A LINEAR INTEGRODIFFERENTIAL EQUATION FOR VISCOELASTIC RODS AND PLATES*

By

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Abstract. It is proved that the resolvent kernel of a certain integrodifferential equation in Hilbert space is absolutely integrable on $(0, \infty)$. The equation arises in the linear theory of isotropic viscoelastic rods and plates.

1. Introduction. We study the (operator-valued) resolvent kernel of an integrodifferential equation in Hilbert space which arises in the linear theory of isotropic viscoelastic rods and plates. As in [5, 2, 3], we find sufficient conditions for certain norms of the resolvent and its derivative to be absolutely integrable, but here the kernel of the equation is made up in a complicated way from the (distinct) moduli of stress relaxation for compression and shear.

Throughout the paper, $\mathbf{L}$ denotes a positive self-adjoint linear operator defined on a dense domain $\mathscr{D}$ of a Hilbert space $\mathscr{H}$. We consider the equation

$$\mathbf{y}'(t) = -A * \mathbf{L}\mathbf{y}(t) + \mathbf{g}(t) \qquad ({}' = d/dt,\ t \in \mathbb{R}^+ \equiv [0, \infty)),\ \mathbf{y}(0) = \mathbf{y}_0, \tag{1.1}$$

where $\mathbf{y}_0$ and $\mathbf{g}(t)$ belong to $\mathscr{H}$, $A\colon \mathbb{R}^+ \to \mathbb{R}$ is locally absolutely continuous, and $*$ denotes the convolution

$$h_1 * h_2(t) = \int_0^t h_1(t-s)h_2(s)\, ds.$$

The integral in (1.1) is a Bochner integral, and a *solution* belongs to $C^1(\mathbb{R}^+, \mathscr{H}) \cap C(\mathbb{R}^+, \mathscr{D})$.

The hypotheses on A will be stated in terms of its Fourier transform $\hat{A}$; the fact that A is locally absolutely continuous will be deduced in Theorem 1.1(i). In this paper, the Fourier transform $\hat{h}$ is defined for a function h such that $h(t)e^{-\sigma t} \in L^1(\mathbb{R}^+)$ for all $\sigma > 0$ by the formula

$$\hat{h}(\tau) = \int_0^\infty e^{-i\tau t} h(t)\, dt \qquad (\operatorname{Im} \tau < 0), \quad \hat{h}(\tau_0) = \lim_{\tau \to \tau_0,\ \operatorname{Im} \tau < 0} \hat{h}(\tau) \qquad (\tau_0 \in \mathbb{R}),$$

whenever the limit exists.

The equation for a dynamic problem in linear viscoelasticity [1] can be obtained by a "correspondence principle" from the equation for the corresponding problem in the purely elastic case. After applying the Fourier transform to the elastic equation, one replaces the elastic moduli of shear and compression (μ and k) by $Y_s(\tau)/2$ and $Y_v(\tau)/3$ respec-

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tively, where Y_s and Y_v are the complex moduli of shear and compression for the viscoelastic material. For example, the transformed equation for transverse vibrations in a viscoelastic plate [1, pp. 109–112] is

$$\left(\frac{\partial^2 y}{\partial t^2}\right)^{\wedge}(x, \tau) = -G_1(\tau)\Delta^2\hat{y}(x, \tau) + H(x, \tau), \tag{1.2}$$

where H is a forcing term and

$$G_1(\tau) = i\tau\hat{a}(\tau)\left(\frac{2\hat{b}(\tau) + \hat{a}(\tau)}{\hat{b}(\tau) + 2\hat{a}(\tau)}\right)$$

with $i\tau\hat{a}(\tau) = Y_s(\tau)$, $i\tau\hat{b}(\tau) = Y_v(\tau)$. Now suppose

$$\hat{A}(\tau) = \hat{a}(\tau)f(\tau) \qquad (\operatorname{Im} \tau < 0), \tag{1.3}$$

