THE DYNAMICS OF SIMPLE FLUIDS IN STEADY CIRCULAR SHEAR*

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Abstract. An isotropic simple fluid of constant density ρ is confined between two infinite horizontal planes which rotate steadily about separate vertical (z) axes with common angular velocity ω. We show that at least one solution to the exact equations of motion is determined by the differential equation

$$\frac{d}{dz}\left(\eta \frac{du}{dz}\right) - i\rho\omega u = 0$$

where $u(z)$ is a complex variable representing the horizontal velocity and $\eta(|du/dz|, \omega)$ is a complex shear modulus. This equation represents the extension to nonlinear viscoelasticity of the previous works of Berker and of Abbott and Walters on linear viscoelastic fluids, for which η reduces to the usual dynamic viscosity $\eta^*(\omega)$.

We cite two examples for the form of η which emerge from particular rheological models; and, without attempting to solve the above equation, we briefly discuss certain of its global features.

1. Introduction. In the science of rheology it is generally useful to know whether a given type of material deformation or motion can be realized in the laboratory for a broad class of materials. If so, then such a motion represents a potential means of investigating material response in the given type of deformation. The related mathematical idea of "controllable motions" has been rather exhaustively explored by Pipkin and coworkers (Pipkin [1], Yin and Pipkin [2], Pipkin and Tanner [3]) for the class of steady, uniaxial simple shearing motions designated as *viscometric flows*. However, such motions represent but a small subset of a larger category of generally unsteady shearing motions which, under the designation "simple shear", have been systematically classified in a recent publication by the present author [4]. In that work, it is shown that simple shear can be classified into two types, (a) *cylindrical shear* or (b) *laminar shear*, according to whether, by choice of an appropriate reference frame, (a) the flow can be made to remain locally parallel to the same (parallel) bundle of material curves, or else (b) the associated velocity gradient can be made to remain locally perpendicular to the same sheaf of material surfaces. In the special case of materially steady motions both of the above simple shears are equivalent to the same viscometric flow.

In general, certain of the above laminar shears can be generated by the relative motion

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of rigid parallel boundaries and, hence, like several of the viscometric flows, may be amenable to a comparatively straightforward experimental realization. One interesting example is the special case of steady circular shear, of the type generated in the so-called orthogonal or eccentric disc rheometer. Under the assumption of negligible edge effects and negligible inertia, this device is known to generate a spatially homongeneous deformation in general materials, a motion which is an important special case of the so-called motions with constant stretch history (Huilgol [5], Walters [6], Goddard [4, 7]). As discussed in previous works, this motion consists of the rigid body rotation of a parallel sheaf of horizontal material planes, $z=$ constant, each of which rotates steadily, with common angular velocity ω, about one of the vertical axes situated along $x=0$, $y=(\kappa/\omega)z$, where κ is a constant shear rate. Thus, the associated velocity gradient $\mathbf{L}=(\nabla\mathbf{v})^T$ has the matrix of Cartesian components

$$[\mathbf{L}]=\begin{bmatrix} 0 & -\omega & \kappa \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.1}$$

Predicated on the condition of negligible inertia, the above type of deformation corresponds to the idea of a "partially controllable motion" put forth by Pipkin and Tanner [3]. However, in practice it is desirable to assess the degree to which inertia causes departures from the idealized flow, especially at high rates of deformation, a point which motivates in large part the previous analysis by Abbott and Walters [8] of steady circular shear for the special case of a Newtonian liquid. We recall that their analysis, which provides an *exact* solution to the Navier-Stokes equation for the flow between infinite parallel planes, indicates that although the flow field becomes vertically nonhomogeneous owing to inertially driven shear components, it can be expressed in the form (1.1) for steady circular shear at any given vertical station z, with vertically uniform angular velocity ω but with variable shear rate $\kappa(z)$. Thus, the kinematics are a special case of the "pseudoplanar" flows discussed by Berker [9, 10], whose applications to the Navier-Stokes equations apparently predates that of Abbott and Walters. Among other things, however, the latter also covers the case of general linear viscoelastic fluids and solids, by means of the device of a "complex viscosity" (Abbott and Walters [8], Waterman [11]).[1]

In the present work, we wish to show that the rather remarkable simplicity inherent in the results of Berker and of Abbott and Walters, being mainly a consequence of kinematical and material symmetry, carries over to general nonlinear isotropic materials. In so doing, we shall derive a dynamical equation accounting both for material inertia and nonlinear viscoelasticity as a notably direct extension of the above works.[2]

2. Reduction of the material functions for laminar shear. We adopt here the terminology employed in [4][3] to describe simple shearing motions. Thus, a *laminar shear* is a

[1] As pointed out by Berker [10], Drouot [12] has given an analysis of the fluids of second grade in pseudo-planar flows, including a treatment of the flow we designate here as circular shear. On the basis of the work of Abbott and Walters and the present work, it becomes apparent that the terms denoted by f'' and g'' in Drouot's equations for circular shear (the second set on p. 302) are in error and should be replaced by f''' and g''' (equivalent to U''' and V''' of the present work).

