passing through its centroid but perpendicular to the plane of buckling. We assume that E is constant but that I is a function of x. We will however assume that all cross sections are similar. This implies that

$$I(x) = KA^2(x) \tag{2}$$

where $A(x)$ is the area of the cross section and K is a constant which depends only on the particular shape of the cross section. The volume v of the column is given by

$$v = \int_0^L A(x)\,dx. \tag{3}$$

An interesting problem with a long history [10, 3, 16, 9, 15, 13, 14] is to determine the shape of a column having given volume v and a buckling load as large as possible. The major purpose of this work is to apply the theory of rearrangements as given by Duff [5], Schwarz [12], Pólya and Szegö [11] and Barnes [1] to the column problem. In particular we will show how to "rearrange" a given column to obtain a larger or smaller buckling load. These results are summarized in Theorem II below and an example is sketched in Figs. 5 and 6.

II. Boundary conditions and differential equations. The boundary conditions we will consider are of the form (see [13, 16])

$$\begin{aligned}&\text{clamped at } x = 0 \text{ and at } x = L\\ &w(0) = w'(0) = w(L) = w'(L) = 0,\end{aligned} \tag{4}$$

* Received January 19, 1982. The author would like to thank Professor J. B. Keller for his help and encouragement with the project.

$$\begin{aligned}&\text{clamped at } x = 0 \text{ and pinned at } x = L\\ &w(0) = w'(0) = w(L) = w''(L) = 0,\end{aligned} \tag{5}$$

$$\begin{aligned}&\text{pinned at } x = 0 \text{ and at } x = L\\ &w(0) = w''(0) = w(L) = w''(L) = 0,\end{aligned} \tag{6}$$

$$\begin{aligned}&\text{clamped at } x = 0 \text{ and free at } x = L\\ &w(0) = w'(0) = EI(L)w''(L) = Pw'(L) - [EI(L)w''(L)]' = 0.\end{aligned} \tag{7}$$

Following Tadjbakhsh and Keller [15] we introduce the bending moment $y(x)$ defined by

$$y(x) = -EI(x)w''(x). \tag{8}$$

Equation (1) becomes

$$y'' = Pw'' \tag{9}$$

Using (8) we obtain

$$y'' + \lambda A^{-2}(x)y = 0 \tag{10}$$

where

$$P = \lambda EK. \tag{11}$$

Integrating (9) twice from 0 to x we find, after setting $x = L$, that

$$y'(L) - y'(0) = P[w'(L) - w'(0)] \tag{12}$$

and

$$y(L) - y(0) - Ly'(0) = P[w(L) - w(0) - Lw'(0)]. \tag{13}$$

Combining (12) and (13) with (4)–(8) we find equivalent boundary conditions on y in each of the four cases:

$$\begin{aligned}&\text{clamped at } x = 0 \text{ and at } x = L:\\ &y'(0) = y'(L), y(L) - y(0) = Ly'(L),\end{aligned} \tag{14}$$

$$\begin{aligned}&\text{clamped at } x = 0 \text{ and pinned at } x = L\\ &y(0) + Ly'(0) = y(L) = 0,\end{aligned} \tag{15}$$

$$\begin{aligned}&\text{pinned at } x = 0 \text{ and at } x = L\\ &y(0) = y(L) = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}&\text{clamped at } x = 0 \text{ and free at } x = L\\ &y'(0) = y(L) = 0.\end{aligned} \tag{17}$$

The differential equation (10) together with one set of boundary conditions (14), (15), (16) or (17) forms a self adjoint eigenvalue problem for the determination of λ which then determines P by (11). Although there will be an infinite sequence of eigenvalues $\lambda_n \geqslant 0$, the critical buckling load P will be determined using the smallest positive eigenvalue. One must note, however, that sometimes the problem of finding y and λ is not equivalent to the problem of finding w and P. In fact $\lambda = 0$ is an eigenvalue of the system (10), (15) but $P = 0$ is not an eigenvalue of the system (1), (5). The eigenfunction of (10), (15) corresponding to $\lambda = 0$ is $y(x) = L - x$. In this case the critical buckling load must be determined using (11) and the second eigenvalue, say λ_2, of system (10), (15).

