AN ALTERNATIVE PROOF OF THE REPRESENTATION THEOREM FOR ISOTROPIC, LINEAR ASYMMETRIC STRESS-STRAIN RELATIONS*

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THEOREM. Let $\mathbb{F} : \mathrm{Lin} \to \mathrm{Lin}$ be linear and isotropic. Then there are scalars λ, μ, and α such that

$$\mathbb{F} : \mathbf{T} = \lambda(\mathrm{tr}\ \mathbf{T})\mathbf{I} + (\mu + \alpha)\mathbf{T} + (\mu - \alpha)\mathbf{T}^* \qquad (\forall \mathbf{T} \in \mathrm{Lin}). \tag{1}$$

Conversely, any such function is linear and isotropic.

This statement, proved in [1], provides a general representation for linear isotropic tensor-valued functions of tensors. The statement (1) was given in coordinate form and proved in a different way earlier in [2]. Along lines of reasoning exploited in [3, 4], we offer in this brief note an alternative proof of the following.

REPRESENTATION THEROEM. For some set $\mathbb{S}\mathrm{et}$ of $\mathbb{L}\mathrm{in}$, the following statements are equivalent:

(i) $\mathbb{S}\mathrm{et} = \mathbb{I}\mathrm{st}$.
(ii) The subspaces Sph, Dev, and Skw of Lin are invariant characteristic spaces of $\mathbb{S}\mathrm{et}$.
(iii) There are scalars $\lambda(\mathbb{F})$, $\mu(\mathbb{F})$, and $\alpha(\mathbb{F})$ such that

$$\mathbb{F} : \mathbf{T} = \lambda(\mathrm{tr}\ \mathbf{T})\mathbf{I} + (\mu + \alpha)\mathbf{T} + (\mu - \alpha)\mathbf{T}^* \qquad (\forall \mathbb{F} \in \mathbb{S}\mathrm{et}, \mathbf{T} \in \mathrm{Lin}). \tag{2}$$

The notations used above and later on are as follows.

Let $\mathscr{R}$ be the space of scalars and $\mathscr{V}$ a 3-dimensional Euclidean space with scalar product $\mathbf{uv}$ and vector product $\mathbf{u} \wedge \mathbf{v}$. Lin is the space of all linear transformations (or simply tensors) on $\mathscr{V}$, with identity $\mathbf{I}$. Given a tensor $\mathbf{T}$, $\mathbf{T}^*$ denotes its transpose and tr $\mathbf{T}$ its trace. The transpose of the tensor product $\mathbf{u} \otimes \mathbf{v}$ is $(\mathbf{u} \otimes \mathbf{v})^* = \mathbf{v} \otimes \mathbf{u}$. Given an orthonormal basis $\{\mathbf{e}_i\}$ or $\mathscr{V}$, any $\mathbf{u} \in \mathscr{V}$ and $\mathbf{T} \in \mathrm{Lin}$ may be expressed in dyadic form:

$$\mathbf{u} = u_i \mathbf{e}_i, \qquad \mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{3}$$

and all the following operations may be interpreted in the language of scalar products of the corresponding basis vectors of the elements concerned, e. g.

$$\mathbf{uv} = (u_i \mathbf{e}_i)(v_j \mathbf{e}_j) = u_i v_j (\mathbf{e}_i \mathbf{e}_j) = u_i v_j \delta_{ij} = u_i v_i, \tag{4}$$

$$\mathbf{Tu} = (T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)(u_k \mathbf{e}_k) = T_{ij} u_k (\mathbf{e}_j \mathbf{e}_k)\mathbf{e}_i = T_{ij} u_j \mathbf{e}_i, \tag{5}$$

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$$\mathbf{uT} = (u_k \mathbf{e}_k)(T_{ji} \mathbf{e}_j \otimes \mathbf{e}_i) = u_k T_{ji} (\mathbf{e}_k \mathbf{e}_j) \mathbf{e}_i = u_j T_{ji} \mathbf{e}_i, \tag{6}$$

$$\mathbf{A} : \mathbf{B} = (A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j) : (B_{mn} \mathbf{e}_m \otimes \mathbf{e}_n) = A_{ij} B_{mn} (\mathbf{e}_i \mathbf{e}_m)(\mathbf{e}_j \mathbf{e}_n) = A_{ij} B_{ij}, \tag{7}$$