[2] On the basis of various past works (cf., Knight [13]), we would not expect a similar degree of success with the related problem involving a lag in angular velocity between the bounding surfaces.

[3] See pp. 165–172, where the third line following Eqs. (2.14) on p. 167 should read "... both $\mathbf{P}_i$ and $\mathbf{P}_j$ the plane of ..." instead of "... both the plane ...".

motion such that for each material particle there exists a frame of reference in which the velocity gradient is given by

$$\mathbf{L} = \mathbf{K}(t) \tag{2.1}$$

where $\mathbf{K}(t)$ has the matrix of components

$$[\mathbf{K}] = \begin{bmatrix} 0 & 0 & \kappa_1(t) \\ 0 & 0 & \kappa_2(t) \\ 0 & 0 & 0 \end{bmatrix} \tag{2.2}$$

relative to an orthonormal basis fixed in the given frame. The scalars $\kappa_1(t)$ and $\kappa_2(t)$ in (2.2) represent orthogonal shears which, like the underlying basis, may depend on the material particle in question, except for the special case of a materially homogeneous deformation.

Whenever the shear rates in (2.1) are simple periodic functions of t, we say the motion is an *oscillatory laminar shear*, the most elementary form of which is the oscillatory uniaxial shear:

$$\kappa_1(t) = \kappa \cos \omega t, \qquad \kappa_2(t) \equiv 0 \tag{2.3}$$

where κ is a constant.

The steady circular shear of (1.1) can also be reduced to the form of an oscillatory laminar shear, by choosing a frame of reference which rotates with angular velocity ω about the z axis, such that the z-component of vorticity is reduced identically to zero. In this frame, the shear rates in (2.1) take on the special forms

$$\kappa_1(t) = \kappa \cos \omega t, \qquad \kappa_2(t) = -\kappa \sin \omega t \tag{2.4}$$

where κ is the constant appearing in (1.1).

It will be noted that both the motions (2.3) and (2.4) are special cases of the most general simply-periodic laminar shear, in which two oscillatory uniaxial shears, having different amplitude and common frequency, are orthogonally superposed. This produces a motion which can appropriately be called "elliptical shear". However, we restrict attention here to the special case (2.4) and to the corresponding stress pattern for an isotropic simple fluid.

As shown in the Appendix, in a corrected version of this author's previous work [4], the components of stress on the orthonormal basis that corresponds to the kinematical representation (1.1) can be written down in terms of the *three* material *functionals* appropriate to a general laminar shear and, hence in terms of *three* material *functions* depending on κ and ω appropriate to steady circular shear.

With the correspondence $(1, 2, 3) = (x, y, z)$, the relevant stresses T_{ij} can be expressed as

$$\begin{gathered} T_{xz} = \tau(\kappa, \omega) = \eta(\kappa, \omega)\kappa, \qquad T_{yz} = \sigma(\kappa, \omega) = -\xi(\kappa, \omega)\kappa, \\ T_{yy} - T_{zz} = -N_2(\kappa, \omega), \qquad T_{xx} - T_{zz} = N_1(\kappa, \omega), \qquad T_{xy} = S(\kappa, \omega), \end{gathered} \tag{2.5}$$

where the notation for the stress functions τ, N_1, N_2, σ, S corresponds to that employed in [7]. In addition, we have for the present purposes introduced "viscosity" functions, or moduli η, ξ.

The functions in (2.5) are given explicitly in terms of the three functionals, ℓ, n_1, n_2, either by Eqs. (A.6) of the Appendix or else by Eqs. (A.10) for the basic stress functions $\hat{\tau}$,

$\hat{N}$, $\hat{S}$, together with the relations given previously in [7]. As discussed in the latter work and in related publications, and as is evident from the relations given here in the Appendix, the limit $\omega \to 0$ with κ fixed corresponds to a *viscometric flow*:

$$\begin{gathered}\tau(\kappa, \omega) \to \tau^{(0)}(\kappa) = \eta^{(0)}(\kappa)\kappa, \qquad N_1(\kappa, \omega) \to N_1^{(0)}(\kappa), \\ N_2(\kappa, \omega) \to N_2^{(0)}(\kappa), \qquad \sigma \to 0 \quad \text{and} \quad S \to 0,\end{gathered} \tag{2.6}$$

where superscript zero denotes the usual viscometric shear-stress, normal-stress and viscosity functions.

