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SOME UNIQUENESS THEOREMS FOR LINEAR WATER WAVES*

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Abstract. Uniqueness theorems, under less restrictive conditions at infinity, for water of uniform finite depth, for the cliff problem and for waves on uniform sloping beaches are proved using the Phragmén-Lindelöf principle.

1. Introduction. The uniqueness theory of the solutions for some linearized water wave problems has been investigated by many researchers using various methods. In particular, the uniqueness of solutions for two-dimensional water of uniform finite depth under the condition of boundedness at infinity was first proved by Weinstein [1] by an eigenvalue method. A similar theory for the three-dimensional case based on the assumption that the waves have cylindrical symmetry was given by Stoker [2]. For the problem of three-dimensional waves against a vertical cliff, a uniqueness theorem was obtained by Weinstein [3], again using the eigenvalue method. For the more interesting problem of progressing waves over uniform sloping beaches, Lewy [4], employing the method of complex variables treated the case of two-dimensional waves for sloping angles $\omega = (p/2n)\pi$, with p an odd integer and n any integer such that $0 < p < n$. Stoker [5] proved, essentially by using Liouville's theorem, uniqueness for $\omega = \pi/2n$ with n an integer. The most general three-dimensional case of periodic waves on sloping beach at any angle was first solved by Peters [6] and Roseau [7], who made use of a certain functional equation derived from a representation of the solution by a Laplace integral. In all the theorems cited above the solutions and their derivatives involved are assumed to be uniformly bounded at infinity. We shall prove, in this paper, uniqueness theorems under weaker conditions at infinity for water of uniform finite depth (two- and three-dimensional), for waves against a vertical cliff (three-dimensional) and for water waves on uniform sloping beach at an arbitrary angle (two- and three-dimensional). In the case of water of uniform finite depth, we have also derived explicit solutions. In fact, dock problems can also be handled in a similar way. Our proofs will be based on the application of the Phragmén-Lindelöf principle given in the book by Protter and Weinberger [8]. This technique has been used by Wen [9] to establish a uniqueness theorem for a water wave of infinite depth. It is to be noted that since all the boundary-value problems involved are homogeneous, by a unique solution we mean a non-trivial solution within a constant multiplying factor.

Phragmén-Lindelöf principle. Since we will repeatedly apply the Phragmén-Lindelöf (P—L) principle we state it here [8].

Let L be a uniformly elliptic second-order operator and D be a domain, bounded or

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unbounded, and let u satisfy

$$L[u] \geq 0 \text{ in } D, \qquad u \leq 0 \text{ in } \Gamma,$$

where Γ is a subset of ∂D. Suppose that there is an increasing sequence of domains $\{D_k\}$ with the properties:

(i) $D_k \subset D, \forall k$;
(ii) $\forall x \in D$ there is an integer N such that $x \in D_N$ (hence $x \in D_k, \forall k \geq N$);
(iii) $\partial D_k = \Gamma_k \cup \Gamma'_k$, where $\Gamma_k \subset \Gamma$ and $\Gamma'_k \subset D$.

Let $\{w_k\}$ be a sequence of functions such that

$$w_k(x) > 0 \quad \text{on} \quad D_k \cup \partial D_k, \tag{2.1}$$

$$L[w_k] \leq 0 \quad \text{in} \quad D_k. \tag{2.2}$$

Assume that there exists a function w with the property that at each point x of D the inequality $w_k(x) \leq w(x)$ holds for k above a certain integer N_x. If $u(x)$ satisfies the growth condition

$$\liminf_{k \to \infty} \left[\sup_{x \in \Gamma_{k'}} \frac{u(x)}{w(x)} \right] \leq 0 \tag{2.3}$$

then $u \leq 0$ in D.

