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ON A CERTAIN CLASS OF ELASTIC MATERIALS WITH NON-ELLIPTIC ENERGY DENSITIES*

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Abstract. In this paper we are concerned with the behavior of a certain class of hyperelastic materials with non-elliptic strain energy density functions that satisfy very weak growth conditions at infinity. It is shown, by means of an example, that solutions of boundary-value problems involving these materials do not exist within the usual admissible spaces in view of the severe discontinuities involved (displacement discontinuities).

1. Introduction. Although the boundary-value problem in finite elasticity was among the first problems to be consistently posed in a proper continuum mechanics framework around the early fifties, most of the associated mathematical problems pertaining to the existence, uniqueness and regularity of the corresponding solutions remained practically unexplored for quite some time. Recently, however, considerable progress has been achieved in this direction. Following some earlier work by Antman [1] on the one-dimensional problem, Ball [2] was apparently the first to prove an existence theorem for the general three-dimensional boundary-value problem in elasticity, using a physically sound form for the strain energy density function. One implication of his constitutive assumptions was that, roughly speaking, the resulting system of equilibrium equations had to be elliptic[1] for all possible displacement fields.

Unfortunately, physically interesting cases exist where the ellipticity condition can be violated for some deformation states. Knowles et al. [3] studied extensively antiplane shear deformations of some non-elliptic materials and Ericksen [4] justified the existence of non-elliptic constitutive equations based on phase transformations observed in certain materials.

One of the most interesting characteristics in solutions of problems of this kind was the possibility of "weak shocks," i.e., discontinuities in the displacement gradient even for homogeneous bodies with smooth boundaries. An existence theorem for the general three-dimensional problem involving the aforementioned materials was recently presented by Dacorogna [5]. The existence proof for this broader class of constitutive laws still requires

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[1] Here and subsequently, by abuse of terminology, we call a material elliptic if the resulting system of equilibrium equations is elliptic for all admissible displacement fields.

some assumptions about the behavior of the energy density functions at large strains. Again, experimental evidence points to the existence of elastic materials whose behavior near infinite strains does not obey the growth conditions of the materials considered in [5]. Experiments conducted by Blatz and Ko [6] concluded forms of the stored energy functions which, for large strains, increase so slowly that the material becomes non-elliptic. In one dimension it can be shown that violation of the coercivity assumptions in [5] is roughly equivalent (if adequate smoothness is required) with the loss of ellipticity (or convexity) of the strain energy function at the neighborhood of infinite strains.

Solutions of boundary-value problems for such materials exhibit some unusual features. For example, numerical investigations by Tvergaard, Needleman and Lo [7] and also Triantafyllidis, Needleman and Tvergaard [8] showed that if a certain maximum level of strain is exceeded, the material unloads everywhere except at vanishingly narrow zones of increasingly localized shear (the finer the mesh subdivision, the narrower the zone and higher the strains). In an attempt to provide some mathematical explanation for these phenomena, we show here by means of a simple model that structures made of these materials (i.e., the ones which for arbitrarily large strains are non-elliptic) admit, in a limiting sense, solutions with displacement discontinuities and zero total strain energy. From a mathematical viewpoint, we present a procedure to construct minimizing (of the total energy) sequences for these materials and show that they can possess limits outside the original admissible function space. Moreover, the energy for the limit function cannot even be defined in an ordinary sense in view of its severe discontinuities.

2. Statement of the problem considered. Instead of considering general (three-dimensional) deformations of a hyperelastic body, we shall restrict our attention to the (essentially one-dimensional) problem of the shear deformation of an infinitely long, homogeneous, isotropic and incompressible layer of constant thickness h under conditions of plane strain. The analytical representation of the kinematical assumptions made above can be put in the form (see also Fig. 1)

$$x = X = u(Y), \qquad y = Y, \qquad z = Z \tag{2.1}$$

where (x, y, z) is the current position of a particle identified with its Cartesian coordinates (X, Y, Z) at the reference state in the three-dimensional Euclidean space $\mathbb{R}^3$.

For an isotropic hyperelastic material, the strain energy density function W depends on the three invariants I_c, II_c, III_c of the Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$ where $F_{ij} =$

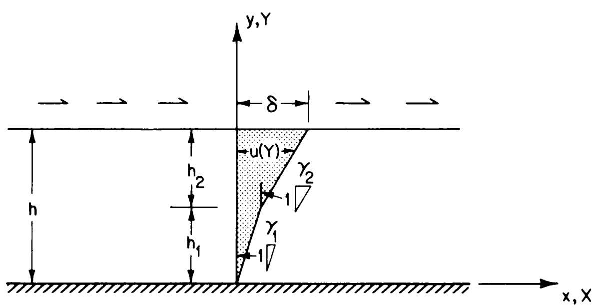

FIG. 1. Layer configuration and graph of a possible equilibrium solution.