Another notable limit, *linear viscoelasticity*, corresponds to $\kappa \to 0$ with ω fixed. Provided the basic functionals are suitably well-behaved, the first two stresses in (2.5) are $O(\kappa)$ and the last three are $O(\kappa^2)$. Thus, if we define for arbitrary ω, κ a complex viscosity function as

$$\eta^*(\kappa, \omega) = \eta(\kappa, \omega) - i\xi(\kappa, \omega) \tag{2.7}$$

with $i = \sqrt{-1}$, then the first two members of (2.5) can be written formally as

$$\tau^* = T_{xz} + iT_{yz} = \eta^*\kappa \tag{2.8}$$

and, in the limit $\kappa \to 0$, the $O(\kappa)$ stresses are determined by

$$\eta^*(\kappa, \omega) \to \eta_0^*(\omega) \tag{2.9}$$

where $\eta_0^*(\omega)$ denotes the usual complex viscosity function of linear viscoelasticity. In Sec. 3 below we shall develop a convenient extension of the notation (2.8). First, however, we consider for illustrative purposes the specific forms of the functions (2.5) appropriate to some special fluid models.

Material functions for special fluids. To gain some idea of their relative complexity for circular shear, let us consider two of the more well-known types of single-integral models for incompressible simple fluids, first a general type of *codeformational* model and then a *corotational* model (cf., Bird et al. [14]). In the first model, we take

$$\mathbf{T} + p\mathbf{I} = \int_0^\infty [M_1(s)\mathbf{C}^{-1}(s) + M_2(s)\mathbf{C}(s)]\, ds \tag{2.10}$$

where $\mathbf{C}(s) = \mathbf{F}_t^T(t-s)\mathbf{F}_t(t-s)$, $\mathbf{F}$ denotes the deformation-gradient tensor, and $M_1(s)$, $M_2(s)$ are memory functions depending on the time lapse s, functions which may also be allowed to depend on the scalar invariants of $\mathbf{C}(s)$. In the special case of simple shear where the velocity gradient has the form (2.1), with $\mathbf{K}(t)$ determined by (2.2) for laminar shear and (A.1) for cylindrical shear, the deformation gradient in (2.10) is readily found to satisfy the following (cf. [4]):

$$\mathbf{F}_t(t-s) = \exp \int_t^{t-s} \mathbf{L}(t')\, dt' = \mathbf{I} + \int_t^{t-s} \mathbf{K}(t')\, dt', \tag{2.11}$$

$$\mathbf{F}_t^{-1}(t-s) = \mathbf{I} - \int_t^{t-s} \mathbf{K}(t')\, dt'. \tag{2.12}$$

Upon substitution of (2.11) and (2.12) into (2.10), one finds for the functionals ℓ, n_1, n_2 of

(A.5), that

$$\begin{aligned}
\ell\{\kappa^{(2)} \mid \mathbf{A}(\cdot)\} &= \int_0^\infty \int_0^s [M_1(s) - M_2(s)]\mathbf{A}(t-s')\, ds'\, ds, \\
n_1\{\kappa^{(2)} \mid \mathbf{K}(\cdot)\mathbf{K}^T(*)\} &= \int_0^\infty \int_0^s \int_0^s M_1(s)\mathbf{K}(t-s')\mathbf{K}^T(t-s'')\, ds''\, ds'\, ds, \\
n_2\{\kappa^{(2)} \mid \mathbf{K}^T(\cdot)\mathbf{K}(*)\} &= \int_0^\infty \int_0^s \int_0^s M_2(s)\mathbf{K}^T(t-s')\mathbf{K}(t-s'')\, ds''\, ds'\, ds.
\end{aligned} \tag{2.13}$$

These are evidently all independent of their first argument $\kappa^{(2)}$ unless M_1 or M_2 are assumed to depend on the invariants of $\mathbf{C}(s)$, which of course are functions of $\kappa_1(t-s)$ and $\kappa_2(t-s)$ for the special case of simple shear.