where

$$f(\tau) = F(\hat{b}(\tau)/\hat{a}(\tau)), \qquad F(w) = \frac{w + m}{pw + q}, \tag{1.4}$$

with $m = p = \frac{1}{2}$, $q = 1$. Then formal differentiation of (1.1) (with $\mathbf{L} = \Delta^2$, $\mathbf{y}(t) = y(\cdot, t)$), followed by an application of the Fourier transform, yields (1.2) with some H. Similarly, the equations for waves in a rod come from (1.1), (1.3), (1.4) with $m = 0$, $p = 2$, $q = 1$; for longitudinal waves, $\mathbf{L} = -\partial^2/\partial x^2$, while for bending waves, $\mathbf{L} = \partial^4/\partial x^4$. In all cases we must take self-adjoint boundary conditions.

Throughout this paper we assume that

$$m, p, \text{ and } q \text{ are nonnegative, } p > 0, \text{ and } q > mp, \tag{1.5}$$

$$\begin{gathered} a(t) \text{ and } b(t) \text{ are continuous, nonnegative, nonincreasing,} \\ \text{convex, and not constant on } \mathbb{R}^+, \text{ with } a'(0) + b'(0) > -\infty. \end{gathered} \tag{1.6}$$

Thus (1.5) includes the examples given above. In the applied literature, the moduli $a(t)$ and $b(t)$ are often assumed to be positive linear combinations of decaying exponentials. Pipkin [7] takes a more general approach, but (1.6), even together with (1.17) below, seems to include many of the plausible models and some implausible ones. If $p = 0$, (1.1) reduces to the problem studied in [5, 2, 3], where weaker hypotheses are imposed; in particular, $A(0)$ need not be finite.

Let $\{\mathbf{E}_\lambda\}$ be the spectral family corresponding to $\mathbf{L}$; without loss of generality we assume that the spectrum of L is contained in $[1, \infty)$. Define

$$\mathbf{U}(t) = \int_1^\infty u(t, \lambda)\, d\mathbf{E}_\lambda, \qquad \mathbf{U}'(t) = \int_1^\infty u_t(t, \lambda)\, d\mathbf{E}_\lambda,$$

where $u(t, \lambda)$ is the solution of

$$u'(t) = -\lambda A * u(t), \qquad u(0) = 1. \tag{1.7}$$

Clearly,

$$\|\mathbf{U}(t)\| \leq \sup_{1 \leq \lambda < \infty} |u(t, \lambda)|, \qquad \|\mathbf{L}^{-1/2}\mathbf{U}'(t)\| \leq \sup_{1 \leq \lambda < \infty} |\lambda^{-1/2}u_t(t, \lambda)|,$$

where $\|\cdot\|$ denotes the norm of bounded operators from $\mathscr{H}$ to itself. Our main results, (1.15), (1.18), (1.19), and (1.20) in Theorem 1.1 below, then imply, respectively,

$$\| \mathbf{U}(t) \| \leq 1 \qquad (t \in \mathbb{R}^+), \tag{1.8}$$

$$\lim_{t \to \infty} \| t\mathbf{U}(t) \| = 0, \tag{1.9}$$

$$\int_0^\infty \| \mathbf{U}(t) \| \, dt < \infty, \tag{1.10}$$

$$\int_0^\infty \| \mathbf{L}^{-1/2} \mathbf{U}'(t) \| \, dt < \infty. \tag{1.11}$$

Existence, uniqueness, and representation results for (1.1) work out just as in [5, 2, 3]. In particular, the conclusions of Theorem 1.1 imply that

$$\mathbf{U}'(t)\mathbf{y} = \frac{d}{dt} [\mathbf{U}(t)\mathbf{y}] \qquad \text{if } \mathbf{L}^{-1/2}\mathbf{y} \in \mathscr{D}; \tag{1.12}$$

moreover, if $\mathbf{y}_0 \in \mathscr{D}$, $\mathbf{g}\colon \mathbb{R}^+ \to \mathscr{H}$ is continuous with $\mathbf{g}(t) \in \mathscr{D}$ for all t, and $\mathbf{Lg}\colon \mathbb{R}^+ \to \mathscr{H}$ is locally Bochner-integrable, then the unique solution of (1.1) is given by

$$\mathbf{y}(t) = \mathbf{U}(t)\mathbf{y}_0 + \mathbf{U} * \mathbf{g}(t). \tag{1.13}$$