3. Simple harmonic motion in water of constant depth. First we shall consider the two-dimensional problem. In the theory of small oscillations for water of uniformly finite depth the velocity potential $e^{i\sigma t}\phi(x, y)$ satisfies [2]:

$$\nabla^2\phi = 0 \text{ in } D = \{(x, y): \quad -\infty < x < \infty, \ -h < y < 0\}, \tag{3.1}$$

$$\phi_y - (\sigma^2/g)\phi = 0 \qquad \text{on } y = 0, \tag{3.2}$$

$$\phi_y = 0 \qquad \text{on } y = -h, \tag{3.3}$$

where σ, g and h are positive constants. At infinity we assume that

$$\phi = O(x^\alpha), \qquad \phi_y = O(x^\alpha) \quad \text{for} \quad |x| \to \infty \tag{3.4}$$

with $0 < \alpha < 1$. We also assume that ϕ_{yy} is uniformly bounded in some strip S with fixed width η, i.e. there is a constant M such that

$$|\phi_{yy}(x, y)| \leq M \quad \text{for} \quad (x, y) \in S \tag{3.5}$$

where

$$S = \{(x, y): -\infty < x < \infty, \ -h \leq y \leq -h + \eta\}. \tag{3.6}$$

2D THEOREM: The solution $\phi(x, y)$ of the boundary-value problem (3.1)–(3.5) is unique and is of the form

$$\phi = \cos(mx + \theta)\cosh m(y + h) \tag{3.7}$$

where θ is a constant and m satisfies

$$\sigma^2/g = m \tanh mh. \tag{3.8}$$

Proof. Let

$$\psi = \phi \operatorname{sech} m(y+h),$$
$$\Phi = \operatorname{sech} m(y+h)[\phi_y - m \tanh m(y+h)\phi].$$

If L is the elliptic operator defined by

$$\Gamma[f] = \nabla^2 f + 2m \tanh m(y+h) f_y + m^2[1 + 2 \operatorname{sech}^2 m(y+h)]f$$

then it can be verified that

$$L\Phi = 0 \quad \text{in } D, \tag{3.9}$$
$$\Phi = 0 \quad \text{on } y = 0, \tag{3.10}$$
$$\Phi = 0 \quad \text{on } y = -h. \tag{3.11}$$

We will apply the P-L principle to the boundary-value problem (3.9)–(3.11). Choose $D_k = \{(x, y): |x| < k, -h < y < 0\}$ $(k = 1, 2, 3, \ldots)$,

$$w_k = w = \operatorname{sech} m(y+h)\{2(y+h+1) + m \tanh m(y+h) \cdot [x^2 + (h+2)^2 - (y+h+1)^2]\} \qquad (k = 1, 2, 3, \ldots),$$
$$\partial D_k = \Gamma_k \cup \Gamma_k',$$

where Γ_k is the union of the horizontal edges of D_k and Γ_k' the union of the two vertical edges of D_k.

It is easily verified that $w_k > 0$ on $D_k \cup \partial D_k$ and $Lw_k = 0$ in D_k. So (2.1) and (2.2) are satisfied. We need to show that (2.3) is also satisfied. To this end, we let

$$A_k = \left\{(x, y): |x| = k, -h + \frac{1}{k} \le y < 0\right\},$$
$$B_k = \left\{(x, y): |x| = k, -h < y \le h + \frac{1}{k}\right\}.$$

Then $\Gamma_k' = A_k \cup B_k$. If k is sufficiently large, i.e. $k > 1/\eta$, we have on A_k

$$\left|\frac{\Phi}{w_k}\right| \le \frac{(M_1 + mM_2)k^\alpha}{mk^2 \tanh(m/k)}$$

where use has been made of (3.4), and M_1 and M_2 are constants. Since $\lim_{k\to\infty} k \tanh(m/k) = m$ and $0 < \alpha < 1$, we get

$$\liminf_{k\to\infty} \left|\frac{\Phi}{w_k}\right| = 0 \quad \text{on } A_k.$$

By (3.5), $|\phi_y| \le M|y + h| \le M/k$ on B_k, and using (3.4) we have for large k (i.e. $k > 1/\eta$)

$$\left|\frac{\Phi}{w_k}\right| \le \frac{|\phi_y|}{2} + \frac{|\phi|}{k^2} < \frac{M}{2k} + \frac{M_1 k^\alpha}{k^2}.$$