From (2.13) together with (A.6) of the Appendix, one readily obtains expressions for the material functions in (2.5). Without recording all of these, we merely note that η^* of (2.7) is given by (A.8) as the Fourier integral

$$\eta^*(\kappa, \omega) = \int_0^\infty [M_1(s) - M_2(s)]e^{i\omega s}\, ds. \tag{2.14}$$

Thus, unless M_1 or M_2 are allowed to depend on $\mathbf{C}(s)$, this function uis identical with its limiting viscoelastic form in (2.9). The corresponding fluid model is therefore relatively simple in form and, conversely, limited in its ability to describe certain differences between laminar and cylindrical shear or other nonlinear effects, such as "shear-thinning", the latter deficiency being already known in the case of viscometric flows.

We turn next to the *corotational* model [7, 14]:

$$\mathbf{T} + p\mathbf{I} = \int_0^\infty M(s)\mathbf{E}(s)\, ds \tag{2.15}$$

where $\mathbf{E}$ is the (relative) "corotational" strain:

$$\begin{aligned}
\mathbf{E}(s) &= \int_0^s \mathbf{Q}^T(s')\mathbf{D}(t-s')\mathbf{Q}(s')\, ds', \\
\frac{d\mathbf{Q}(s)}{ds} &= -\mathbf{\Omega}(t-s)\mathbf{Q}(s)
\end{aligned} \tag{2.16}$$

with $\mathbf{Q}(0) = \mathbf{I}$. Here, $\mathbf{D}(t)$ is the deformation rate or symmetric part of $\mathbf{L}(t)$, $\mathbf{\Omega}(t)$ the vorticity or the antisymmetric part of $\mathbf{L}(t)$, and $\mathbf{Q}(s)$ the (relative) mean-rotation tensor.

The above corotational model appears sufficiently complex to prevent a general reduction like (2.13) for an arbitrary simple shear. However, for the steady circular shear in (1.1) both $\mathbf{D}$ and $\mathbf{\Omega}$ are represented by elementary constant matrices. Hence, it follows that

$$\mathbf{Q}(s) = \mathbf{I} - \alpha(s)\mathbf{\Omega} + \beta(s)\mathbf{\Omega}^2, \tag{2.17}$$

where

$$\alpha(s) = \frac{\sin \lambda s}{\lambda}, \qquad \beta(s) = \frac{1 - \cos \lambda s}{\lambda^2}, \quad \text{and} \quad \lambda^2 = \omega^2 + \tfrac{1}{4}\kappa^2.$$

With some straightforward matrix algebra, it is possible to work out the stress components in (2.15) and, hence, the material functions in (2.5). Once again, we record only those functions associated with (2.7):

$$\begin{aligned}\eta(\kappa, \omega) &= \eta_0[1 - \tfrac{1}{2}\langle\alpha^2\rangle\kappa^2 - \langle\beta\rangle\omega^2]\\ \xi(\kappa, \omega) &= \eta_0[\langle\alpha\rangle - \tfrac{1}{2}\langle\alpha\beta\rangle\kappa^2]\omega\end{aligned} \tag{2.18}$$

where we employ the notation

$$\eta_0 = \int_0^\infty \int_{s'}^\infty M(s)\, ds\, ds', \tag{2.19}$$

for the "zero-shear" viscosity, and

$$\langle f\rangle = \frac{1}{\eta_0}\int_0^\infty \int_{s'}^\infty M(s) f(s')\, ds'\, ds. \tag{2.20}$$

To obtain a better appreciation of (2.18), it is useful to consider the special case where the memory function has the form corresponding to the "corotational Maxwell model" [14]:

$$M(s) = \frac{\mu}{\tau}\, e^{-s/\tau} \tag{2.21}$$

where μ denotes a modulus and τ a relaxation time, both constants. After further algebra, (2.18) can be reduced to

$$\begin{aligned}\eta(\kappa, \omega) &= \frac{\eta_0}{1 + 4(\lambda\tau)^2}\left\{1 + \frac{3(\omega\tau)^2}{1 + (\lambda\tau)^2}\right\},\\ \xi(\kappa, \omega) &= \frac{\eta_0\, \omega\tau}{1 + (\lambda\tau)^2}\left\{1 - \frac{3}{2}\,\frac{(\kappa\tau)^2}{1 + 4(\lambda\tau)^2}\right\},\end{aligned} \tag{2.22}$$

where we recall that λ is given by (2.17). It is evident that the material functions of (2.22), which could be generalized to reflect a sum of terms like (2.21), exhibit a variety of nonlinear shear effects.