$$\mathbf{I} : \mathbf{T} = T_{ii} = \operatorname{tr} \mathbf{T}, \tag{8}$$

where $\mathbf{u}, \mathbf{v} \in \mathscr{V}$ and $\mathbf{A}, \mathbf{B}, \mathbf{T} \in$ Lin. Every $\mathbf{T}$ possesses at least one right proper direction $\mathbf{r}$ $(\neq \mathbf{0})$,

$$\mathbf{Tr} = \lambda \mathbf{r}, \tag{9}$$

and one left proper direction $\mathbf{l}$ $(\neq \mathbf{0})$,

$$\mathbf{lT} = \lambda \mathbf{l}, \tag{10}$$

λ being the associated proper value. If $\mathbf{r} = \mathbf{l}$, we call $\mathbf{r}$ simply a proper direction. Further, let Sym, Skw, Orth, Sph, and Dev denote, respectively, the sets of symmetric, skew, orthogonal, spherical (scalar multiples of $\mathbf{I}$), and deviatoric (traceless symmetric) tensors. Given noncoplanar $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathscr{V}$, every $\mathbf{A} \in$ Skw may be expressed as (with $\xi, \eta, \zeta \in \mathscr{R}$):

$$\mathbf{A} = \xi(\mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u}) + \eta(\mathbf{v} \otimes \mathbf{w} - \mathbf{w} \otimes \mathbf{v}) + \zeta(\mathbf{w} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{w}). \tag{11}$$

The space Lin may be viewed as the direct sum of subspaces Sph, Dev, and Skw:

$$\text{Lin} = \text{Sph} \oplus \text{Dev} \oplus \text{Skw}. \tag{12}$$

It means that every $\mathbf{T} \in$ Lin has a unique additive decomposition in the form

$$\mathbf{T} = \mathbf{S} + \mathbf{D} + \mathbf{A}, \tag{13}$$

where

$$\mathbf{S}[=\tfrac{1}{3}(\operatorname{tr} \mathbf{T})\mathbf{I}] \in \text{Sph}, \tag{14}$$

$$\mathbf{D}[=\tfrac{1}{2}(\mathbf{T} + \mathbf{T}^*) - \tfrac{1}{3}(\operatorname{tr} \mathbf{T})\mathbf{I}] \in \text{Dev}, \tag{15}$$

$$\mathbf{A}[=\tfrac{1}{2}(\mathbf{T} - \mathbf{T}^*)] \in \text{Skw}. \tag{16}$$

Now let $\mathbb{L}$in be the space of all linear transformations on Lin. We shall call them simply mappings, omitting the adjective "linear". The dyadic form of $\mathbb{F} \in \mathbb{L}$in is

$$\mathbb{F} = F_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \tag{17}$$

and the action of $\mathbb{F}$ upon $\mathbf{T} \in$ Lin is effectuated by means of a double dot (7) as follows:

$$\mathbb{F} : \mathbf{T} = (F_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) : (T_{mn} \mathbf{e}_m \otimes \mathbf{e}_n) = F_{ijkl} T_{kl} \mathbf{e}_i \otimes \mathbf{e}_j. \tag{18}$$

For the proof of the stated representation theorem, some definitions and lemmata are needed. Those lemmata already proved by other authors are indicated in every case.

Definition 1: A mapping $\mathbb{F}$ is isotropic if and only if

$$\mathbf{Q}(\mathbb{F} : \mathbf{T})\mathbf{Q}^* = \mathbb{F} : (\mathbf{QTQ}^*) \qquad (\forall \mathbf{T} \in \text{Lin}, \mathbf{Q} \in \text{Orth}). \tag{19}$$

The set of all isotropic mappings is denoted by $\mathbb{I}$st.

Definition 2: A scalar λ is called a proper value of a mapping $\mathbb{F}$ if there is a tensor $\mathbf{R}$ such that

$$\mathbb{F} : \mathbf{R} = \lambda \mathbf{R}. \tag{20}$$

We call $\mathbf{R}$ a proper direction corresponding to λ. The characteristic space for $\mathbb{F}$ corresponding to λ is the subspace of Lin consisting of all tensors satisfying (20). Generally, the characteristic spaces of different mappings are different. If every mapping from some set of $\mathbb{L}$in has the same characteristic space (the corresponding proper values may be distinct), we say that this characteristic space is invariant for this set of mappings.

LEMMA 1 (Rivlin-Ericksen, Serrin, Noll). Every proper direction of the symmetric or skew argument $\mathbf{T} \in$ Lin of an $\mathbb{F} \in \mathbb{I}$st is also a proper direction of the value $\mathbb{F} : \mathbf{T}$.