Hence $\liminf_{k\to\infty} |\Phi/w_k| \le 0$ on B_k. It then follows that

$$\liminf_{k\to\infty}\left(\sup_{\Gamma_{k'}} \frac{\Phi}{w_k}\right) \le 0$$

which establishes (2.3). Then, by the P-L principle, $\Phi \leq 0$ in D. Since $-\Phi$ also satisfies (3.9)–(3.11), we get $\Phi \geq 0$ in D, Since $\Phi \equiv 0$ in D, i.e.

$$\phi_y - m \tanh m(y + h)\phi \equiv 0 \text{ in } D.$$

Solving this, we get $\phi = A(x)\cosh m(y + h)$. But $\nabla^2\phi = 0$; this implies $A''(x) + m^2 A(x) = 0$. So $A(x) = \cos(mx + \theta)$, for some constants m and θ. Therefore

$$\phi = \cos(mx + \theta)\cosh m(y + h)$$

For the three-dimensional case, the formulation is the same except now D in (3.1) is replaced by

$$D = \{(x, y, z): -\infty < x < \infty, \ -\infty < z < \infty, \ -h < y < 0\}$$

and S in (3.6) is

$$S = \{(x, y, z): -\infty < x < \infty, \ -\infty < z < \infty, \ -h \leq y \leq h + \eta\}.$$

Also we replace x in (3.4) by $r = (x^2 + z^2)^{1/2}$ and note that ∇^2 in (3.1) is now a three-dimensional Laplace operator.

3D THEOREM. The solution $\phi(x, y, z)$ of the boundary-value problem (3.1)–(3.5), with D and S as indicated above, is unique and of the form:

$$\phi = \cosh m(y + h)\cos(x\sqrt{m^2 - k^2} + \theta_1)\cos(kz + \theta_2) \qquad (k^2 < m^2)$$

where θ_1 and θ_2 are constants and m is given by (3.8).

Proof. Since all the equations are the same, the proof is similar to that given for the two-dimensional case except now the D_k are rectangular boxes instead of rectangles. It suffices just to indicate how to construct w_k. It can be seen that

$$w_k = \operatorname{sech} m(y + h)\{2(y + h + 1) - m[(y + h + 1)^2 - \tfrac{1}{2}(x^2 + z^2) - h + 2)^2]\tanh m(y + h)\}$$

will work.

It is worth noting that in cylindrical coordinates ($r = (x^2 + x^2)^{1/2}$, $\theta = \tan^{-1}(z/x)$), if ϕ is independent of θ the unique solution is of the form

$$\phi = \cosh m(y + h)J_0(mr)$$

where J_0 is the Bessel function of the first kind of order zero.

4. Water waves against a vertical cliff. The problem of three-dimensional waves against a vertical cliff can be formulated by seeking solutions of velocity potential in the form $\exp\{i(\sigma t + kz + \beta)\}\phi(x, y)$, with ϕ satisfying the following non-dimensionalized boundary-value problem [2]:

$$\nabla^2_{(x,y)}\phi - k^2\phi = 0 \quad \text{in } D = \{(x, y): x > 0, \ y < 0\}, \tag{4.1}$$

$$\phi_y - \phi = 0 \quad \text{on } y = 0, \ x > 0, \tag{4.2}$$

$$\phi_x = 0 \quad \text{on } x = 0, \ y < 0, \tag{4.3}$$

$$\phi = \bar{\phi} \log r + \bar{\bar{\phi}} \quad \text{for } r = \sqrt{x^2 + y^2} \ll 1, \tag{4.4}$$

$$\phi = O(r^\alpha), \qquad \phi_x = O(r^\alpha), \qquad \phi_{xy} = O(r^\alpha) \quad \text{as } r \to \infty, \tag{4.5}$$

where k and α are constants with $0 \le k \le 1$ and $0 < \alpha < 1$, and $\bar{\phi}$ and $\bar{\bar{\phi}}$ are certain given bounded functions with bounded first and second derivatives in a neighborhood of the origin. The conditions (4.5) we have at infinity are weaker than those imposed by Weinstein [3] and Stoker [2]. They required $|\phi| + |\phi_x| + |\phi_{xy}| < M$ for large r.