We turn now to a consideration of the dynamical equations for steady circular shear.[4]

3. Dynamics. Consider now an homogeneous, incompressible and isotropic fluid sandwiched between solid surfaces in the form of infinite parallel planes at $z = -h/2$ and $z = h/2$, which, viewed from an inertial frame of reference, rotate steadily with common angular velocity ω about vertical axes situated at $(x, y) = (0, -\gamma h/2)$ and $(0, \gamma h/2)$, respectively. With u, v, w denoting the Cartesian components of velocity on the system x, y, z, we assume that the fluid adheres to the bounding surfaces, so that

$$u = -\omega\left(y \pm \frac{\gamma h}{2}\right), \qquad v = \omega x, \qquad w = 0, \quad \text{at} \quad z = \mp h/2 \tag{3.1}$$

In view of (3.1) and the previous analysis of Abbott and Walters [8], we shall assume

[4] It should become evident from the following analysis and the preceding works [8–12] that the dynamical equations for any simple fluid exhibiting linear-viscoelastic shear response in laminar shear can be solved exactly for all pseudo-planar flows and several other laminar shearing flows [9, 10, 12]. The solution is readily effected by means of a Fourier analysis involving the shear modulus of (2.14).

that the velocity field is steady and can be represented by the simple form

$$\begin{aligned} u(x, y, z) &= U(z) - \omega y \\ v(x, y, z) &= V(z) + \omega x \qquad \text{for } -h/2 \le z \le h/2 \\ w &\equiv 0 \end{aligned} \tag{3.2}$$

where, because of (3.1),

$$V(\pm h/2) = 0, \qquad U(\pm h/2) = \pm \gamma \omega h/2. \tag{3.3}$$

We note that the velocity components (U, V) are equivalent to the set (A, B) of Abbott and Walters.

The velocity gradient **L** associated with (3.2) obviously has a matrix of (xyz) components

$$[\mathbf{L}] = \begin{bmatrix} 0 & -\omega & U'(z) \\ \omega & 0 & V'(z) \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.4}$$

In a frame rotating with the bounding surfaces, it is easy to show that this velocity gradient can be reduced to the form of the periodic laminar shear given by (2.1), (2.2) and (2.4), with vertically non-homogeneous shear rate:

$$\kappa = \kappa(z) = \sqrt{(U')^2 + (V')^2}. \tag{3.5}$$

Furthermore, at any station z, it is possible to choose a new set of Cartesian coordinates $\hat{x}$, $\hat{y}$, z, related to the above set x, y, z by rotation about the common axis z through an angle $\hat{\theta}(z)$, such that (3.4) can be reduced to the form (1.1). By well-known rules of tensor transformation, the stresses (2.5) can then also be transformed to the coordinates $\hat{x}\hat{y}z$. As it turns out, these transformation rules are concisely summarized by means of complex-variable notation, with

$$u^*(z) = U(z) + iV(z), \qquad \kappa^*(z) = du^*/dz, \qquad \kappa = |\kappa^*| = \kappa(z), \tag{3.6}$$

where $\kappa(z)$ is defined by (3.5) and where $\hat{\theta} = \arg\{u^*(z)\}$ is the above-mentioned angle between the coordinates xy and $\hat{x}\hat{y}$. Then, the appropriate generalization of (2.8) is quite simply

$$\tau^* = \eta^* \kappa^* \tag{3.7}$$

with η^* defined by (2.7) and (3.6) and τ^* by the left-hand equality in (2.8).

Although we shall have no direct need of them here, we note that the normal stresses in the last three members of (2.5) can similarly be transformed through the remarkably simple relations

$$\begin{aligned} N^* &\equiv \tfrac{1}{2}(T_{xx} - T_{yy}) + iT_{xy} = \psi^*(\kappa^*)^2, \\ \Delta &\equiv \tfrac{1}{2}(T_{xx} + T_{yy}) - T_{zz} = \chi\kappa^2, \\ \psi^* &= \Psi(\kappa, \omega) + i\Phi(\kappa, \omega) \end{aligned} \tag{3.8}$$

valid for *any* set of xy coordinates, where, we recall, κ^* is defined by (3.6). The (real) material functions χ, Ψ, Φ in (3.8) are defined in terms of a set, Ψ_1, Ψ_2, Φ, derived from the stress functions of (2.5) as follows:

$$\begin{aligned} N_1(\kappa, \omega) &= \Psi_1(\kappa, \omega)\kappa^2, \qquad N_2(\kappa, \omega) = \Psi_2(\kappa, \omega)\kappa^2, \\ S(\kappa, \omega) &= \Phi(\kappa, \omega)\kappa^2, \qquad \Psi = \tfrac{1}{2}(\Psi_1 + \Psi_2), \qquad \chi = \tfrac{1}{2}(\Psi_1 - \Psi_2). \end{aligned} \tag{3.9}$$

