ON STATIC SIMILARITY DEFORMATIONS FOR ISOTROPIC MATERIALS*

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Abstract. For static deformations of isotropic materials satisfying the principle of material indifference, the governing equations remain invariant under arbitrary orthogonal rotations of the material and spatial coordinate systems. Moreover, the basic equations are also invariant under the same change of length scale for both coordinate systems. The general functional form of the similarity deformation corresponding to these invariances is deduced. Although these invariances involve seven arbitrary constants, it is shown that by an appropriate selection of coordinates only three arbitrary constants are involved in an essential way.

1. Introduction. Many of the known solutions in finite elasticity and fluid mechanics are in fact similarity solutions which arise due to simple invariances of the governing equations, such as invariance under rotations, stretchings and translations (see for example, Klingbeil and Shield [1], Wesolowski [2] and Langlois [3]). With this in mind it would seem worthwhile to deduce the general functional form for static three-dimensional deformations which is applicable to all isotropic materials satisfying the principle of material indifference. For these materials we show that, with an appropriate selection of material and spatial spherical polar coorndinates (R, Θ, Φ) and (r, θ, ϕ), respectively such deformations are given simply as

$$\begin{aligned} r &= Rf(\Theta, \alpha\Phi + \beta \log R), \\ \theta &= g(\Theta, \alpha\Phi + \beta \log R), \\ \phi &= \gamma\Phi + h(\Theta, \alpha\Phi + \beta \log R), \end{aligned} \tag{1}$$

where α, β and γ denote arbitrary constants and f, g and h are functions of Θ and $\alpha\Phi + \beta \log R$ only. The deformation (1) arises as a consequence of the invariance of the equations under arbitrary orthogonal rotations of the coordinate axes and changes of the length scale employed. These invariances actually involve seven arbitrary constants. The thrust of this note is that only three combinations of the seven constants are relevant (resulting in α, β and γ in Eq. (1)) if the spherical polar coordinates are selected in an appropriate manner. This fact is by no means obvious and only becomes apparent in the process of deducing the functional form (1).

The results obtained in this note were originally obtained for finite elastic deformations of the isotropic incompressible neo-Hookean material and the technqiue of constructing solutions of partial differential equations by means of one-parameter groups (see Bluman and Cole [4]). For the neo-Hookean material it can be shown that the "classical" procedure of Bluman and Cole [4] yields only the known rotation and stretching invariances.

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These calculations are extremely long and tedious and can be found in Hill [5]. Here we restrict our attention to only the functional forms of the deformation resulting from these groups which are then applicable to all isotropic materials satisfying the principle of material indifference.

With a symbolic notation, suppose that the known one-parameter group of transformations given by

$$\begin{aligned} X^* &= F(X, x, \varepsilon) = X + \varepsilon U(X, x) + O(\varepsilon^2), \\ x^* &= G(X, x, \varepsilon) = x + \varepsilon u(X, x) + O(\varepsilon^2), \end{aligned} \tag{2}$$

leaves invariant the partial differential equation denoted symbolically by

$$L[x] = 0, \tag{3}$$

where X is the independent variable. Then if $x = x(X)$ is a solution of (3) we have that $x^* = x(X^*)$ and therefore

$$x + \varepsilon u = x(X + \varepsilon U) + O(\varepsilon^2). \tag{4}$$

With an obvious notation we deduce from Eq. (4) on equating terms of order ε that

$$(U.\ \nabla_X)x = u, \tag{5}$$

and it is this equation which gives rise to the appropriate similarity variables and the functional form of the solution $x = x(X)$. Moreover, we observe that the "global" form of the one-parameter group (2) is obtained formally from the "infinitesimal" form by integrating the following autonomous system of ordinary differential equations:

$$dX^*/d\varepsilon = U(X^*, x^*), \qquad dx^*/d\varepsilon = u(X^*, x^*), \tag{6}$$

subject to the initial conditions

$$X^* = X, \qquad x^* = x \qquad \text{when} \qquad \varepsilon = 0. \tag{7}$$

In the following section we illustrate the procedure for plane deformations. For two dimensions the solution of (5) for the rotation and stretching groups is relatively straightforward. In the final section we consider fully three-dimensional deformations and deduce Eq. (1).