Under weaker hypotheses, (1.13) gives the unique solution of (1.1) in a weak sense. Clearly, (1.8) through (1.13) can be used to study the asymptotic behavior of $\mathbf{y}(t)$ as $t \to \infty$ under various assumptions on $\mathbf{g}$. We refer the reader to [3] for further discussion.

THEOREM 1.1. Suppose (1.5) and (1.6) hold.

(i) There exists $B \in L^2(\mathbb{R}^+)$ such that the function

$$A(t) \equiv f(\infty)a(t) + \int_0^t B(s) \, ds$$

satisfies (1.3).

(ii) For $\lambda > 0$, $u(t, \lambda)$ satisfies

$$\hat{u}(\tau, \lambda) = [i\tau + \lambda \hat{A}(\tau)]^{-1} \qquad (\operatorname{Im} \tau < 0), \tag{1.14}$$

$$|u(t, \lambda)| \leq 1 \qquad (t \geq 0); \tag{1.15}$$

moreover,

$$\sup_{0 < \lambda < \infty,\, 0 \leq t \leq T} \lambda^{-1/2} |u_t(t, \lambda)| < \infty \qquad (T < \infty). \tag{1.16}$$

(iii) If, in addition,

$$-a' \text{ and } -b' \text{ are convex on } \mathbb{R}^+, \tag{1.17}$$

then

$$\lim_{t \to \infty} \sup_{1 \leq \lambda < \infty} |tu(t, \lambda)| = 0, \tag{1.18}$$

$$\int_0^\infty \sup_{1 \leq \lambda < \infty} |u(t, \lambda)| \, dt < \infty, \tag{1.19}$$

$$\int_0^\infty \sup_{1 \leq \lambda < \infty} \lambda^{-1/2} |u_t(t, \lambda)| \, dt < \infty. \tag{1.20}$$

Remark. In (i), it will be seen below in (2.11) that $f(\infty)$ exists and is equal to $F(b(0)/a(0))$.

2. Proof of Theorem 1.1. We first recall some consequences of (1.6). By [4],

$$\begin{aligned}&\hat{a}(\tau) \equiv \phi(\tau) - i\tau\theta(\tau) \text{ and } \hat{b}(\tau) \text{ are analytic in } \{\operatorname{Im} \tau < 0\}\\ &\text{and continuous in } S \equiv \{\tau \in \mathbb{C}: \operatorname{Im} \tau \leq 0, \tau \neq 0\}. \text{ Moreover,}\\ &\text{if } \operatorname{Im} \tau < 0 \text{ and } \operatorname{Re} \tau > 0,\ \hat{a}(\tau) \text{ and } \hat{b}(\tau) \text{ lie in}\\ &\{-\pi/2 < \arg w < 0\}; \text{ if (1.17) holds, this}\\ &\text{conclusion remains true when } \operatorname{Im} \tau = 0, \tau > 0.\end{aligned} \tag{2.1}$$

In this paper, $-\pi < \arg w \leq \pi$ $(w \in \mathbb{C})$.
Integration by parts and the Riemann-Lebesgue theorem show that

$$\hat{a}(\tau) = \frac{a(0)}{i\tau} + O(\tau^{-2}), \qquad \hat{b}(\tau) = \frac{b(0)}{i\tau} + O(\tau^{-2}) \qquad (\tau \to \infty, \tau \in S); \tag{2.2}$$

if (1.17) holds, this can be strengthened to

$$\begin{aligned}\hat{a}(\tau) &= \frac{a(0)}{i\tau} - \frac{a'(0)}{\tau^2} + o(\tau^{-2}),\\ \hat{b}(\tau) &= \frac{b(0)}{i\tau} - \frac{b'(0)}{\tau^2} + o(\tau^{-2}) \qquad (\tau \to \infty, \tau \in S).\end{aligned} \tag{2.3}$$

