ON THE NONEXISTENCE OF SOLUTION OF A DIFFERENTIAL SYSTEM GOVERNING AXISYMMETRIC FLOW OF A STRATIFIED FLUID*

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The differential equation governing steady axisymmetric flow of an incompressible fluid was first derived by Yih [3]. If cylindrical coordinates (r, y, z) are used with z measured vertically upward, and if g denotes the gravitational acceleration, ρ denotes the density, and ρ_0 denotes a constant reference density, the equation is [3, 4, 5]

$$\frac{1}{r^2}\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\psi' + \frac{g}{\rho_0}\frac{d\rho}{d\psi'}z = \frac{1}{\rho_0}\frac{dH}{d\psi'} = h(\psi'). \tag{1}$$

Here ψ' is a stream function related to the usual stream function ψ of Stokes by

$$\psi' = \int \left(\frac{\rho}{\rho_0}\right)^{1/2} d\psi, \tag{2}$$

and H is the Bernoulli quantity defined by

$$H = p + \frac{\rho}{2}(u^2 + w^2) + g\rho z, \tag{3}$$

p being the pressure and u and w being the velocity components in the directions of increasing r and z, respectively. From ψ' we obtain

$$u' = -\frac{1}{r}\frac{\partial\psi'}{\partial z} = \left(\frac{\rho}{\rho_0}\right)^{1/2} u, \qquad w' = \frac{1}{r}\frac{\partial\psi'}{\partial r} = \left(\frac{\rho}{\rho_0}\right)^{1/2} w. \tag{4a,b}$$

In order to solve (1), it is necessary to determine the functions $\rho(\psi')$ and $h(\psi')$, and for this determination one goes to an upstream section, where the density and velocity distributions are assumed. (Note that for steady flows ρ does not change along a streamline and hence is a function of ψ'.) Let us consider an axisymmetric flow between two horizontal planes at a distance d apart, and let the fluid flow into a point sink located on the axis of symmetry. For the problem of selective withdrawal of a stratified fluid the sink is usually assumed to be located on the upper or lower boundary.

If the flow far upstream, i.e. at large r, is assumed to be purely radial, then ψ' is a function of z only at $r = \infty$, and assuming that ρ, ψ', and $\partial\psi'/\partial z$ all exist at $r = \infty$ and are differentiable with z, we see from (1) that

$$\frac{g}{\rho_0}\frac{d\rho}{d\psi'} = h(\psi') \quad \text{at} \quad r = \infty. \tag{5}$$

The only linear case is obtained if we assume that at $r = \infty$ the density is given by

$$\rho = \rho_0 - \beta z \tag{6}$$

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and ψ' is given by

$$\psi' = U'z, \tag{7}$$

a constant having been omitted in (7) because it is of no consequence. The coefficients β and U' in (6) and (7) are constants. Then

$$d\rho/d\psi' = -\beta/U', \tag{8}$$

and

$$h(\psi') = -g\beta/\rho_0 U'^2. \tag{9}$$

Defining

$$(\xi, \zeta) = \left(\frac{r}{d}, \frac{z}{d}\right), \qquad \Psi = \frac{\psi'}{U'd}, \qquad F^2 = \frac{U'^2}{g'd}, \qquad g' = \frac{g\beta d}{\rho_0}, \tag{10}$$

we can write Eq. (1) as

$$\frac{1}{\xi^2}\left(\frac{\partial^2}{\partial \xi} - \frac{1}{\xi}\frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \zeta^2}\right)\Psi - F^{-2}\zeta = -F^{-2}\Psi. \tag{11}$$

The boundary conditions are

$$\Psi = 0 \quad \text{at} \quad \zeta = 0, \tag{12}$$

$$\Psi = 1 \quad \text{at} \quad \zeta = 1 \quad \text{and at} \quad \xi = 0. \tag{13}$$

The jump of the value of Ψ from 0 to 1 at the origin corresponds to the point source. The differential system consisting of (11), (12), and (13) was treated by Hino and Onishi in a series of papers starting in 1968 [1]). But, as will be seen below, their solution is fallacious**, and the differential system they treated has no solution.