We note that the viscometric limit (2.6) $\omega \to 0$ is now distinguished by

$$\begin{aligned} &\eta^* \to \eta^{(0)}(\kappa), \qquad \psi^* \to \Psi(\kappa) \\ &\Psi_1 \to \Psi_1^{(0)}(\kappa), \qquad \Psi_2 \to \Psi_2^{(0)}(\kappa), \qquad \Phi \to 0, \end{aligned} \tag{3.10}$$

in (3.7)–(3.9), where $\eta^{(0)}$, $\Psi_1^{(0)}$, and $\Psi_2^{(0)}$ denote respectively the corresponding *viscosity*, and *primary normal-stress* and *secondary normal-stress* coefficients (cf. Bird et al. [14]. With the understanding that complex quantities are involved, a somewhat more economical notation could be achieved here by dropping the asterisks employed above to distinguish complex quantities, except for those associated with the linear viscoelastic limit, and by writing $|\kappa|$ for the modulus of κ wherever appropriate.)

It remains now to establish the compatibility of the assumed kinematics (3.2) with the (Cauchy) equations of motion:

$$\nabla \cdot \mathbf{T} = \rho\left(\mathbf{v} \cdot \nabla\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}\right), \qquad \nabla \cdot \mathbf{v} = 0. \tag{3.11}$$

The second of these, the continuity equation, is satisfied identically. Because the velocity gradient and, hence, the deviatoric stress do not depend on the position (x, y) in a horizontal plane, the first member of (3.11) can be reduced to the set

$$\begin{aligned} \frac{dT_{xz}}{dz} &= \frac{\partial p}{\partial x} + \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial p}{\partial x} - \rho\omega(V + \omega x), \\ \frac{dT_{yz}}{dz} &= \frac{\partial p}{\partial x} + \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \frac{\partial p}{\partial y} + \rho\omega(U - \omega y), \\ \frac{dT_{zz}}{dz} &= 0, \end{aligned} \tag{3.12}$$

where p is a (dynamical) pressure. Following Abbott and Walters [8], we write for the latter

$$p = ax + by + \rho\frac{\omega^2}{2}(x^2 + y^2) \tag{3.13}$$

where a and b are constants, so that the first two equations of (3.12) can be expressed concisely in terms of complex variables as

$$d\tau^*/dz = g + i\rho\omega u^*, \tag{3.14}$$

where $g = a + ib$ and τ^* and u^* are the quantities appearing in (3.6) and (3.7).

The third equation of (3.12) indicates that the vertical normal stress T_{zz} is dynamically inoperative and also independent of elevation z. Therefore, in light of (3.3)–(3.7), the complete dynamical equations and boundary conditions reduce to the compact form

$$\frac{d}{dz}\left(\eta^*\frac{du^*}{dz}\right) - i\rho\omega u^* = g \tag{3.15}$$

with

$$u^* = \pm\omega\gamma h/2 \quad \text{at} \quad z = \pm h/2$$

where $\eta^* = \eta^*(|du^*/dz|, \omega)$ is the complex viscosity defined by (2.7).

Eq. (3.15), the main result of the present work, provides a pair of coupled, generally nonlinear second-order differential equations for the velocity functions $U(z)$ and $V(z)$ of (3.2) and (3.6). We shall not discuss any solutions for specific material models but, rather, content ourselves with some general remarks on the form of (3.15).

As in the previous work of Abbott and Walters, the complex constant g in (3.14) and (3.15) represents an externally imposed horizontal pressure gradient which can be set equal to zero in the usual application to the orthogonal rheometer. In this case, the most salient non-dimensional parameters associated with (3.15) are the characteristic strain or eccentricity γ, a Reynolds number Re, and a Weissenberg number Ws:

$$\mathrm{Re} \equiv \frac{\rho\omega h^2}{\eta_0}, \qquad \mathrm{Ws} \equiv \omega\tau \tag{3.16}$$

where η_0 and τ denote, respectively, a characteristic fluid viscosity and relaxation time such as those in (2.18)–(2.22). As already evident from the linear-viscoelasticity theory of Abbott and Walters, where the strain γ can be effectively scaled out of the problem, the magnitudes of parameters in (3.16) dictate the global importance of inertia and fluid elasticity. In the more general theory encompassed by (3.15), the strain parameter γ reflects the importance of rheological nonlinearities, particularly for $\mathrm{Re} = 0$.

Although we have heretofore limited attention to fluids, the dynamical equations (3.15) can also be applied to arbitrary isotropic simple materials, so long as the material function η^* remains well defined. Of course, in the case of a solid material η^* could more appropriately be interpreted in terms of a complex elastic modulus μ^*, and the nominal "Reynolds number" in (3.16) should also be reformulated in terms of a modulus or characteristics stress μ_0 by substitution of μ_0/ω for η_0.