From [8] (with slight modifications) we know that $\hat{a}(\tau)$ and $\hat{b}(\tau)$ are differentiable for $\tau > 0$ and

$$\begin{aligned}\frac{1}{2^{3/2}} \int_0^{1/\tau} a(t)\, dt &\leq |\hat{a}(\tau)| \leq 4 \int_0^{1/\tau} a(t)\, dt,\\ |\hat{a}'(\tau)| &\leq 40 \int_0^{1/\tau} ta(t)\, dt \qquad (\tau > 0),\end{aligned} \tag{2.4}$$

$$\int_0^1 \frac{\int_0^{1/\tau} ta(t)\, dt}{(\int_0^{1/\tau} a(t)\, dt)^2}\, d\tau < \infty, \tag{2.5}$$

with similar estimates for b. By [5, Lemma 2.2],

$$\theta'(\tau) < 0 \qquad (\tau > 0). \tag{2.6}$$

If (1.17) holds, [2, Lemma 5.1] shows that $\hat{a}$ and $\hat{b}$ belong to $C^2(0, \infty)$ with

$$|a''(\tau)| \leq 6000 \int_0^{1/\tau} t^2 a(t)\, dt; \tag{2.7}$$

together with (2.4) and (2.5), this implies

$$\int_0^1 \frac{|\tau\hat{a}''(\tau)| + |\hat{a}'(\tau)|}{|\hat{a}(\tau)|^2}\, d\tau < \infty. \tag{2.8}$$

The analogous inequalities for b are, of course, valid.

The fractional linear transformation F maps $\{\operatorname{Re} w \geq 0\}$ onto the disk with diameter $[m/q, 1/p]$. Moreover,

$$\begin{aligned} &0 < \arg F(w) < \arg w && (0 < \arg w < \pi/2), \\ &F(\bar{w}) = \overline{F(w)} && (w \in \mathbb{C}). \end{aligned} \tag{2.9}$$

It follows that $\arg \hat{a}(\tau) f(\tau)$ lies between $\arg \hat{a}(\tau)$ and $\arg \hat{b}(\tau)$, and strictly between them when $\arg \hat{a}(\tau) \neq \arg \hat{b}(\tau)$ $(\tau \in S)$. In particular, if we write $\hat{a}(\tau) f(\tau) = \Phi(\tau) - i\tau\Theta(\tau)$, then in $\{\operatorname{Im} \tau < 0\}$ (and in S when (1.17) holds),

$$\Phi(\tau) > 0. \tag{2.10}$$

By (2.2),

$$f(\tau) = f(\infty) + O(\tau^{-1}) \qquad (\tau \to \infty, \tau \in S), \tag{2.11}$$

with $f(\infty) = F(b(0)/a(0)) \equiv L$.

To prove (i), define

$$G(\tau) = [f(\tau) - f(\infty)][\hat{a}'(\tau) + a(0)].$$

By (2.1), (2.2), and the properties of F, G belongs to the Hardy class H^2 in $\{\operatorname{Im} \tau < 0\}$, so $G = \hat{B}$ for some B in $L^2(\mathbb{R}^+)$. But

$$\frac{1}{i\tau} G(\tau) + f(\infty)\hat{a}(\tau) = f(\tau)\hat{a}(\tau),$$

so (1.3) holds with A as in (i), as asserted.

By (2.10) and elementary transform theory, (1.14) holds, and [6, Theorem 1 *i* and its proof] yields (1.15). Differentiation of (1.7) gives us

$$u''(t) = -\lambda[A(0)u(t) + A' * u(t)] \qquad (t > 0),$$

so by (1.15),

$$\sup_{0 \leq t \leq T} |u_{tt}(t, \lambda)| \leq \lambda \left[A(0) + \int_0^T |A'(t)|\, dt \right];$$

this estimate and (1.15) yield (1.16).