THEOREM. The solution of the boundary-value problem (4.1)–(4.5) is unique.

Proof. Suppose ϕ_1 and ϕ_2 are two solutions of (4.1)–(4.5). Let $\phi = \phi_1 - \phi_2$. Then ϕ satisfies (4.1)–(4.3) and $\phi = 0$ near and at the origin and it also satisfies (4.5). Now let $\psi = \partial/\partial x(\partial/\partial y - 1)\phi$; then

$$(\nabla^2 - k^2)\psi = 0 \quad \text{in } D, \tag{4.6}$$

$$\psi = 0 \quad \text{on } y = 0,\ x > 0, \tag{4.7}$$

$$\psi = 0 \quad \text{on } x = 0,\ y < 0, \tag{4.8}$$

$$\psi = 0 \quad \text{for } r \ll 1, \tag{4.9}$$

$$\psi = O(r^\alpha) \quad \text{as } r \to \infty. \tag{4.10}$$

We will apply the P-L principle to show that $\psi \equiv 0$ in D. Choose an increasing sequence $\{R_n\}$ of integers such that $R_n \to \infty$ as $n \to \infty$. Let $\theta = \tan^{-1}(x/-y)$ and

$$D_n = \{(r, \theta): 0 < r < R_n,\ 0 < \theta < \pi/2\};$$

then ∂D_n is the boundary of D_n, Γ_n is the ray of the sector D_n where $\theta = 0$ and $\theta = \pi/2$, and Γ_n' is the circular arc of D_n ($r = R_n$). Let

$$w_n = 1 + (2/\pi)R_n \tan^{-1}[2R_n r \cos\theta/(R_n^2 - r^2)].$$

Note that $w_n = 1 + R_n$ on Γ_n' and $w_n > 0$ in $\bar{D}_n$. Also $\lim_{n\to\infty} w_n = 1 + (4/\pi)r \cos\theta$.

Choose $w = 2 + (4/\pi)r \cos\theta$. Then $w \ge w_n$ for sufficiently large n. So

$$\psi/w_n \le MR_n^\alpha/(1 + R_n) \quad \text{on } \Gamma_n'$$

for large n. But this implies

$$\liminf_{n\to\infty}\left[\sup_{\Gamma_{n'}} \frac{\psi}{w_n}\right] \le 0.$$

Hence by the P-L principle $\psi \le 0$ in D. Since $-\psi$ also satisfies (4.6)–(4.10), we can repeat the above argument with $-\psi$ and get $-\psi \le 0$ in D. Therefore, we conclude that $\psi \equiv 0$ in D, i.e.

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} - \right)\phi \equiv 0 \quad \text{in } D.$$

This means that $\phi = F(y) + e^y A(x)$ for arbitrary functions F and A. The conditions $\nabla^2\phi - k^2\phi = 0$, $\phi_x = 0$ on $x = 0$, $\phi_y - \phi = 0$ on $y = 0$, $\phi = O(r^\alpha)$ as $r \to \infty$ and $\phi = 0$ in a neighborhood of the origin imply $\phi \equiv 0$ in D. Hence $\phi_1 \equiv \phi_2$ in D.

5. Waves on sloping beaches. We consider first the problem of two-dimensional progressive waves over a uniformly sloping beach at an angle ω with $0 < \omega < \pi/2$. We have found it advantageous to employ polar coordinates. Let (r, θ) be any point in $D = \{(x, y): x > 0. -x \tan \omega < y < 0\}$ where $r = (x^2 + y^2)^{1/2}$ and the angle θ is measured from the position x-axis and increased in the *clockwise* sense, i.e. $x = r \cos \theta, y = -r \sin \theta$. The beach problem can be formulated in terms of the velocity potential $e^{i\sigma t}\phi(x, y) = e^{i\sigma t}\phi(r, \theta)$ (the formulation in Cartesian coordinates can be found in [2]):

$$\nabla^2\phi = 0 \quad \text{in } D = \{(r, \theta): r > 0, 0 < \theta < \omega\}, \tag{5.1}$$