2. Plane deformations. With material and spatial rectangular cartesian coordinates (X, Y, Z) and (x, y, z) respectively we consider the plane deformation

$$x = x(X, Y), \qquad y = y(X, Y), \qquad z = Z. \tag{8}$$

For the isotropic incompressible neo-Hookean material the "classical" procedure of Bluman and Cole [4] yields the following one-parameter group (see Hill [5])

$$\begin{aligned} X^* &= X + \varepsilon(aX + BY) + O(\varepsilon^2), \qquad & Y^* &= Y + \varepsilon(-BX + aY) + O(\varepsilon^2), \\ x^* &= x + \varepsilon(ax + by) + O(\varepsilon^2), \qquad & y^* &= y + \varepsilon(-bx + ay) + O(\varepsilon^2), \end{aligned} \tag{9}$$

where a, b and B denote arbitrary constants and we have neglected additive constants which reflect the invariance of the equations under translations of the coordinate axes. The constant a represents the independence of the equations with respect to the length scale employed while B and b reflect the invariance of the equations under rotations of the initial and final coordinates respectively.

In order to solve the first-order partial differential equation corresponding to (5) we introduce a characteristic parameter σ in the usual way and obtain

$$\begin{aligned} dX/d\sigma &= aX + BY, \qquad dY/d\sigma = -BX + aY, \\ dx/d\sigma &= ax + by, \qquad dy/d\sigma = -bx + ay. \end{aligned} \tag{10}$$

In terms of the usual cylindrical polar coordinates (R, Θ) and (r, θ) defined by

$$\begin{aligned} R &= (X^2 + Y^2)^{1/2}, \qquad \Theta = \tan^{-1}(Y/X), \\ r &= (x^2 + y^2)^{1/2}, \qquad \theta = \tan^{-1}(y/x), \end{aligned} \tag{11}$$

Eqs. (10) become simply

$$\begin{aligned} dR/d\sigma &= aR, \qquad d\Theta/d\sigma = -B, \\ dr/d\sigma &= ar, \qquad d\theta/d\sigma = -b. \end{aligned} \tag{12}$$

From (12) it is a simple matter to deduce the following functional form for plane deformations:

$$r = Rf(a\Theta + B \log R), \qquad \theta = (b/B)\Theta + g(a\Theta + B \log R), \tag{13}$$

where $\xi = a\Theta + B \log R$ is the similarity variable and f and g are functions of ξ only which are determined from the appropriate governing equations. We note that for finite elastic deformations the case $b = 0$ is considered by Klingbeil and Shield [1] and that the important solution of Ericksen's problem given by Klingbeil and Shield [1] and Singh and Pipkin [6] is also a special case of Eq. (13). Thus the more general three-dimensional deformation (1) may also be relevant to Ericksen's problem (Ericksen [7]).

Finally in this section we note that on solving the equations corresponding to (6) and (7) we obtain in a straightforward manner the global form of the one-parameter groups, namely

$$\begin{aligned} \begin{pmatrix} X^* \\ Y^* \end{pmatrix} &= e^{a\varepsilon} \begin{pmatrix} \cos B\varepsilon & \sin B\varepsilon \\ -\sin B\varepsilon & \cos B\varepsilon \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \\ \begin{pmatrix} x^* \\ y^* \end{pmatrix} &= e^{a\varepsilon} \begin{pmatrix} \cos b\varepsilon & \sin b\varepsilon \\ -\sin b\varepsilon & \cos b\varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned} \tag{14}$$