In the remainder of this paper, we assume that (1.17) holds. We first develop estimates on $\hat{A}$. Using (2.3) and (1.5), one shows with a little calculation that

$$\lim_{\tau \to +\infty} \tau^2 \Phi(\tau) = \frac{b'(0)(mp - q) - a'(0)[p\alpha^2 + 2mp\alpha + mq]}{(p\alpha + q)^2} > 0, \tag{2.12}$$

$$\lim_{\tau \to +\infty} \tau^2 \Theta(\tau) = a(0) f(\infty) > 0, \tag{2.13}$$

where $\alpha = b(0)/a(0)$.

If a [or b] belongs to $L^1(\mathbb{R}^+)$, then $\hat{a}(0) = \int_0^\infty a(t)\, dt$ [or $\hat{b}(0) = \int_0^\infty b(t)\, dt$]; otherwise, $1/\hat{a}(\tau) \to 0$ $(\tau \to 0, \tau \in S$ [and similarly for b] (see [4]). By examining the various cases (a or b in or not in L^1, $m = 0$ or $m \neq 0$), one deduces that

$$\lim_{\substack{\tau \to 0 \\ \tau \in S}} \frac{1}{\hat{A}(\tau)} \text{ exists and is real and nonnegative.} \tag{2.14}$$

Thus for each $\lambda > 0$

$$\hat{u}(\tau, \lambda) \text{ is continuous in } \{\operatorname{Im} \tau \leq 0\}, \tag{2.15}$$

Since $\hat{a}, \hat{b} \in C^2(0, \infty)$ when (1.17) holds, $\hat{A} \in C^2(0, \infty)$. We see from (2.1), (1.5), (2.4), (2.7), (2.8), and (2.3) that

$$\int_0^1 \frac{|\tau \hat{A}''(\tau)| + |\hat{A}'(\tau)|}{|\hat{A}^2(\tau)|}\, d\tau < \infty, \tag{2.16}$$

$$\frac{|\tau \hat{A}'(\tau)| + |\tau^2 \hat{A}''(\tau)|}{|\hat{A}(\tau)|} = O(1) \qquad (\tau \to 0+), \tag{2.17}$$

$$\tau^{j+1} \hat{A}^{(j)}(\tau) = O(1) \qquad (\tau \to +\infty, j = 0, 1, 2). \tag{2.18}$$

Furthermore, $\hat{u} \in C^2(0, \infty)$ as a function of τ, and

$$\frac{\partial^j u(\tau, \lambda)}{\partial \tau^j} = O(\tau^{-j-1}) \qquad (\tau \to +\infty, j = 0, 1, 2, \lambda > 0). \tag{2.19}$$

By (2.3),

$$\hat{u}(\tau, \lambda) = O(\tau^{-1}) \qquad (\tau \to \infty, \tau \in S, \lambda > 0). \tag{2.20}$$

By (1.14), (2.15), (2.20), (2.16), and the argument of [8, pp. 323–324],

$$u(\cdot, \lambda) \in L^1(\mathbb{R}^+) \qquad (0 < \lambda < \infty). \tag{2.21}$$

We now follow the scheme of [5, 2]. Let $D(\tau, \lambda) = \hat{A}(\tau) + (i\tau/\lambda)$, and choose $\rho > 0$ such that

$$|\hat{A}(\tau)| \geq 2\tau \qquad (0 < \tau \leq \rho); \tag{2.22}$$

ρ exists, by (2.4) and (2.14). Using the complex inversion formula for Laplace transforms, together with a contour shift and integration by parts, we establish the representation

$$\pi u(t, \lambda) = \operatorname{Im}\{\lambda^{-1} u_1(t) + i\lambda^{-2} u_2(t) + \lambda^{-3} u_3(t) + u_4(t, \lambda) + u_5(t, \lambda)\}, \tag{2.23}$$