To solve (11), let

$$\Psi = \eta + \phi. \tag{14}$$

Then ϕ must satisfy

$$\frac{1}{\xi^2}\left(\frac{\partial^2}{\partial \xi^2} - \frac{1}{\xi}\frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \zeta^2}\right)\phi - F^{-2}\phi = 0, \tag{15}$$

$$\phi = 0 \quad \text{at} \quad \zeta = 0, \tag{16}$$

$$\phi = 0 \quad \text{at} \quad \zeta = 1, \tag{17}$$

$$\phi = 1 - \zeta \quad \text{at} \quad \xi = 0. \tag{18}$$

Using the method of separation of variables, one obtains

$$\phi = \sum_{n=1}^{\infty} A_n f_n(\xi) \sin n\pi\zeta, \tag{19}$$

** The fallacy is akin to the assertion that $e^{-x+x} = e^{-x}(1 + x + (x^2/2) + \cdots)$ approaches zero as x approaches infinity, on the ground that each term of the expansion does so.

where f_n satisfies

$$f''_n - \frac{1}{\xi} f'_n + (F^{-2}\xi^2 - n^2\pi^2) f_n = 0, \tag{20}$$

and

$$\sum_{n=1}^{\infty} A_n f_n(0) \sin n\pi\zeta = 1 - \zeta. \tag{21}$$

It is necessary, for (7) to hold at infinity, that

$$f_n(\infty) = 0. \tag{22}$$

We shall see that no solution of (20) can possibly satisfy (22).

Let

$$\eta = iF^{-1}\xi^2. \tag{23}$$

Then (20) becomes

$$\frac{d^2 f_n}{d\eta^2} + \left(-\frac{1}{4} + \frac{k}{\eta} \right) f_\eta = 0, \tag{24}$$

where

$$k = -\frac{in^2\pi^2 F}{4}. \tag{25}$$

This is a special case of the Whittaker equation [2, p. 337]

$$\frac{d^2 W}{d\eta^2} + \left(-\frac{1}{4} + \frac{k}{\eta} + \frac{1 - 4m^2}{4\eta^2} \right) W = 0, \tag{26}$$

since (25) reduces to (24) if $4m^2 = 1$. One of the solutions is [2, p. 340], apart from a constant multiplier which is inconsequential in our case,

$$W_{k,\,1/2}(\eta) = e^{-\eta/2}\eta^k \int_0^\infty t^{-k} \left(1 + \frac{t}{\eta} \right)^k e^{-t}\, dt. \tag{27}$$

Note that the integral is convergent for the k given by (25). The other solution is [2, p. 343] $W_{-k,\,1/2}(-\eta)$.

For large $|\eta|$,

$$W_{k,\,1/2}(\eta) = e^{-\eta/2}\eta^k [1 + O(\eta^{-1})], \tag{28}$$

$$W_{-k,\,1/2}(-\eta) = e^{\eta/2}(-\eta)^{-k}[1 + O(\eta^{-1})]. \tag{29}$$

Since both η and k are imaginary, it is evident from (28) and (29) that both are complex, oscillatory, and undamped at $\xi = \infty$. Hence neither of them, nor any linear combination thereof, vanishes at infinite ξ.

Hence (22) can never be satisfied by any solution of (20). Thus, a solution by separation of variables does not exist. Furthermore, if a solution for ϕ in (14) exists at all, at any fixed r (or ξ) it must be expressible as a Fourier series in terms of $\sin n\pi\zeta$, because the set of functions $\sin n\pi\zeta$ $(n = 1, 2, \ldots)$ is complete and because the series satisfies the boundary conditions on ϕ. The coefficients are functions of r, since the solution is a function of r. If the differential system does not allow separation of variables, we have no easy way to determine these r-dependent coefficients. But the differential system under consideration does allow separation of variables, and thereby also allows a conclusive demonstration of the asymptotic behavior of $f_n(\xi)$. This is fortunate, but in no way restricts the validity of the conclusion of nonexistence of the solution that has been reached. In other words, the expansion (19) is valid if a solution exists, whether or not the variables can be separated. That the variables can indeed be separated is the lucky circumstance that allows the conclusion of nonexistence of the solution to be reached beyond doubt.

Thus if the density distribution at $r = \infty$ is given by (6), the ψ' at infinity cannot be given by (7), and the differential equation (1) will be nonlinear. If (6) does not hold at infinity, then (1) will be nonlinear, whether or not (7) holds. In either case separation of variables is out of the question. Hence in any case a nonlinear partial differential equation has to be solved, and furthermore one can only *assume* ψ' to be some function of z at infinite r, and see whether a solution can be found.

The problem, then, is extremely difficult, and even a powerful computer is not likely to be of much help. For a computer is no more, indeed very much less, able than the human mind to decide the distribution of ψ' at infinity, and, when a solution does not exist for an assumed ψ'-distribution at infinity, is not only unable to find the solution, but also unable to see the futility of such an attempt and thus to decide to abandon the effort. I should like, by this note, to call attention to the necessity of continuing the classical discipline traditional in fluid mechanics, even if, and perhaps especially because, the computer has been so helpful to us in the solution of problems in fluid mechanics.

References

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