This lemma holds also for non-linear mappings. For the proof the reader is referred to to [5, p. 167].

LEMMA 2 (Lew). Let $\mathbf{D} \in$ Dev. Then there is an orthonormal basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ such that

$$\mathbf{D} = \xi(\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u}) + \eta(\mathbf{v} \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{v}) + \zeta(\mathbf{w} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{w}) \tag{21}$$

with $\xi, \eta, \zeta \in \mathscr{R}$.

The proof of this lemma was furnished by J. Lew to M. E. Gurtin in a private communication in 1968 (cf. [3, pp. 36–37]).

LEMMA 3. Sph is an invariant characteristic space for $\mathbb{I}$st.

Proof. Taking into account that every $\mathbf{S} \in$ Sph is a scalar multiple of $\mathbf{I}$, and the mapping is linear, it suffices at first to use the isotropy definition (19) for unity argument to get

$$\mathbf{Q}(\mathbb{F} : \mathbf{I}) = (\mathbb{F} : \mathbf{I})\mathbf{Q} \qquad (\forall \mathbb{F} \in \mathbb{I}\text{st}, \mathbf{Q} \in \text{Orth}). \tag{22}$$

Assume that $\mathbf{r}$ is a proper direction of the value $\mathbb{F} : \mathbf{I}$:

$$(\mathbb{F} : \mathbf{I})\mathbf{r} = \beta\mathbf{r}. \tag{23}$$

Then, from (22),

$$(\mathbb{F} : \mathbf{I})\mathbf{Q}\mathbf{r} = \mathbf{Q}(\mathbb{F} : \mathbf{I})\mathbf{r} = \beta\,\mathbf{Q}\mathbf{r}. \tag{24}$$

Because $\mathbf{Q}$ is arbitrary, (24) leads to the conclusion that every direction is a proper direction for $\mathbb{F} : \mathbf{I}$ with the same proper value $\beta(\mathbb{F})$; in other words,

$$\mathbb{F} : \mathbf{I} = \beta\,\mathbf{I}, \tag{25}$$

or, by virtue of the linearity of $\mathbb{F}$,

$$\mathbb{F} : \mathbf{S} = \beta(\mathbb{F})\mathbf{S} \qquad (\forall \mathbb{F} \in \mathbb{I}\text{st}, \mathbf{S} \in \text{Sph}). \tag{26}$$

Lemma 4. Let $\mathbb{F} \in \mathbb{I}$st. Then there are scalars $\varphi_{\pm}$(in general $\varphi_+ \neq \varphi_-$) such that

$$\mathbb{F} : (\mathbf{u} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{u}) = 2\varphi_{\pm}(\mathbf{u} \otimes \mathbf{v} \pm \mathbf{v} \otimes \mathbf{u}) \qquad (\forall \text{ non-collinear } \mathbf{u}, \mathbf{v} \in \mathscr{V}). \tag{27}$$

Proof. The argument $\mathbf{u} \otimes \mathbf{v} \pm \mathbf{v} \otimes \mathbf{u}$ has a proper direction

$$\mathbf{w} := \mathbf{u} \wedge \mathbf{v}. \tag{28}$$

By virtue of Lemma 1, we have

$$[\mathbb{F} : (\mathbf{u} \otimes \mathbf{v} \pm \mathbf{v} \otimes \mathbf{u})]\mathbf{w} = \mathbf{w}\mathbb{F} : (\mathbf{u} \otimes \mathbf{v} \pm \mathbf{v} \otimes \mathbf{u}) = \mathring{\beta}_{\pm}\,\mathbf{w}. \tag{29}$$

Referred to the basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, the value $\mathbb{F} : (\mathbf{u} \otimes \mathbf{v} \pm \mathbf{v} \otimes \mathbf{u})$ may be expressed in the dyadic form:

$$\begin{aligned}\mathbb{F} : (\mathbf{u} \otimes \mathbf{v} \pm \mathbf{v} \otimes \mathbf{u}) = {} & \beta_{11}\mathbf{u} \otimes \mathbf{u} + \beta_{12}\mathbf{u} \otimes \mathbf{v} + \beta_{13}\mathbf{u} \otimes \mathbf{w} + \beta_{21}\mathbf{v} \otimes \mathbf{u} \\ & + \beta_{22}\mathbf{v} \otimes \mathbf{v} + \beta_{23}\mathbf{v} \otimes \mathbf{w} + \beta_{31}\mathbf{w} \otimes \mathbf{u} + \beta_{32}\mathbf{w} \otimes \mathbf{v} + \beta_{33}\mathbf{w} \otimes \mathbf{w}.\end{aligned} \tag{30}$$