$\partial x_i/\partial X_j$ is the deformation gradient. For the assumed deformation in (2.1) one has

$$I_c = 3 + [u'(Y)]^2, \qquad II_c = 3 + [u'(Y)]^2, \qquad III_c = 1 \tag{2.2}$$

where $u' \equiv du/dY$ is the shear strain of the layer and will be subsequently denoted for simplicity by γ. Here we also note that the assumed deformation of the layer satisfies incompressibility (see (2.2)).

Therefore, if only pure shear deformations of the type (2.1) are considered, the strain energy density function W depends solely on the shear strain γ.

We can now set the boundary-value problem for the sheared layer described above (in the case where thermal effects and body forces are ignored) as follows: If $E(\gamma)$ is the total potential energy of the layer (per unit area in the XZ plane)

$$E(\gamma) = \int_0^h W(\gamma)\, dY, \qquad \gamma = du/dY \tag{2.3}$$

then the function $u(Y)$ which satisfies the boundary conditions

$$u(0) = 0, \qquad u(h) = \delta \tag{2.4}$$

belongs to a certain functional space S (whose exact nature will be specified later) and minimizes the total potential energy E; it will be called "stable equilibrium" (or simple "stable") solution of the boundary-value problem.

Obviously the existence of a solution to the above problem, as well as its behavior, depends crucially on the properties of the strain energy density function W. In the rest of this section we shall motivate the particular choice of W which will be used subsequently. At this point we also note that without loss of generality we can assume $W(\gamma)$ to be an even function of γ and also that $\tau = dW/d\gamma > 0$ for $\gamma > 0$ ($\tau = 0$ iff $\gamma = 0$). This last condition is an analytical statement of the requirement that the shear strain γ should be of the same sign as the shear stress τ.

For the one-dimensional class of deformations considered here (see 2.1)), one can show existence of a unique solution $u(Y)$ to the boundary-value problem described by (2.3), (2.4) when the strain energy density function satisfies the condition

$$d^2W/d\gamma^2 > 0, \qquad \forall \gamma \in \mathbb{R}. \tag{2.5}$$

The condition (2.5) ensures the convexity of W. It can also be shown (see Ericksen [4]) that the same condition expresses the ellipticity of W in the case of plane-strain, incompressible deformations of an isotropic elastic body.

For strain energy density functions W that violate the ellipticity condition (2.5), one can show that a minimization problem of the type (2.3), (.24) can still admit a solution (but not necessarily a unique one) in an appropriate sense (see for example Ekeland and Temam [9] for the one-dimensional case and Dacorogna [5] for a more general three-dimensional version of the pertinent theorems) once the following coercivity condition is satisfied:

$$a + b|\gamma|^p \le W(\gamma) \le c + d|\gamma|^p \quad a, c \in \mathbb{R}, \quad b, d > 0, \quad p > 1. \tag{2.6}$$

Interesting cases exist, however, where (2.6) is no longer applicable. Knowles (see [3] and the references cited therein), for example, has extensively used for his studies in antiplane-shear deformation of various elastic bodies a strain energy density of the form

$$W(\gamma) = E_1[(1 + \gamma^2)^p - 1], \qquad 0 < p < \tfrac{1}{2}, \qquad E_1 > 0. \tag{2.7}$$

Motivated by studies in large deformations of metals, Hutchinson et al. [10] (see also [7], [8]) uses a strain energy density function which under a uniaxial deformation exhibits a power law behavior of the type $\sigma = K\varepsilon^{1/m}$ where σ is the uniaxial Cauchy stress ε is the logarithmic strain and $m > 1$ is a hardening exponent. For plane-strain isochoric deformations the corresponding strain energy density can be written as:

$$\begin{aligned} W(u) &= \frac{E}{2}\,\varepsilon_e^2 \quad \text{for} \quad 0 \le \varepsilon_e \le \varepsilon_Y \\ &= \frac{m}{m+1}\,E\varepsilon_Y^2\left(\frac{\varepsilon_e}{\varepsilon_Y}\right)^{(m+1)/m} + \frac{1-m}{1+m}\,\frac{E}{2}\,\varepsilon_r^2 \quad \text{for} \quad \varepsilon_e \ge \varepsilon_Y \end{aligned} \tag{2.8}$$

where $\varepsilon_e = (3)^{-1/2}\ln\,[1 + \gamma^2/2 + (\gamma^4/4 + \gamma^2)^{1/2}]$.