where (see [2], Eq. (4.34) for the computation)

$$u_1(t) = \frac{1}{t} \int_0^\rho e^{i\tau t} \frac{\hat{A}'(\tau)\, d\tau}{\hat{A}^2(\tau)},$$

$$u_2(t) = \frac{1}{t} \int_0^\rho \frac{e^{i\tau t}}{\hat{A}^2(\tau)} \left[1 - \frac{2\hat{A}'(\tau)}{\hat{A}(\tau)}\right] d\tau,$$

$$u_3(t) = \frac{1}{t} \int_0^\rho e^{i\tau t} \frac{2\tau\, d\tau}{\hat{A}^3(\tau)},$$

$$u_4(t, \lambda) = \frac{-1}{\lambda^3 t} \int_0^\rho e^{i\tau t} \frac{\tau^2 D_\tau(\tau, \lambda)}{\hat{A}^3(\tau) D(\tau, \lambda)} \left[\frac{2}{\hat{A}(\tau)} + \frac{1}{D(\tau, \lambda)}\right] d\tau,$$

$$u_5(t, \lambda) = \frac{1}{t\lambda} \int_0^\infty e^{i\tau t} \frac{D_\tau(\tau, \lambda)}{D^2(\tau, \lambda)}\, d\tau.$$

The validity of this representation and the absolute convergence of the integrals are ensured by estimates (2.16) through (2.20) and (2.22).

We show that

$$|u_4(t,\lambda)| \le Mt^{-2} \qquad (\lambda, t \ge 1), \tag{2.24}$$

$$|u_5(t,\lambda)| \le Mt^{-2} \qquad (\lambda, t \ge 1). \tag{2.25}$$

(Here and below, M denotes a finite constant, independent of t and λ; its value can change from line to line.) Then by (2.23), (2.21), and (1.15), (1.19) follows. Moreover, (1.18) is a consequence of (2.23), (2.24), (2.25), and the Riemann-Lebesgue theorem.

For (2.24), integrate by parts in the definition of u_4 (using (2.17)) and then use (2.17) and (2.22).

For (2.25), first define $\omega_1 = \omega_1(\lambda) > 0$ by

$$\theta(\omega_1) = 1/\lambda;$$

by (2.6), ω_1 is unique when it exists. Now let $\omega(\lambda) = \omega_1(\lambda)$ when ω_1 exists and $\omega_1 \ge \rho$, and let $\omega(\lambda) = \rho$ otherwise. As shown in [5], ω is continuous and nondecreasing, and by (2.3),

$$\omega^2 = \lambda a(0) + o(\lambda^{1/2}) \qquad (\lambda \to \infty). \tag{2.26}$$

Moreover ([5, Eq. (6.8)]), there exists $Q > 0$ such that

$$\frac{|\omega(\lambda) - \tau|[\omega(\lambda) + \tau]}{\tau^2 \lambda} \le Q\left|\frac{1}{\lambda} - \theta(\tau)\right| \qquad (\tau \ge \rho, \lambda \ge 1). \tag{2.27}$$

By (2.11) and (2.2),

$$\Theta(\tau) = L\theta(\tau) + O(\tau^{-3}) \qquad (\tau \to +\infty). \tag{2.28}$$

Using (2.27), we see that there exists $R > 0$ such that

$$Q\left|\Theta(\tau) - \frac{L}{\lambda}\right| \ge \frac{|\omega(\lambda) - \tau|[\omega(\lambda) + \tau]L}{\tau^2\lambda} - \frac{R}{\tau^3} \qquad (\tau \ge \rho). \tag{2.29}$$