Generally, the β_{ij} $(i, j = 1, 2, 3)$ depend on $\mathbf{u}$ and $\mathbf{v}$, and of course on the sign "$\pm$". Inserting (30) into (29), we obtain

$$\beta_{13}\mathbf{u} + \beta_{23}\mathbf{v} + (\beta_{33} - \beta_{\pm})\mathbf{w} = \mathbf{0}, \tag{31}$$

$$\beta_{31}\mathbf{u} + \beta_{32}\mathbf{v} + (\beta_{33} - \beta_{\pm})\mathbf{w} = \mathbf{0}. \tag{32}$$

The linear independence of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, leads to

$$\beta_{13} = \beta_{31} = \beta_{23} = \beta_{32} = 0, \tag{33}$$

$$\beta_{33} = \beta_{\pm}(\mathbf{u}, \mathbf{v}) = \mathring{\beta}_{\pm}/|\mathbf{w}|^2. \tag{34}$$

Taking these results and

$$\mathbf{Q}(\mathbf{u} \otimes \mathbf{v} \pm \mathbf{v} \otimes \mathbf{u})\mathbf{Q}^* = (\mathbf{Qu}) \otimes (\mathbf{Qv}) \pm (\mathbf{Qv}) \otimes (\mathbf{Qu}), \tag{35}$$

$$(\mathbf{Qu}) \wedge (\mathbf{Qv}) = \mathbf{Q}(\mathbf{u} \wedge \mathbf{v}) = \mathbf{Qw} \tag{36}$$

into account, and applying the isotropy definition (19) to the expression (30), we have, $\forall \mathbf{Q} \in \mathrm{Orth}$,

$$\begin{aligned}(\beta_{11} - \beta_{11}^{Q})(\mathbf{Qu}) \otimes (\mathbf{Qu}) + (\beta_{12} - \beta_{12}^{Q})(\mathbf{Qu}) \otimes (\mathbf{Qv}) \\ + (\beta_{21} - \beta_{21}^{Q})(\mathbf{Qv}) \otimes (\mathbf{Qu}) + (\beta_{22} - \beta_{22}^{Q})(\mathbf{Qv}) \otimes (\mathbf{Qv}) \\ + [\beta_{\pm}(\mathbf{u}, \mathbf{v}) - \beta_{\pm}(\mathbf{Qu}, \mathbf{Qv})](\mathbf{Qw}) \otimes (\mathbf{Qw}) = \mathbf{0},\end{aligned} \tag{37}$$

where $\beta_{ij}^{Q} = \beta_{ij}(\mathbf{Qu}, \mathbf{Qv})$. By virtue of the linear independence of the dyadic basis above, we get from (37)

$$\beta_{ij}(\mathbf{u}, \mathbf{v}) = \beta_{ij}(\mathbf{Qu}, \mathbf{Qv}) \qquad (i, j = 1, 2), \tag{38}$$

$$\beta_{\pm}(\mathbf{u}, \mathbf{v}) = \beta_{\pm}(\mathbf{Qu}, \mathbf{Qv}). \tag{39}$$

Since $\mathbf{Q}$ is arbitrary, these coefficients depend on the sign "$\pm$" only. Thus, (30) reduces to

$$\begin{aligned}\mathbb{F} : (\mathbf{u} \otimes \mathbf{v} \pm \mathbf{v} \otimes \mathbf{u}) = \beta_{11}\mathbf{u} \otimes \mathbf{u} + \beta_{12}\mathbf{u} \otimes \mathbf{v} + \beta_{21}\mathbf{v} \otimes \mathbf{u} + \beta_{22}\mathbf{v} \otimes \mathbf{v} \\ + \beta_{\pm}(\mathbf{u} \wedge \mathbf{v}) \otimes (\mathbf{u} \wedge \mathbf{v}).\end{aligned} \tag{40}$$

The assumed linearity of $\mathbb{F}$ makes it also linear relative to $\mathbf{u}$ and $\mathbf{v}$; therefore

$$\beta_{11} = \beta_{22} = \beta_{\pm} = 0, \tag{41}$$

and, consequently,

$$\mathbb{F} : (\mathbf{u} \otimes \mathbf{v} \pm \mathbf{v} \otimes \mathbf{u}) = \beta_{12}\mathbf{u} \otimes \mathbf{v} + \beta_{21}\mathbf{v} \otimes \mathbf{u}. \tag{42}$$