It is not very difficult to verify that both (2.7) and (2.8) are not coercive in the sense of (2.6). Moreover, it can be readily shown that for each of the above models one can find a number γ_m such that

$$\begin{aligned} \frac{d\tau}{d\gamma} &= \frac{d^2W}{d\gamma^2} \ge 0 \qquad \text{for} \qquad 0 \le \gamma \le \gamma_m, \\ \frac{d\tau}{d\gamma} &= \frac{d^2W}{d\gamma^2} \le 0 \qquad \text{for} \qquad \gamma \ge \gamma_m, \end{aligned} \tag{2.9}$$

$$\tau = dw/d\gamma = 0 \text{ at } \gamma = 0, \qquad \tau \to 0 \quad \text{as} \quad \gamma \to \infty.$$

The fact that the aforementioned energy densities are noncoercive have very interesting implications in the behavior of the boundary-value problem (2.3), (2.4). For the rest of our discussion we can without loss of generality consider strictly increasing functions $W(\gamma)$, $W \in C^2[0, \infty)$ which satisfy (2.9). No further specification about W is needed for our purpose. Typical graphs for both W and its derivative τ in this case are depicted in Fig. 2.

3. Admissible equilibrium solutions. The equilibrium equation for the layer can be obtained from the Euler-Lagrange equation of the minimization problem (2.3), (2.4) and for

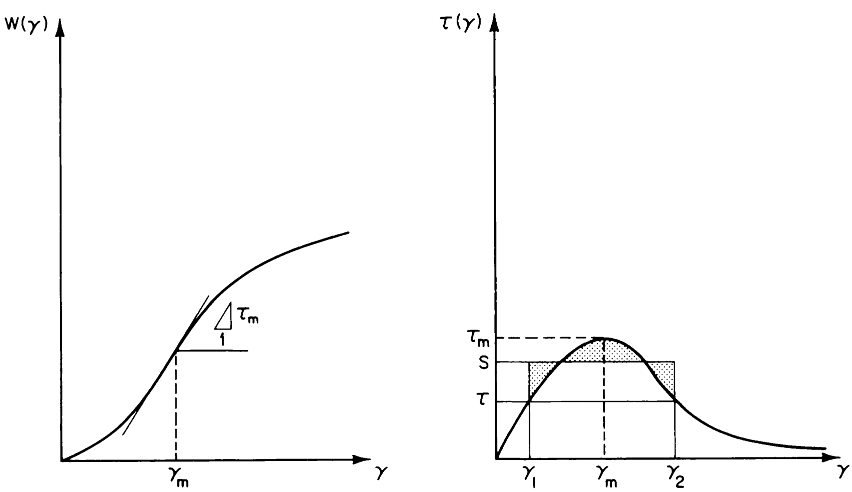

FIG. 2. Graphs of the assumed strain energy density W and its derivative τ (the shear stress) as functions of the shear strain γ.

piecewise continuous functions $u(Y)$ reduces to

$$dW/d\gamma \equiv \tau(\gamma) = \text{const on } [0, h] \qquad (u(0) = 0, \qquad u(Y) = \delta). \tag{3.1}$$

In view of the non-convexity of the energy density W, we will show that in fact (3.1) admits a multitude of solutions. Of course, one does not expect them all to be minimizers of the total energy E of the layer. In the rest of this section we examine the stability (in the sense of minimizing E) of the admissible equilibrium solutions for the class of energy density functions introduced in (2.9).

For simplicity let us assume that $u(Y)$ has only one point of discontinuity in its derivative at $Y = h_1$. From (3.1) we obtain

$$\begin{aligned} u(Y) &= \gamma_1 Y && \text{for} \quad 0 \le Y \le h_1 \\ &= \gamma_1 h_1 + \gamma_2(Y - h_1) && \text{for} \quad h_1 \le Y \le h_1 + h_2 = h \end{aligned} \tag{3.2}$$

where $\tau(\gamma_1) = \tau(\gamma_2)$. The total energy of the layer E is found from (3.2) and (2.3) to be

$$E = h_1 W(\gamma_1) + h_2 W(\gamma_2). \tag{3.3}$$

From (3.3) and the boundary condition $u(h) = \delta$ we obtain the following alternative expression for E:

$$\frac{E}{h} = \frac{\gamma_2 - \gamma_0}{\gamma_2 - \gamma_1} W(\gamma_1) + \frac{\gamma_0 - \gamma_1}{\gamma_2 - \gamma_1} W(\gamma_2), \qquad \gamma_1 \le \gamma_0 = \frac{\delta}{h} \le \gamma_2. \tag{3.4}$$