In a similar way $\lambda = 0$ is an eigenvalue of multiplicity 2 of system (10), (14). The two eigenfunctions are of the form $y = mx + b$. The critical buckling load must be determined using (11) and the third eigenvalue, say λ_3, of system (10), (14). It is also possible for the higher eigenvalues, λ_3, λ_4, etc., to be of multiplicity 2 in this case.

We summarize these observations in the following theorem:

THEOREM I. $\lambda = 0$ is an eigenvalue of multiplicity 2 for (10), (14).

$\lambda = 0$ is an eigenvalue of multiplicity 1 for (10), (15).

$\lambda = 0$ is not an eigenvalue of (10) subject to either (16) or (17).

$P = 0$ is never an eigenvalue of (1) subject to any of the boundary conditions (4), (5), (6) or (7).

A nonzero number λ is an eigenvalue of (10) subject to any of the boundary conditions (14), (15), (16) or (17) if and only if $P = \lambda EK$ is an eigenvalue of (1) subject to the corresponding boundary conditions (4), (5), (6) or (7). In any case the eigenfunctions $y(x)$ and $w(x)$ are related by

$$y(x) - y(0) - xy'(0) = P[w(x) - w(0) - xw'(0)].$$

Most of this theorem has already been proved. We just remark that the last equation is obtained by integrating (9) twice from 0 to x. Then if $P \neq 0$ we can solve for $w(x)$ terms of $y(x)$ which provides the equivalence of λ and P in the nonzero case.

III. Rearrangements of columns. Two columns having similar cross sectional shapes will be called equimeasurable if the corresponding area functions, say $A_1(x)$ and $A_2(x)$, are equimeasurable. That is, for all $t \geqslant 0$

$$\text{measure of } \{x \mid A_1(x) \geqslant t\} = \text{measure of } \{x \mid A_2(x) \geqslant t\}.$$

In this connection see [7, p. 276]. If two columns are equimeasurable then both will have the same amount of mass located within any given distance from the axis of the column, but they may have quite different overall shapes. In fact equimeasurability will be maintained if mass is moved parallel to the axis of the column in any manner which yields a similar cross sectional shape.

The problem which we consider in this work can now be stated in terms of equimeasurability as follows:

> Given a column having area function $A(x)$, find an equimeasurable column having similar cross sectional shape for which the critical buckling load is as large as possible and find also an equimeasurable column for which the critical buckling load is as small as possible.

For some types of boundary conditions this problem has been solved by Schwarz [12] using certain rearrangements of functions which we now recall (see also [1, 5]).

Given a function $f(x)$ defined on an interval $J = [0, L]$, we define some rearrangements of $f(x)$ which we denote by $f_+(x), f_-(x), f_{\pm n}(x), f_{-n}(x)$ and $\tilde{f}_{-n}(x)$ as follows:

(I) The functions f, f_+, f_-, f_{+n} and f_{-n} are all equimeasurable on J,

(II) f_+ is increasing and f_- is decreasing on J,

(III) $f_{\pm n}$ and $\tilde{f}_{-n}$ are periodic on J with period L/n,

(IV) $f_{\pm n}$ and $\tilde{f}_{-n}$ are symmetric on $0 \leqslant x \leqslant L/n$,

$$f_{\pm n}\left[L(2n)^{-1} - x\right] = f_{\pm n}\left[L(2n)^{-1} + x\right]$$

for all x with $0 \leqslant x \leqslant L/2n$.

(V) $f_{\pm n}(x)$ is decreasing for $0 \leqslant x \leqslant L/2n$,

(VI) $f_{-n}(x)$ and $\tilde{f}_{-n}(x)$ are increasing for $0 \leqslant x \leqslant L/2n$,

(VII) $\tilde{f}_{-n}(x)$ is only defined if df/dx is piecewise continuous and in this case $|df/dx|$ and $|d\tilde{f}_{-n}/dx|$ are equimeasurable.

(VIII) $\tilde{f}_{-n}(x)$ is concave on $0 \leqslant x \leqslant L/n$,

(IX) $\tilde{f}_{-n}(0) = \tilde{f}_{-n}(L/n) = 0$.