3. Three-dimensional deformations. For three-dimensional deformations

$$x = x(X, Y, Z), \qquad y = y(X, Y, Z), \qquad z = z(X, Y, Z), \tag{15}$$

of the isotropic incompressible neo-Hookean material the classical procedure gives (see Hill [5])

$$\begin{aligned} X^* &= X + \varepsilon(aX + BY + CZ) + O(\varepsilon^2), \qquad Y^* = Y + \varepsilon(-BX + aY + DZ) + O(\varepsilon^2), \\ Z^* &= Z + \varepsilon(-CX - DY + aZ) + O(\varepsilon^2), \qquad x^* = x + \varepsilon(ax + by + cz) + O(\varepsilon^2), \\ y^* &= y + \varepsilon(-bx + ay + dz) + O(\varepsilon^2), \qquad z^* = z + \varepsilon(-cx - dy + az) + O(\varepsilon^2), \end{aligned} \tag{16}$$

where a, b, c, d, B, C and D denote seven arbitrary constants. In order to find the functional forms of the similarity solutions we need to deduce two integrals of the following system of ordinary differential equations:

$$\begin{aligned} dX/d\sigma &= aX + BY + CZ, \qquad dY/d\sigma = -BX + aY + DZ, \\ dZ/d\sigma &= -CX - DY + aZ. \end{aligned} \tag{17}$$

With the usual spherical polar coordinates (R, Θ, Φ) defined by

$$R = (X^2 + Y^2 + Z^2)^{1/2},\ \Theta = \tan^{-1}[(X^2 + Y^2)^{1/2}/Z],\ \Phi = \tan^{-1}(Y/X), \tag{18}$$

Eq. (17) becomes

$$dR/d\sigma = aR, \qquad d\Theta/d\sigma = E \cos \Psi, \qquad d\Psi/d\sigma = -B - E \cot \Theta \sin \Psi, \tag{19}$$

where $E = (C^2 + D^2)^{1/2}$ and Ψ is defined by

$$\Psi = \Phi - \tan^{-1}(D/C). \tag{20}$$

From $(19)_2$ and $(19)_3$ we have

$$E \cos \Psi \sin \Theta\, d\Psi + E \sin \Psi \cos \Theta\, d\Theta + B \sin \Theta\, d\Theta = 0, \tag{21}$$

from which it is apparent that one integral of (19) is given by

$$\zeta = B \cos \Theta - E \sin \Theta \sin \Psi. \tag{22}$$

From $(19)_1$ and $(19)_2$ we have

$$\frac{d\Theta}{E \cos \Psi} = \frac{dR}{aR}, \tag{23}$$

and on using (22) to express Ψ as a function of Θ and ζ we can deduce that a second independent integral of (19) is given by

$$\eta = a \sin^{-1}\left\{\frac{E \cos \Theta + B \sin \Theta \sin \Psi}{[M^2 - (B \cos \Theta - E \sin \Theta \sin \Psi)^2]^{1/2}}\right\} + M \log R, \tag{24}$$

where $M = (B^2 + C^2 + D^2)^{1/2}$. If we introduce new polar angles (Θ_1, Ψ_1) defined by

$$\begin{aligned} \cos \Theta_1 &= (B/M)\cos \Theta - (E/M)\sin \Theta \sin \Psi, \\ \sin \Theta_1 \sin \Psi_1 &= (E/M)\cos \Theta + (B/M)\sin \Theta \sin \Psi, \end{aligned} \tag{25}$$

then the two integrals (22) and (24) of (19) are essentially Θ_1 and $a\Psi_1 + M \log R$ respectively. Further, from (20) and (25) it is clear that the appropriate spherical polar system to be employed (R, Θ_1, Ψ_1) is obtained by first rotating the (X, Y) plane about the Z axis through an angle $\tan^{-1}(D/C)$ and then rotating the (Y, Z) plane about the X axis through an angle $\tan^{-1}[(C^2 + D^2)^{1/2}/B]$.