The above equations show that there exists an infinity of solutions to the boundary-value problem in (3.1). The uniform solution is also included in (3.2) as a special case when $\gamma_0 = \gamma_1$ or $\gamma_0 = \gamma_2$. It must also be noted at this point that the assumption of only one point of discontinuity for du/dY does not restrict the generality of our analysis. In fact (3.3) and (3.4) are valid for any piecewise continuous $u(Y)$ when h_1 and h_2 are interpreted as the total thicknesses of the layers that are currently under shear strains γ_1 and γ_2, respectively, where γ_1 and γ_2 are such that $\tau(\gamma_1) = \tau(\gamma_2)$.

Next the stability of the equilibrium solution of the type (3.2) will be examined. More specifically, we shall show that for $\gamma_0 < \gamma_m$ the uniform solution is a minimizer of the total energy E while for $\gamma_0 \ge \gamma_m$ this property is no longer valid.

We start with the observation that for a given $\gamma_0 < \gamma_m$ we can find γ_1 adequately close to γ_0 that

$$\frac{W(\gamma_0) - W(\gamma_1)}{\gamma_0 - \gamma_1} < \frac{W(\gamma_2) - W(\gamma_0)}{\gamma_2 - \gamma_0}, \qquad \tau(\gamma_1) = \tau(\gamma_2), \qquad \gamma_1 < \gamma_0. \tag{3.5}$$

Indeed, by considering each of the functions appearing in the above inequality as functions of τ we note that at the limit as $\tau \to \tau(\gamma_0)$, (3.5) reduces to the inequality

$$\tau(\gamma_1) = \left.\frac{dW}{d\gamma}\right|_1 < \frac{W(\gamma_2) - W(\gamma_1)}{\gamma_2 - \gamma_1} = s(\gamma_1), \qquad \gamma_0 = \gamma_1, \tag{3.6}$$

which is true in view of the assumed properties of W in (2.9) (see also Fig. 2). Then (3.5) follows from (3.6) and the continuity properties of W. After some manipulation it follows directly from (3.5) that

$$\frac{E}{h} = \frac{\gamma_2 - \gamma_0}{\gamma_2 - \gamma_1} W(\gamma_1) + \frac{\gamma_0 - \gamma_1}{\gamma_2 - \gamma_1} W(\gamma_2) > W(\gamma_0), \tag{3.7}$$

which proves that the uniform equilibrium solution is a local minimum for the total energy E and therefore a "stable" solution.

In an analogous way we can show that for $\gamma_0 > \gamma_m$ the uniform solution is "unstable," i.e., there exist equilibrium solutions of the type (3.2) arbitrarily close[2] to it with less total strain energy.

Thus for $\gamma_0 < \gamma_m$ the uniform solution is the stable one while for $\gamma_0 > \gamma_m$ the uniform solution is unstable. This last remark could have also been concluded from the property of the uniform solution that $d^2W/d\gamma^2 < 0$ for $\gamma_0 > \gamma_m$ and thus the corresponding $u(Y)$ cannot be an energy minimizer since the ellipticity condition is violated (see for example Morrey [11], Ball [2]). In fact, in view of the above remark any equilibirum solution of the type (3.2) with $\gamma_1 \neq \gamma_2$ will be "unstable" since it contains a finite zone (the zone under shear strain γ_2) where the ellipticity condition is violated. Moreover, for $\gamma_0 > \gamma_m$ we can show that there exist a sequence of equilibirum solutions to the problem (3.1) with corresponding energies tending to zero. Indeed, for fixed $\gamma_0 > \gamma_m$ we have from (3.4) and the continuity assumptions on W

$$\begin{aligned}\lim_{\tau\to 0}\left(\frac{E}{h}\right) &= \lim_{\tau\to 0}\left\{\frac{\gamma_2(\tau)-\gamma_0}{\gamma_2(\tau)-\gamma_1(\tau)}\,W(\gamma_1(\tau)) + \frac{\gamma_0-\gamma_1(\tau)}{\gamma_2(\tau)-\gamma_1(\tau)}\,W(\gamma_2(\tau))\right\}\\ &= \gamma_0\lim_{\tau\to 0}\frac{W(\gamma_2(\tau))}{\gamma_2(\tau)} = \gamma_0\lim_{\gamma\to\infty}\frac{W(\gamma)}{\gamma} = \gamma_0\lim_{\gamma\to\infty}\frac{dW}{d\gamma} = 0. \end{aligned} \tag{3.8}$$