Basic properties of rearrangements are well known and may be collected from several sources (in particular [1, 5, 11, 12]). We will expound a few of the more important properties necessary for understanding the application to columns.

Give $f(x)$ defined on $0 \leqslant x \leqslant L$ then, $f_-(x)$ its rearrangement into decreasing order, is a decreasing function for which $f(x)$ and $f_-(x)$ are equimeasurable, that is for all $t \geqslant 0$,

$$\text{measure}\left\{x \mid f(x) \geqslant t\right\} = \text{measure}\left\{x \mid f_-(x) \geqslant t\right\}.$$

The rearrangement into increasing order $f_+(x)$ can be defined by

$$f_+(x) = f_-(L - x).$$

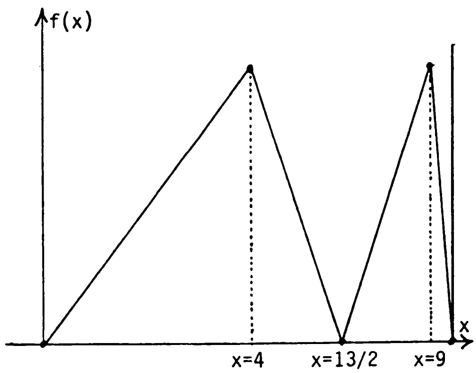

FIG. 1.

We will illustrate the various rearrangements of $f(x)$ using an example consisting of a piecewise linear function having four segments whose graph is in Fig. 1. This function $f(x)$ is defined for $0 \leqslant x \leqslant 10$ by

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant 4, \\ 8/10\,(13 - 2x), & 4 \leqslant x \leqslant 13/2, \\ 8/10\,(2x - 13), & 13/2 \leqslant x \leqslant 9, \\ 4(10 - x), & 9 \leqslant x \leqslant 10. \end{cases} \tag{18}$$

We see that $f(x)$ is continuous since $f(4 \pm 0) = f(9 \pm 0) = 4$ and $f(13/2 \pm 0) = 0$. It is easy to see that the measure of the set $\{x \mid f(x) \geqslant t\}$ is a linear function of t. It follows that $f_+(x) = (4/10)x$ and $f_-(x) = (4/10)(10 - x)$. Graphs of $f_+(x)$ and $f_-(x)$ are drawn as dotted lines in Fig. 2 and 3. From the definition of $f_{\pm n}(x)$ it follows that

$$f_{\pm n}(x) = f_{\mp}(2nx) \quad \text{for } 0 \leqslant x \leqslant L/2n. \tag{19}$$

Indeed (19) holds for any function $f(x)$ not only for our example. Using (19) and the symmetry and periodicity properties III and IV it is easy to draw the graphs of $f_{\pm 2}(x)$ and these are given in Figs. 2 and 3.

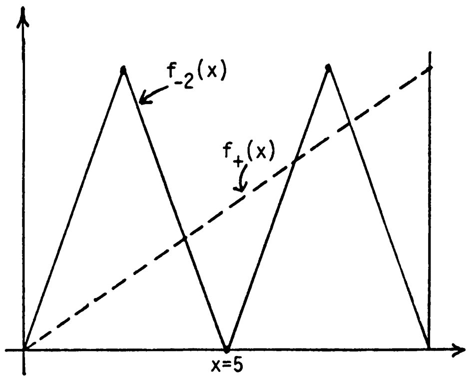

FIG. 2.

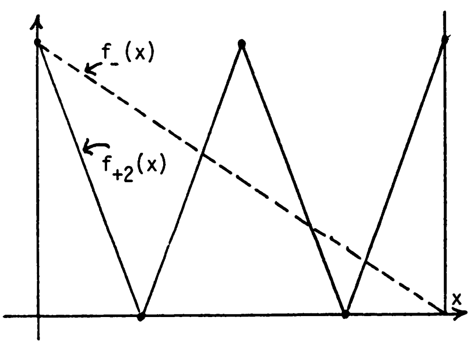

FIG. 3.

In a similar fashion the uniformly tapered column is rearranged and graphed in Figs. 5 and 6.