Similarly for the spatial coordinates we have

$$dr/d\sigma = ar, \qquad d\theta/d\sigma = e \cos \psi, \qquad d\psi/d\sigma = -b - e \cot \theta \sin \psi, \tag{26}$$

where $e = (c^2 + d^2)^{1/2}$, (r, θ, ϕ) are the usual spherical polars and ψ is defined by

$$\psi = \phi - \tan^{-1}(d/c). \tag{27}$$

Further, the appropriate polar angles (θ_1, ψ_1) are defined by the relations

$$\begin{aligned} \cos \theta_1 &= (b/m)\cos \theta - (e/m)\sin \theta \sin \psi, \\ \sin \theta_1 \sin \psi_1 &= (e/m)\cos \theta + (b/m)\sin \theta \sin \psi, \end{aligned} \tag{28}$$

where $m = (b^2 + c^2 + d^2)^{1/2}$. With this notation it is now a simple matter to deduce from (19) and (26) that the functional form of the deformation resulting from the rotation and stretching invariances and in terms of the spherical polar coordinates (R, Θ_1, Ψ_1) and

(r, θ_1, ψ_1) become

$$
\begin{aligned}
r &= Rf(\Theta_1, a\Psi_1 + M \log R), \\
\theta_1 &= g(\Theta_1, a\Psi_1 + M \log R), \\
\psi_1 &= (m/M)\Psi_1 + h(\Theta_1, a\Psi_1 + M \log R),
\end{aligned}
\tag{29}
$$

where f, g and h are functions of the arguments indicated. We see that Eq. (1) arises from Eq. (29) and that the deformation depends in an essential way only on the three constants a, m and M.

Finally, we note the following expressions for the global form of the one-parameter groups (16) which are obtained from the equations corresponding to (6) and (7). We find that

$$
\mathbf{X}^* = e^{a\varepsilon} \underset{\sim}{Q}(\varepsilon)\mathbf{X}, \quad \mathbf{x}^* = e^{a\varepsilon} \underset{\sim}{q}(\varepsilon)\mathbf{x}, \tag{30}
$$

where $\mathbf{X}$, $\mathbf{x}$, $\mathbf{X}^*$ and $\mathbf{x}^*$ denote the obvious column vectors and where $\underset{\sim}{Q}(\varepsilon)$ and $\underset{\sim}{q}(\varepsilon)$ are orthogonal matrices given by

$$
\begin{aligned}
\underset{\sim}{Q}(\varepsilon) &= \underset{\sim}{\delta} + \sin M\varepsilon\, \underset{\sim}{Q}_1 + (1 - \cos M\varepsilon)\underset{\sim}{Q}_1^2, \\
\underset{\sim}{q}(\varepsilon) &= \underset{\sim}{\delta} + \sin m\varepsilon\, \underset{\sim}{q}_1 + (1 - \cos m\varepsilon)\underset{\sim}{q}_1^2,
\end{aligned}
\tag{31}
$$

where $\underset{\sim}{\delta}$ is the unit matrix and

$$
\underset{\sim}{Q}_1 = \begin{bmatrix} 0 & B/M & C/M \\ -B/M & 0 & D/M \\ -C/M & -D/M & 0 \end{bmatrix}, \qquad \underset{\sim}{q}_1 = \begin{bmatrix} 0 & b/m & c/m \\ -b/m & 0 & d/m \\ -c/m & -d/m & 0 \end{bmatrix}. \tag{32}
$$

The orthogonality relations

$$
\underset{\sim}{Q}\underset{\sim}{Q}^T = \underset{\sim}{\delta}, \qquad \underset{\sim}{q}\underset{\sim}{q}^T = \underset{\sim}{\delta}, \tag{33}
$$

can be checked directly from (31) making use of the relations

$$
\underset{\sim}{Q}_1^4 = -\underset{\sim}{Q}_1^2, \qquad \underset{\sim}{q}_1^4 = -\underset{\sim}{q}_1^2. \tag{34}
$$

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