In general, one expects of course to encounter further dimensionless parameters associated with the rheological description of the material, for example, the ratio of intrinsic time constants for viscoelastic materials, or the ratio of yield stress to some other characteristic stress in the case of a material exhibiting plastic behavior.

We believe that Eqs. (3.15) open the door to a comprehensive exploration of several interesting effects involving nonlinear viscoelasticity including, for example, the structure of hydrodynamic boundary layers that develop for $\mathrm{Re} \gg 1$ (cf. Abbott and Walters [8]).[5] In the context of rheological testing one would of course like to evaluate the presumably unknown material functions η^*, ψ^* and χ, in principle by deriving the stresses τ^*, N^*, Δ from experimental measurements. Based on the previous works aimed at assessing the complicating effects of inertia in rheologically linear or weakly nonlinear materials, one can discern the possibility of a general and internally self-consistent scheme for carrying out such a program with arbitrary isotropic materials.

To end with a question concerning the possible extensions and limitations of the preceding analysis, we note that the high degree of kinematical symmetry may render (3.15) directly applicable to certain anisotropic materials. For that matter, it is plausible that the above type of analysis might be extended to other laminar shears, such as elliptical shear.

[5] Note added in proof: In a work which has recently come to the author's attention, Joseph [15] has already discussed such boundary layers for the linear-viscoelastic problem.

One should not, however, overlook an interesting question as to the uniqueness or dynamic stability of such shearing motions, a question which for Newtonian fluids has been partially answered by a global stability theorem in the recent work of Berker [10].

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Appendix. The stress pattern for simple shearing of isotropic fluids. I wish to amend here the treatment of simple shearing motions given in an earlier work [4, pp. 174–177], for which we shall herein designate the equation and page numbers by prefix "G". In that work, primary attention was devoted to the special case of cylindrical shear, for which $\mathbf{K}(t)$ in (2.1) has matrix[6]:

$$[\mathbf{K}(t)] = \begin{bmatrix} 0 & \kappa_1(t) & \kappa_2(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{A.1}$$

The symmetry argument on p. G175 involving the orthogonal transformation (G2.43) is incomplete in that it does not allow for products of shear rates at different instants of past time. Thus, *all* bilinear or quadratic forms involving products of the type

$$\kappa_1^2(\cdot), \qquad \kappa_1(\cdot)\kappa_2(\cdot), \qquad \text{etc.} \tag{A.2}$$

which appear in (G2.43) to (G2.50) and which involve the values of the product functions evaluated at a single past time, denoted by $(\cdot)$, should instead read:

$$\kappa_1(\cdot)\kappa_1(*), \qquad \kappa_1(\cdot)\kappa_2(*), \qquad \text{etc.} \tag{A.3}$$

[6] Accordingly, and as the correction of a typographical error, the subscripts "1" and "3" on the basis vectors $\mathbf{g}_i$ should be interchanged in the sentence preceding Equation (G2.41).

where, now, two different instants of past time denoted here by $(\cdot)$ and $(*)$ are involved. As a conquence, (G2.48) should be replaced by the quadratic form

$$\kappa^{(2)}(\cdot\,,\,*) = \mathrm{tr}\{\mathbf{K}^T(\cdot)\mathbf{K}(\,*\,)\} = \kappa_1(\cdot)\kappa_1(\,*\,) + \kappa_2(\,\cdot\,)\kappa_2(\,*\,). \tag{A.4}$$

Then, the principal result (G2.49), giving the stress tensor $\mathbf{T}$ in all simple shears of a Noll (simple) fluid, should read:

$$\begin{aligned}\mathbf{T} + p\mathbf{l} = {}& \ell\{\kappa^{(2)}(\,\cdot\,,\,*\,)\,|\,\mathbf{K}(\,\cdot\,) + \mathbf{K}^T(\cdot)\}\\ &+ n_1\{\kappa^{(2)}(\,\cdot\,,\,*\,)\,|\,\mathbf{K}(\cdot)\mathbf{K}^T(\,*\,)\}\\ &+ n_2\{\kappa^{(2)}(\,\cdot\,,\,*\,)\,|\,\mathbf{K}^T(\cdot)\mathbf{K}(\,*\,)\}\end{aligned} \tag{A.5}$$

where $\mathbf{K}$ is the velocity gradient of (2.1). Here, ℓ, n_1 and n_2, which are isotropic linear forms in their second arguments, $\mathbf{K}(\cdot) + \mathbf{K}^T(\cdot)$, $\mathbf{K}(\cdot)\mathbf{K}^T(\,*\,)$, etc., are essentially the same as the functionals introduced in [4], except they are now defined on bilinear forms in the values of functions evaluated at distinct instants of past time.