The functions $\tilde{f}_{-2}(x)$ involve more difficult manipulation but the basic idea is quite simple. Starting with a function $f(x)$ on $0 \leqslant x \leqslant L$ which satisfies $f(0) = f(x_1) = f(L) = 0$ and $f(\alpha_1) = f(\alpha_2)$ where $0 < \alpha_1 < x_1 < \alpha_2 < L$ we first compute $|df/dx|$. We next rearrange this function and then integrate it to obtain a function which we compare to the original $f(x)$. It turns out that a useful way to do this is to define $\tilde{f}_{-2}(x)$ by

$$\tilde{f}_{-2}(x) = \int_0^x |df/dx|_{+2}\, dx.$$

As Theorem 1 below shows it follows that $\tilde{f}_{-2}(x) \geqslant f_{-2}(x)$. Now $|df/dx|_{+2}$ has its largest values located at $x = 0$, $L/2$ and L. Thus the geometrical meaning of Theorem 1 is that to make a function grow most rapidly we put the large values of its derivative first. In terms of our example we see

$$|df/dx| = \begin{cases} 1, & 0 \leqslant x \leqslant 4, \\ 8/5, & 4 \leqslant x \leqslant 9, \\ 4, & 9 \leqslant x \leqslant 10. \end{cases}$$

Thus we see, for $0 \leqslant x \leqslant 10/4$, that

$$|df/dx|_{+2} = \begin{cases} 4, & 0 \leqslant x \leqslant 1/4, \\ 8/5, & 1/4 \leqslant x \leqslant 6/4, \\ 1, & 6/4 \leqslant x \leqslant 10/4. \end{cases}$$

Integrating this defines $\tilde{f}_{-2}(x)$ for $0 \leqslant x \leqslant 10/4$. Periodicity and symmetry are used to define $\tilde{f}_{-2}(x)$ for $x \geqslant 10/4$. Its graph is drawn in Fig. 4 as a solid line and compared with $f_{-2}(x)$ which is the dotted line. In general it follows that,

THEOREM II. Suppose $f(x)$ has a piecewise continuous derivative on J and has $n+1$ zeros $x_j \in J$, $0 = x_0 < x_1 < x_2 < \cdots < x_n = L$. Further suppose that in each interval $[x_{i-1}, x_i]$ that $f(x)$ increases to its maximum value at $x = \alpha_i$ and decreases in $[\alpha_i, x_i]$ and that $f(x)$ has the same maximum value in each interval so that

$$f(\alpha_i) = f(\alpha_j) \quad \forall i, j.$$

Then

$$\tilde{f}_{-n}(x) \geqslant f_{-n}(x).$$

For a proof of this result see [1].

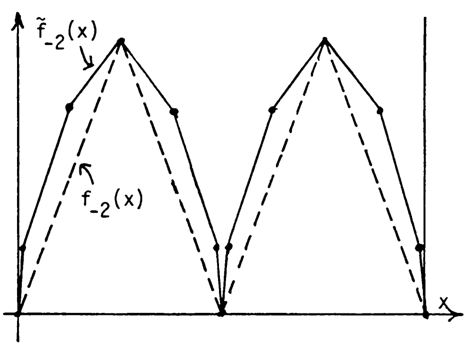

FIG. 4.

There are many integral inequalities dealing with rearrangements. One of the most important (given in [7, p. 278]) is

$$\int_J fg\, dx \leqslant \int_J f_+ g_+\, dx.$$

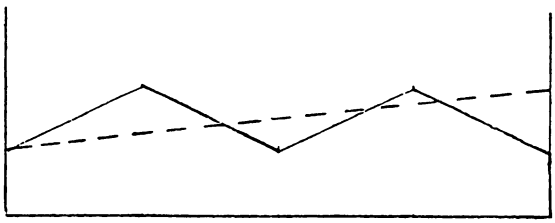

FIG. 5. The shape of a weaker column.

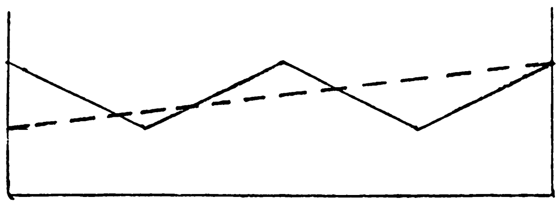

FIG. 6. The shape of a stronger column.