It must be noted from the above limiting procedure that the pointwise limit, say $\bar{u}$, of the corresponding sequence of displacements $u(Y)$ in (3.2) will be a discontinuous function of Y. The corresponding stress in the layer is $\bar{\tau} = 0$ and the sum of jumps of $\bar{u}$ at its points of discontinuity will be equal to $\gamma_0 h = \delta_0$. On physical grounds one might reject the piecewise constant function $\bar{u}$ in view of its discontinuity with respect to Y. Also note that since $\bar{\tau} = 0$, the material is elliptic at all points of smoothness of $\bar{u}$, as one had to expect from a stable solution. In the next section we will adopt an alternative approach to our minimization problem (2.3), (2.4) which will justify the answer obtained here.

4. An alternative approach for the minimization problem—relaxation methods. In the previous section we have obtained closed-form solutions to the Euler-Lagrange equations of the minimization problem (Eqs. (3.1)), and we have discussed their stability. In this section we will follow a different approach to find stable solutions by using the idea of relaxation methods discussed by Ekeland and Temam [9] and quasiconvexification methods introduced by Dacorogna [5].

Let us again assume that we want to minimize the total energy of the layer E in (2.3) under the assumption (2.4) for W obeying (2.9). As far as the space S over which the minimization will take place, it should be a proper functional space for which (2.3), (2.4) make sense. For the example discussed here, we take the space S to be $W^{1,p}[0, h]$ with $p > 1$. We consider the following procedure in order to construct minimizing sequences for the total energy E in the case of non-coercive energy density functions W. Let us assume that $W_\varepsilon(\gamma) \in C^2[0, \infty]$ is a decreasing sequence (in ε) approaching W (for $\varepsilon \to 0$) for every real γ. Moreover, each W_ε possesses the coercivity property (2.6) where, of course, a, b, c, d

[2] It can be easily shown that if γ_1 is close to γ_0, then $u(Y; \gamma_1, \gamma_2)$ and $u(Y, \gamma_0)$ are also close to each other in some norm sense, e.g., $L^2[0. h]$, $H^1[0, h]$.

are constants dependent on ε. Without loss of generality we can also assume that

$$W(\gamma) \leq W_\varepsilon(\gamma) \leq W_{\varepsilon'}(\gamma), \qquad \varepsilon < \varepsilon', \qquad \forall \gamma \in \mathbb{R}, \qquad W(\gamma) = W_\varepsilon(\gamma), \qquad \gamma \in [0, 1/\varepsilon] \quad (4.1)$$

Pictorial representations of both W_ε and $\tau_\varepsilon = dW_\varepsilon/d\gamma$ are found in Fig. 3. From the aforementioned properties of W_ε we deduce

$$\inf_{u \in S} E(u) = \lim \{\lim_{u \in S} E_\varepsilon(u)\} \quad \text{where} \quad \begin{cases} E_\varepsilon(u) = \displaystyle\int_0^h W_\varepsilon\left(\frac{du}{dY}\right) dY \\ S \equiv W^{1,p}[0, h] \end{cases} \qquad (4.2)$$

But in view of the coercivity property satisfied by W_ε we can ensure the existence of a function $u_\varepsilon \in S$ that minimizes E_ε. Therefore, from (4.2) we obtain

$$\inf_{u \in S} E(u) = \lim_{\varepsilon \to 0} E_\varepsilon(u_\varepsilon).$$

The minimizing function u_ε can be found by using the relaxation theorem of Ekeland and Temam [9], according to which u_ε is the solution of the minimization problem (2.3), (2.4) where $W(\gamma)$ is replaced by $W_\varepsilon^*(\gamma)$ with W_ε^* being the lower convex envelope of W_ε. Pictorial representations of both W_ε^* and $dW_\varepsilon^*/d\gamma$ are also found in Fig. 3. Note that $\bar{\tau}^*(\varepsilon)$, the shear stress corresponding to the plateau region of $dW_\varepsilon^*/d\gamma$, separates the $dW_\varepsilon/d\gamma$ curve in such a way that the parts above and below that curve are isoareal.