Using the linearity again with (42), we have

$$\mathbb{F} : (\mathbf{u} \otimes \mathbf{v} \pm \mathbf{v} \otimes \mathbf{u}) = \pm\mathbb{F} : (\mathbf{v} \otimes \mathbf{u} \pm \mathbf{u} \otimes \mathbf{v}) = \pm(\beta_{21}\mathbf{v} \otimes \mathbf{u} + \beta_{21}\mathbf{u} \otimes \mathbf{v}). \tag{43}$$

A comparison of (42) and (43), by virtue of the linear independence of $\mathbf{u} \otimes \mathbf{v}$ and $\mathbf{v} \otimes \mathbf{u}$, gives

$$\beta_{12} = \pm\beta_{21} \equiv 2\varphi_{\pm} \tag{44}$$

and, finally, from (42) the representation (27).

Introducting the notation

$$\mu(\mathbb{F}) = \varphi_{+}, \qquad \alpha(\mathbb{F}) = \varphi_{-}, \tag{45}$$

we apply (27) in sequence to the expressions (21) and (11); because $\mathbb{F}$ is linear, we get the following

COROLLARY. Dev and Skw are invariant characteristic spaces for $\mathbb{I}$st, i.e., $\forall\ \mathbb{F} \in \mathbb{I}$st,

$$\mathbb{F} : \mathbf{D} = 2\mu(\mathbb{F})\mathbf{D} \qquad (\forall \mathbf{D} \in \mathrm{Dev}), \tag{46}$$

$$\mathbb{F} : \mathbf{A} = 2\alpha(\mathbb{F})\mathbf{A} \qquad (\forall \mathbf{A} \in \mathrm{Skw}), \tag{47}$$

At this point we are already able to prove the representation theorem. On the basis of Lemma 3 and corollary, (i) implies (ii) directly. In order to show the implication (iii) by (ii), we assume that for $\mathbb{F} \in \mathbb{S}$et, $3k(\mathbb{F})$, $2\mu(\mathbb{F})$ and $2\alpha(\mathbb{F})$ are the proper values corresponding to the characteristic spaces Sph, Dev and Skw, respectively. Introducing the scalar $\lambda(\mathbb{F})$:

$$3k(\mathbb{F}) = 2\mu(\mathbb{F}) + 3\lambda(\mathbb{F}), \tag{48}$$

and taking the linearity of $\mathbb{F}$ and (14–16) into account, we use Lemma 3 and the Corollary to $\mathbf{T} \in$ Lin expressed in (13) and obtain

$$\begin{aligned}\mathbb{F} : \mathbf{T} &= \tfrac{1}{3}(\mathrm{tr}\ \mathbf{T})(2\mu + 3\lambda)\mathbf{I} + \mu(\mathbf{T} + \mathbf{T}^*) - \frac{2\mu}{3}(\mathrm{tr}\ \mathbf{T})\mathbf{I} + \alpha(\mathbf{T} - \mathbf{T}^*) \\ &= \lambda(\mathrm{tr}\ \mathbf{T})\mathbf{I} + \mu(\mathbf{T} + \mathbf{T}^*) + \alpha(\mathbf{T} - \mathbf{T}^*).\end{aligned} \tag{49}$$

This is just the expression (2) in (iii). Finally, to verify the implication (iii) $\Rightarrow$ (i) means to check whether every $\mathbb{F}$ of the form (2) satisfies the isotropy condition (19). To this end, taking $\mathrm{tr}\ (\mathbf{QTQ}^*) = \mathrm{tr}\ \mathbf{T}\ \forall\ \mathbf{Q} \in$ Orth into account, from (2) we have

$$\begin{aligned}\mathbb{F} : (\mathbf{QTQ}^*) &= \lambda(\mathrm{tr}\,\mathbf{T})\mathbf{I} + (\mu + \alpha)\mathbf{QTQ}^* + (\mu - \alpha)\mathbf{QT}^*\mathbf{Q}^* \\ &= \mathbf{Q}[\lambda(\mathrm{tr}\ \mathbf{T})\mathbf{I} + (\mu + \alpha)\mathbf{T} + (\mu - \alpha)\mathbf{T}^*]\mathbf{Q}^* = \mathbf{Q}(\mathbb{F} : \mathbf{T})\mathbf{Q}^*,\end{aligned} \tag{50}$$

which shows that (iii) $\Rightarrow$ (i).

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