Now choose $\rho_1 \ge \rho$ so large that when $\omega(\lambda) \ge \rho_1$ we have

$$\omega^2 L/4\lambda > R/\rho_1, \tag{2.30}$$

$\frac{1}{2}\lambda a(0) < \omega^2 < 2\lambda a(0)$ (see (2.26)). $L \le \lambda$, and

$$\varepsilon \equiv 16R/a(0)L < \omega/2.$$

Then if $\omega \ge \rho_1$, $\tau > \frac{1}{2}\omega$,

$$Q\left|\Theta(\tau) - \frac{L}{\lambda}\right| \ge \frac{1}{\tau\lambda}\left[L|\omega - \tau|\left(\frac{\omega + \tau}{\tau}\right) - \frac{8R}{a(0)}\right].$$

Thus

$$Q\left|\Theta(\tau) - \frac{L}{\lambda}\right| \ge \frac{|\tau - \omega|L}{2\tau\lambda} \qquad \left(\tau \ge \frac{\omega}{2}, |\tau - \omega| \ge \varepsilon, \omega \ge \rho_1\right). \tag{2.31}$$

Integrate by parts in the definition of u_5 (using (2.18)) to see that

$$-\frac{i\lambda t^2}{L} u_5\left(t, \frac{\lambda}{L}\right) = e^{i\rho t}\frac{D_\tau(\rho, \lambda/L)}{D^2(\rho, \lambda/L)} + \int_\rho^\infty e^{i\tau t}\left[\frac{\hat{A}''(\tau)}{D^2(\tau, \lambda/L)} - \frac{2D_\tau^2(\tau, \lambda/L)}{D^3(\tau, \lambda/L)}\right] d\tau.$$

Then by (2.10) and (2.18)

$$\frac{t^2\lambda}{L}\left|u_5\left(t, \frac{\lambda}{L}\right)\right| \leq M\left[1 + \int_{\rho_1}^{\infty}\left(\frac{\tau^{-3}}{|D^2(\tau, \lambda/L)|} + \frac{\tau^{-4} + \lambda^{-2}}{|D^3(\tau, \lambda/L)|}\right) d\tau\right]. \tag{2.32}$$

In view of (2.10) and (2.18) it is clear that the right-hand side of (2.32) is bounded on $\{\lambda \geq L: \rho \leq \omega(\lambda) \leq 2\rho_1\} \equiv \Lambda$. For each $\lambda \in [L, \infty)\backslash\Lambda$, let $E_1 = [\rho_1, \omega/2]$, $E_2 = [\omega/2, \omega - \varepsilon]$, $\omega_3 = [\omega - \varepsilon, \omega + \varepsilon]$, $E_4 = [\omega + \varepsilon, \infty)$. On E_1, (2.29) and (2.30) imply that

$$Q\,|D(\tau, \lambda/L)| \geq Q\tau\,|\Theta(\tau) - L/\lambda| \geq \frac{(\omega + \tau)\omega L}{2\lambda\tau}\,\frac{R}{\tau\rho_1}$$
$$\geq \frac{(\omega + \tau)\omega L}{4\lambda\tau}.$$

On $E_2 \cup E_4$, (2.31) shows that

$$Q\,|D(\tau, \lambda/L)| \geq \frac{L\,|\tau - \omega|}{2\lambda}.$$

On E_3, (2.3) and (2.26) show that $\lambda^{-1} = \theta(\tau) + O(\omega^{-3})$ ($\lambda \to \infty$, uniformly for $|\tau - \omega| \leq \varepsilon$). Then by (2.12) and (2.28),

$$\sup_{\omega(\lambda) \geqslant 2\rho_1,\, |\tau - \omega| \leqslant \varepsilon} \frac{1}{\lambda\Phi(\tau)} < \infty.$$