It is not difficult to show that for the minimization problem (2.3), (2.4) with strain density W_ε^* being convex, the solution u_ε is of the form (3.12) where $\gamma_1 = \gamma_1^*(\varepsilon)$, $\gamma_2 = \gamma_2^*(\varepsilon)$ both correspond to stress $\bar{\tau}^*$ (see Fig. 3). Although $u_\varepsilon \in S$ the sequence is not uniformly bounded in that space. Fortunately, as can be easily observed in this case, the monotonic and continuous u_ε are bounded in any $L^p[0, h]$ space with $1 \leq p < \infty$. Since L^p is reflexive we can extract a weakly convergent subsequence to an element of L^p. Without difficulty we can see that functions of the type $\delta \cdot H(x_0)$, for any $x_0 \in (0, h)$ are limits of minimizing

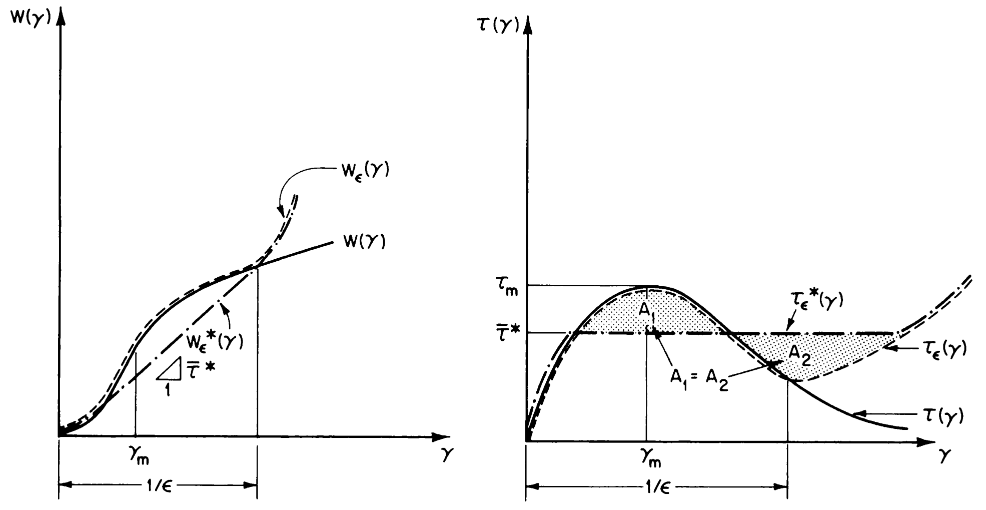

FIG. 3. Graphs of the strain energy density W (——), the coercive density W_ε (– – –) and its lower convex envelope W_ε^* (– · – · –). Also drawn are the graphs of their corresponding derivatives.

sequences of E. Note that $\delta \cdot H(x_0)$ does not belong to $W^{1,p}[0, h]$ and moreover for this particular function E cannot be defined, at least in an ordinary sense. (For this one-dimensional example there is a possibility of extending the space S of admissible functions of E using Young's [12] method, but this approach will not concern us here. By enlarging the class of admissible functions, the infimum is not going to change anyway since it is already zero.) As an additional remark, we observe the nonuniqueness of the limit of the minimizing sequences.

In conclusion, the example discussed here shows that for energy densities not obeying the coercivity condition (2.6) the problem might not have a (meaningful) solution in the space of admissible functions originally considered.

5. Discussion. So far we have shown that boundary-value problems in finite elasticity involving materials with non-coercive strain energy densities do not necessarily admit solutions, at least within some physically meaningful functional space. In proving our assertion we used an essentially one-dimensional example, but we also indicated an approach for constructing minimizing sequences for the potential energy functional in the three-dimensional problem. In this more general case, the sequence u_ε^* of solutions to the quasiconvexified problem involving W_ε^* might not be convergent in any sense.

The most important reservation that one should have for the use of non-coercive strain energy densities is their physical relevance. We have to remember that the supporting experimental evidence for all the suggested energy density functions W involves a rather limited range of strains. Although loss of ellipticity for W can be observed in certain cases, it is impossible to experimentally verify non-coercivity since any material can sustain only finite amounts of straining. Moreover, at these high strains, other physical mechanisms start becoming important and the path-independent idealization for any material becomes questionable.

Finally, on the mathematical side the conclusion that can be drawn here is that for non-coercive materials not even a meaningful existence problem can be posed.

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