An example of Theorem II. A uniformly tapered column, clamped at both ends (indicated by the dotted lines — — — —) can be arranged to give either a stronger or weaker column.

Now f, f_+ and f_{-n} are all equimeasurable but f_{-n} is periodic. Similarly g_{-n} is periodic and the product $f_{-n}g_{-n}$ matches the large values of f with the large values of g, just like $f_+ g_+$ does. It follows that

$$\int_J fg\,dx \leq \int_J f_+ g_+ \,dx = \int_J f_{-n}g_{-n}\,dx. \tag{20}$$

A formal proof of this can be given based on (19).

Using the definition of equimeasurability and rearrangement it is not difficult to show that

$$\text{if } Q(x) = [A(x)]^{-2} \quad \text{then } Q_{\pm n}(x) = [A_{\mp n}(x)]^{-2}. \tag{21}$$

There are a number of other interesting facts involving rearrangements but these will be sufficient for our purposes. We now return to the column problem.

If we consider a column which is pinned at both ends then the buckling load is determined by (10), (16). It follows from the results of Schwarz [12] and (21) that the buckling load of a pinned column having area function $A(x)$ is bounded above by the buckling load of an equimeasurable column having area function $A_{-1}(x)$ and it is bounded below by the buckling load of an equimeasurable column having area function $A_{+1}(x)$.

In the same way it follows that the buckling load of a column having area function $A(x)$ which is clamped at $x = 0$ and free at $x = L$ is bounded above by that of an equimeasurable column having area function $A_-(x)$ and below by one having area function $A_+(x)$.

We now give a corresponding result in case the column is clamped at both ends.

THEOREM III. The buckling load of a column clamped at both ends and having area function $A(x)$ is bounded above by the buckling load of an equimeasurable column having area function $\bar{A}_{+2}(x)$.

It is bounded below by the buckling load of an equimeasurable column having area function $\bar{A}_{-2}(x)$.

We postpone the proof of Theorem III until Sec. V.

Other boundary conditions may also be considered. In particular the case of a column clamped at $x = 0$ and pinned at $x = L$ is of interest. However the results [15] indicate that the extremal functions for the second eigenvalue will not have the symmetry properties which might be expected. It appears however that one might be able to show that a point $x = \xi$ and a function, call it $A_\xi(x)$, exist which is an extremal of the second eigenvalue and which is symmetric about $x = \xi$ and $x = (L + \xi)/2$. In [15] the value of ξ used was $\xi = 0.22617L$. However in our case we must expect that even if such a function $A_\xi(x)$ exists that ξ will depend on the form of the function $A(x)$.

All of our results may be easily generalized to the equation

$$y'' + \lambda A^{-\alpha}(x)y = 0.$$

Various values of α other than 2 are of interest [13, p. 136]. In particular Theorem III is valid for any $\alpha > 0$.

Tadjbakhsh and Keller [15, p. 163] give a proof that the stationary value is a maximum. That proof however fails in case the column is clamped at both ends or in case it is clamped at one end and pinned at the other end. In these cases $\lambda = 0$ is the smallest eigenvalue and the variational methods used in the proof are invalid. It may be possible to adapt their method to deal with the higher eigenvalues but it is not at all clear how such a task might proceed.

On the other hand Theorem III provides (at least in the clamped clamped cased) an alternative proof since it allows us to restrict our search for the maximum buckling load to columns which are symmetric about $x = L/2$ and also about $x = L/4$ and $x = 3L/4$. The corresponding eigenfunction (even if the eigenvalue has multiplicity 2) may be taken to satisfy the boundary conditions

$$U'(0) = U(L/4) = U'(L/2) = U(3L/4) = U'(L) = 0.$$

This allows us to consider the clamped symmetric column as being composed of 4 distinct congruent columns each of which is clamped at one end and free at the other end. The methods used in [15] are valid for such columns and the proof is complete.