Thus for $\lambda \notin \Lambda$ we can estimate the integral on the right in (2.32) as follows ((2.26) is used):

$$\left|\int_{\rho_1}^{\infty}\right| \leq M\left\{\int_{E_1}\left[\frac{\tau^{-3}}{(\omega/4\lambda)^2} + \frac{\lambda^{-2} + \tau^{-4}}{(\omega/4\lambda)^3(1 + \omega/\tau)}\right] d\tau\right.$$
$$+ \int_{E_2}\left[\frac{\lambda^{-3/2}}{(\tau - \omega)^2/4\lambda^2} + \frac{\lambda^{-2}}{(\tau - \omega)^3/8\lambda^3}\right] d\tau$$
$$+ \int_{E_3}\left[\frac{\lambda^{-3/2}}{\Phi^2(\tau)} + \frac{\lambda^{-4}}{\Phi^3(\tau)}\right] d\tau$$
$$\left. + \int_{E_4}\left[\frac{\tau^{-3}}{(\tau - \omega)^2/4\lambda^2} + \frac{\tau^{-4}}{(\tau - \omega)^3/8\lambda^3} + \frac{8\lambda}{(\tau - \omega)^3}\right] d\tau\right\}$$
$$\leq M\left\{\lambda\int_{\rho_1}^{\infty}\tau^{-3}\,d\tau + \lambda^{-1/2}\int_0^{\omega/2} d\tau + \int_{\varepsilon}^{\infty}\left(\frac{\lambda^{1/2}}{\sigma^2} + \frac{\lambda}{\sigma^3}\right) d\sigma + 2\varepsilon\lambda\right\}$$
$$\leq M\lambda.$$

This establishes (2.25) and completes the proof of (1.18) and (1.19).

Finally, we sketch the proof of (1.10). In place of (2.23), we have

$$\pi u_t(t, \lambda) = \mathrm{Re}\left\{\frac{1}{t\lambda}\int_0^{\infty} e^{i\tau t}\,\frac{\tau\hat{A}'(\tau) - \hat{A}(\tau)}{D^2(\tau, \lambda)}\,d\tau\right\} \qquad (t > 0); \tag{2.33}$$

the integral converges, by (2.16) and (2.19). Integration by parts, together with (2.16) and

(2.19), shows that

$$\left|\int_0^\infty e^{i\tau t}\,\frac{\tau\hat{A}'(\tau)-\hat{A}(\tau)}{D(\tau,\lambda)}\,d\tau\right| \le \frac{M}{t}\left[1+\int_{\rho_1}^\infty \frac{\lambda^{-1}+\tau^{-2}}{\tau\,|D^2(\tau,\lambda)|}\,d\tau\right]. \tag{2.34}$$

Estimating the denominator as above when $\omega \ge 2\rho_1$, we get

$$\begin{aligned}\int_{\rho_1}^\infty \frac{\lambda^{-1}+\tau^{-2}}{\tau\,|D^3(\tau,\lambda/L)|}\,d\tau &\le M\left\{\int_{E_1}\frac{(\lambda^{-1}+\tau^{-2})\,d\tau}{\tau(\omega/4\lambda)^3(1+\omega/\tau)}+\int_{E_2}\frac{\lambda^{-3/2}\,d\tau}{(\tau-\omega)^3/\lambda^3}\right.\\ &\qquad\left.+\int_{E_3}\frac{\lambda^{-3/2}\,d\tau}{\Phi^3(\tau)}+\int_{E_4}\frac{(\lambda^{-1}+\tau^{-2})\,d\tau}{\tau(\tau-\omega)^3/\lambda^3}\right\}\\ &\le M\left\{\int_{\rho_1}^{\omega/2}\frac{\lambda^{1/2}\,d\tau}{\rho_1}+\int_{\rho_1}^\infty\frac{\lambda\,d\tau}{\tau^2}+\int_\varepsilon^\infty\frac{\lambda^{3/2}\,d\tau}{\sigma^3}\,d\sigma+2\varepsilon\lambda^{3/2}\right\}\\ &\le M\lambda^{3/2}.\end{aligned}$$

Thus by (2.33) and (2.34),

$$t^2\sup_{\lambda\geqslant 1}|\lambda^{-1/2}u_t(t,\lambda)| < \infty.$$

Since (1.16) holds, (1.20) is proved.

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