Upon substituting the expression (2.2) and (2.4) into (A.5), one obtains the stress for steady circular shear expressed in terms of time-periodic stress components associated with the rotating frame introduced in Sec. 2. Upon transforming back to the original frame, associated with the representation (1.1), one obtains an amended version of (G2.55) for the steady stresses and, hence, the material functions for steady circular shear[7]. With a vertical bar denoting, as in (A.5), linearity in the functional arguments immediately following the bar, the stress components are thus given by the following relations:

$$\begin{gathered}T_{13} = \kappa\ell\{\kappa^{(2)}\,|\,C(\cdot)\}, \qquad T_{23} = -\kappa\ell\{\kappa^{(2)}\,|\,S(\cdot)\}, \qquad T_{12} = -\kappa^2 n_1\{\kappa^{(2)}\,|\,S(\cdot)C(\,*\,)\},\\ T_{11} - T_{33} = \kappa^2 n_1\{\kappa^{(2)}\,|\,C(\cdot)C(\,*\,)\} - n_2\{\kappa^{(2)}\,|\,\kappa^{(2)}\},\\ T_{22} - T_{33} = \kappa^2 n_1\{\kappa^{(2)}\,|\,S(\cdot)S(\,*\,)\} - n_2\{\kappa^{(2)}\,|\,\kappa^{(2)}\},\end{gathered} \tag{A.6}$$

where

$$\begin{aligned}C(\cdot) &= \cos\,\underset{s=0}{\overset{\infty}{\omega s}} \equiv \{\cos\,\omega s\colon 0 \le s < \infty\},\\ S(\cdot) &= \sin\,\underset{s=0}{\overset{\infty}{\omega s}} \equiv \{\sin\,\omega s\colon 0 \le s < \infty\},\\ \kappa^{(2)} \equiv \kappa^{(2)}(\cdot\,,\,*\,) &= \kappa^2[C(\cdot)C(\,*\,) + S(\cdot)S(\,*\,)]\\ &\equiv \kappa^2\cos\,\omega[(\cdot) - (\,*\,)],\end{aligned} \tag{A.7}$$

with s denoting the time lapse $s = t - t'$ at past time $t' \le t$.

We note that (A.6) serves to define the complex material function of (2.7) as

$$\eta^*(\kappa, \omega) = \ell\{\kappa^{(2)}\,|\,E(\cdot)\} \tag{A.8}$$

where $E(\cdot)$ stands for the set $\{E(s)\colon 0 \le s < \infty\}$ and

$$E(s) = e^{i\omega s}. \tag{A.9}$$

[7] The stress components in (A.6) are identical with those in Eq. (2) of [7], which in turn are identical with those distinguished by an affixed asterisk in Eq. (2.55) of [4].

Thus, as the generalization of a well-known form appropriate to linear viscoelasticity, we see that η^* represents a kind of Fourier transform of the functional ℓ.

In closing here, we further note that the expressions (A.6) serve also to define the three fundamental material functions for steady circular shear which were introduced in [4]:

$$
\begin{aligned}
\hat{\tau}(\kappa, \omega) &\equiv (T_{13} - T_{23})/\sqrt{2} \\
&= \kappa\ell\{\kappa^{(2)} \,|\, \hat{C}(\cdot)\}, \\
\hat{N}(\kappa, \omega) &\equiv (T_{11} + T_{22})/2 - T_{33} - T_{12} \\
&= \kappa^2 n_1\{\kappa^{(2)} \,|\, \hat{C}(\cdot)\hat{C}(\,*\,)\} - n_2\{\kappa^{(2)} \,|\, \kappa^{(2)}\}, \\
\hat{S}(\kappa, \omega) &= (T_{11} - T_{22})/2 \\
&= \tfrac{1}{2}\kappa^2 n_1\{\kappa^{(2)} \,|\, \cos \omega[(\,*\,) + (\cdot)]\},
\end{aligned}
\tag{A.10}
$$

where $\kappa^{(2)}$ is defined by (A.7) and $\hat{C}(\cdot)$ stands for $\{\hat{C}(s): 0 \leq s < \infty\}$, with

$$\hat{C}(s) \equiv \cos(\omega s - \pi/4). \tag{A.11}$$

As shown already in [7], all the members of (A.6) can be derived from the material functions in (A.10), which relations further serve to define the parity in κ and ω for the various stresses.