In addition the work by Earl R. Barnes [2] gives a general method for finding the maximum of the first eigenvalue λ_1 of equation 10 subject to various kinds of constraints on $A(x)$. It seems to be difficult to generalize his method to treat the higher eigenvalues λ_2, λ_3, etc. Now $\lambda_1 = 0$ for the column which is clamped at one end and either clamped or pinned at the other end. Thus the methods [2] do not apply to the buckling problem in these cases. However a combination of our Theorem II with his work [2, Thm. 4.2] shows that the shape of the strongest column clamped at both ends and satisfying $a \leqslant A(x) \leqslant b$ is symmetric about $x = L/4$, $L/2$ and $3L/4$. In each of the four intervals $[0, L/4]$, $[L/4, L/2]$, $[L/2, 3L/4]$, $[3L/4, L]$ it has the shape of the strongest clamped free column as determined in [2]. The determination of the strongest column clamped at one end and pinned at the other end satisfying $a \leqslant A(x) \leqslant b$ must still be regarded as an unsolved problem.

V. Proof of Theorem III. In order to indicate their dependence on the area function $A(x)$ we denote the nth eigenvalue λ_n of (10), (14) by $\lambda_n(A)$. Using this notation, Theorem III will follow (by letting $n = 2$) from the more general theorem:

THEOREM IV. Let $\lambda_n(A)$ be the nth eigenvalue of (10), (14). Then

$$\lambda_{n+1}(A_{-n}) \leqslant \lambda_{n+1}(A) \leqslant \lambda_{n+1}(A_{+n}).$$

We will give a proof of the upper bound on $\lambda_{n+1}(A)$ for all $n = 1, 2, 3, \ldots$. In the case of the lower bound $\lambda_{n+1}(A) \geqslant \lambda_{n+1}(A_{-n})$ we will give a proof only in the case $n = 2$. I have constructed a proof valid for all n but it is much more difficult than the one given here for $n = 2$ and will not be included since the higher eigenvalues have no physical significance in this context.

We will first prove $\lambda_{n+1}(A_{+n}) \geqslant \lambda_{n+1}(A)$. In addition to the results [12] we will need the following interlacing theorem due to Ettlinger [6]. See also Ince [8, pp. 252–253]. Let $\mu_n(A)$ be the nth eigenvalue of the system

$$U'' + \mu A^{-2}(x)U = 0, \qquad U'(0) = U'(L) = 0.$$

Then there are exactly 4 possible ways of interlacing the eigenvalues $\lambda_n(A)$ of (10), (14) and $\mu_n(A)$:

(Ia) $\mu_1 \leqslant \lambda_1 < \mu_2 < \lambda_2 \leqslant \mu_3 \leqslant \lambda_3 < \mu_4 < \lambda_4 < \cdots$,
(ib) $\lambda_1 \leqslant \mu_1 \leqslant \lambda_2 < \mu_2 < \lambda_3 \leqslant \mu_3 \leqslant \lambda_4 < \mu_4 < \cdots$,
(IIa) $\lambda_1 < \mu_1 < \lambda_2 \leqslant \mu_2 \leqslant \lambda_3 < \mu_3 < \lambda_4 \leqslant \mu_4 \leqslant \cdots$,
(IIb) $\mu_1 < \lambda_1 \leqslant \mu_2 \leqslant \lambda_2 < \mu_3 < \lambda_3 \leqslant \mu_4 \leqslant \lambda_4 < \cdots$.

However $\lambda_1 = \lambda_2 = 0$ and also $\mu_1 = 0$. Furthermore $\lambda_3 > 0$ and $\mu_2 > 0$. These conditions rule out cases Ia, IIa and IIb above. The only possible case is then 1b and we have $\lambda_n(A) \leqslant \mu_n(A)$ for $n = 1, 2, 3, \ldots$. We now appeal to the result of Schwarz [12, p. 417] which shows $\mu_{n+1}(A) \leqslant \mu_{n+1}(A_{+n})$. Now the function A_{+n} has symmetry properties which imply that if $W(x)$ is the eigenvalue corresponding to $\mu_{n+1}(A_{+n})$ then

$$W'(0) = W'(L) = 0, \qquad W(0) = W(L).$$

Therefore W will satisfy the boundary conditions (14) for a clamped column. Thus the symmetry of A_{+n} implies

$$\lambda_{n+1}(A_{+n}) = \mu_{n+1}(A_{+n}). \tag{22}$$

This yields $\lambda_{n+1}(A) \leqslant \lambda_{n+1}(A_{-n})$.

We now prove that $\lambda_3(A) \geqslant \lambda_3(A_{-2})$. Let y_n be the nth eigenfunction of system (10), (14). In particular we will take

$$y_1(x) = 1, \qquad y_2(x) = B - x.$$

The constant B is chosen so that y_1 and y_2 are orthogonal with respect to the function

$A^{-2}(x)$. Now the functions y_n are all orthogonal,

$$\int_0^L y_n(x) y_m(x) A^{-2}(x)\,dx = 0 \quad \text{if } n \neq m. \tag{23}$$

Define a new function $z(x)$ by

$$z = ay_1 + by_2 + y_3 \tag{24}$$

where a and b are constants to be determined later. The function $y_3(x)$ will have two zeros in $[0, L]$ say x_1 and x_2. Let α be the point in the interval $[x_1, x_2]$ at which $|y_3(x)|$ has its maximum value. Select the constants a and b so that

$$z(0) = z(L) \quad \text{and} \quad z(\alpha) = -z(0). \tag{25}$$

The function $z(x)$ will satisfy (14) since all $y_n(x)$ do. It follows that

$$z'(0) = z'(L) = 0. \tag{26}$$

Since $y_3(x)$ satisfies (10) with $\lambda = \lambda_3$ we see that (24) implies

$$z'' + \lambda_3 A^{-2} z = \lambda_3 A^{-2}[ay_1 + by_2].$$

We now multiply this equation by z and integrate the result from 0 to L;

$$\int_0^L zz''\,dx + \lambda_3 \int_0^L z^2 A^{-2}\,dx = \lambda_3 \int_0^L [ay_1 + by_2][ay_1 + by_2 + y_3] A^{-2}\,dx. \tag{27}$$

The orthogonality (23) implies that the right hand side of (27) simplifies to

$$\lambda_3 \int_0^L [ay_1 + by_2]^2 A^{-2}\,dx.$$

This is a nonnegative quantity so (27) implies that

$$\lambda_3 \int_0^L z^2 A^{-2}\,dx \geqslant -\int_0^L zz''\,dx.$$

Integrating the right hand side of this relationship by parts and using (26) we find that,

$$\lambda_3(A) \geqslant \frac{\int_0^L (z')^2\,dx}{\int_0^L z^2 A^{-2}\,dx}. \tag{28}$$

Now (25) implies that the function $z^2(x)$ has two zeros in $[0, L]$, say β, γ and that

$$z^2(0) = z^2(\alpha) = z^2(L). \tag{29}$$

We now cut off the right hand end of the column and weld it back on the left hand end. More precisely if we have a function $f(x)$ defined for $0 \leqslant x \leqslant L$ we define a new function $f_*(x)$ by

$$f_*(x) = \begin{cases} f(x+\gamma), & 0 \leqslant x \leqslant L - \gamma, \\ f(x + \gamma - L), & L - \gamma \leqslant x \leqslant L. \end{cases}$$

Now $f(x)$ and $f_*(x)$ are equimeasurable. Applying the $_*$ operation to both $z(x)$ and $A(x)$ and using (28) yields

$$\lambda_3(A) \geqslant \frac{\int_0^L (z_*')^2\,dx}{\int_0^L z_*^2 (A_*)^{-2}\,dx}. \tag{30}$$

Now the function z_* satisfies the hypothesis of Theorem I with $n = 2$. Thus

$$\left(\tilde{z}_*\right)_{-2} \geqslant (z)_{-2}, \qquad \int_0^L \left(\tilde{z}_*\right)'^2\,dx = \int_0^L \left(z_*\right)'^2\,dx.$$

In addition (20) yields

$$\int_0^L z_*^2\left(A_*\right)^{-2} dx \leqslant \int_0^L \left(z_*\right)_{-2}^2\left[\left(A_*\right)_{+2}\right]^{-2} dx \leqslant \int_0^L \left(\tilde{z}_*\right)_{-2}^2\left[\left(A_*\right)_{+2}\right]^{-2} dx.$$

Thus (30) implies

$$\lambda_3(A) \geqslant \frac{\int_0^L \left(\tilde{z}_*\right)'^2\,dx}{\int_0^L \left(\tilde{z}_*\right)_{-2}^2\left[\left(A_*\right)_{+2}\right]^{-2} dx}. \tag{31}$$

Since $\tilde{z}_*$ is continuous with a piecewise continuous derivative and vanishes at $L/2$, it follows that [4, p. 463] the right hand side of (31) is not less than the second eigenvalue say $\nu_2(A)$ of the system

$$V'' + \nu_2 Q(x) V = 0, \qquad V(0) = V(L) = 0, \tag{32}$$

where to simplify notation we have taken $Q(x) = [(A_*)_{+2}]^{-2}$.

We now cut off the left hand end of the column and weld it back onto the right hand end. That is given a function $f(x)$ defined for $0 \leqslant x \leqslant L$ we define a new function $f^*(x)$ by

$$f^*(x) = \begin{cases} f(x + L/4), & 0 \leqslant x \leqslant 3L/4, \\ f(x - 3L/4), & 3L/4 \leqslant x \leqslant L. \end{cases}$$

Applying this operation to (32) yields

$$(V^*)'' + \nu_2 Q^*(x) V^* = 0$$

and the symmetry of V implies that V^* satisfies (14). Therefore $\nu_2(Q) = \lambda_3(Q^*)$. Thus (31) implies $\lambda_3(A) \geqslant \nu_2(Q) = \lambda_3(Q^*)$. The symmetry of Q implies that $A_{-2} = Q^*$. Therefore $\lambda_3(A) \geqslant \lambda_3(A_{-2})$ which finishes the proof.

References

[1] D. C. Barnes, *Rearrangements of functions and lower bounds for eigenvalues of differential equations*, J. Appl. Anal. (to appear)

[2] E. R. Barnes, *The shape of the strongest column and some related extremal eigenvalue problems*, Quart. Appl. Math. **34**, 393–409 (1977)

[3] T. Clausen, *Uber die Form Architektonisher Saulen*, Bull. Phys.-Math. Astronum., Tome 1, 1849–1853

[4] R. Courant and D. Hilbert, *Methods of mathematical physics*, Interscience, New York, 1953

[5] G. F. D. Duff, *Differences, derivatives and decreasing rearrangements*, Canad. J. Math. **19**, 1153–1178 (1967)

[6] H. J. Ettlinger, *Existence theorems for the general real self-adjoint linear system of second order*, Trans. Amer. Math. Soc. **19**, 79 (1918)

[7] Hardy Littlewood and Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1964

[8] E. L. Ince, *Ordinary differential equations*, Dover, New York, 1956

[9] J. B. Keller, *The shape of the strongest column*, Arlch. Rational Mech. Anal. **5**, 275–285 (1960)

[10] J. L. Lagrange, *Sur la figure des colonnes*, Miscellanea Tourinesia (Royal Society of Turin), Tomus V 123 (1770–1773); Also Oevures Vol. 2, 125–170

[11] Pólya and Szegö, *Isoperimetric inequalities in mathematical physics*, Princeton, 1951

[12] B. Schwarz, *On the extrema of the frequencies of nonhomogeneous strings with equimeasurable density*, J. Math. Mech. **10**, 401 (1961)

[13] G. J. Simitses, *An introduction to the elastic stability of structures*, Prentice-Hall, Englewood Cliffs, N.J., 1976

[14] G. J. Simitses, M. P. Kamat and C. Y. Smith, *Strongest column by the finite element displacement method*, AIAA J. **11**, 1231 (1961)

[15] I. Tadjbakhsh and J. B. Keller, *Strongest columns and isoperimetric inequalities for eigenvalues*, J. Appl. Mech. **29**, 159 (1962)

[16] S. P. Timeshenko and J. M. Gere, *Theorey of elastic stability*, McGraw-Hill, New York, 1961