$$-\phi_\theta - \phi = 0 \quad \text{on } \theta = 0, \tag{5.2}$$

$$\phi_\theta = 0 \quad \text{on } \theta = \omega, \tag{5.3}$$

$$\phi = \bar{\phi} \log r + \bar{\bar{\phi}} \quad \text{for } r \ll 1, \tag{5.4}$$

$$\phi = O(r^\alpha), \qquad \phi_\theta = O(r^\alpha) \quad \text{with } 0 < \alpha < 1, \tag{5.5}$$

where $\bar{\phi}$, $\bar{\bar{\phi}}$ are given bounded functions with bounded first and second derivatives in a neighborhood of the origin. It is well known that in order to allow a physically important class of solutions with singularities at the origin some kind of singularity is assumed at the origin. Here we follow Stoker by assuming it has a logarithmic singularity at the origin.

THEOREM. The solution of the boundary-value problem (5.1)–(5.5) is unique.

Proof. Let ϕ_1 and ϕ_2 be two solutions of (5.1)–(5.5). Let $\phi = \phi_1 - \phi_2$. Then ϕ satisfies (5.1)–(5.3) and (5.5), and $\phi = 0$ in a neighborhood of the origin.

Let $\Phi = \operatorname{sech} m(\omega - \theta)[\phi_\theta + m \tanh m(\omega - \theta)\phi]$, where m is the positive root of $m \tanh m\omega = 1$. Then

$$L\Phi = \nabla^2\Phi - \frac{2m}{r} \tanh m(\omega - \theta)\Phi_\theta + \frac{m^2}{r^2} [2 \operatorname{sech}^2 m(\omega - \theta) + 1]\Phi = 0 \quad \text{in } D, \tag{5.6}$$

$$\Phi = 0 \quad \text{on } \theta = 0, \tag{5.7}$$

$$\Phi = 0 \quad \text{on } \theta = \omega, \tag{5.8}$$

$$\Phi = 0 \quad \text{for } r \ll 1, \tag{5.9}$$

$$\Phi = O(r^\alpha) \quad \text{as } r \to \infty. \tag{5.10}$$

We shall apply the P-L principle to (5.6)–(5.10).

Let $D_k = \{(r, \theta): 1/k < r < k, 0 < \theta < \omega\}$, Γ_k be the union of the two rays of D_k and Γ_k' be the two circular arcs of D_k. Let

$$w_k = w = r \operatorname{sech} m(\omega - \theta)[\cos \theta + m \sin \theta \tanh m(\omega - \theta)]. \tag{5.11}$$

It can be verified that $w_k > 0$ on $\bar{D}_k$ and

$$L[w_k] = 0 \quad \text{in } D_k. \tag{5.12}$$

In view of (5.2) we see that there exists a $\delta > 0$ such that

$$\left|\frac{\Phi}{w_k}\right| < \frac{1}{2r} \quad \text{for } 0 < \theta < \delta.$$

For $\delta < \theta < \omega$ we have

$$\left|\frac{\Phi}{w_k}\right| \le \frac{|\phi_\theta|}{r\cos\theta} + \frac{|\phi|}{r\sin\theta} \le \frac{|\phi_\theta|}{r\cos\omega} + \frac{|\phi|}{r\sin\delta}.$$

It follows then that

$$\liminf_{k\to\infty}\left[\sup_{\Gamma_{k'}}\frac{\Phi}{w_k}\right] \le 0.$$

Hence by the P-L principle we get $\Phi \le 0$ in D. Similarly, repeating the above procedure with $-\Phi$ in place of Φ, we get $-\Phi \le 0$ in D. Therefore $\Phi \equiv 0$ in D. This implies that

$$\phi_\theta + m\tanh m(\omega-\theta)\phi \equiv 0 \quad \text{in } D, \qquad \text{or } \phi = A(r)\cosh m(\omega-\theta) \quad \text{in } D.$$

But $\nabla^2\phi = 0$. This implies $r^2A'' + rA' + m^2A = 0$, so

$$\phi = \cosh m(\omega-\theta)[c_1\cos(m\log r) + c_2\sin(m\log r)].$$

Since $\phi \equiv 0$ for $r \ll 1$, $c_1 = c_2 = 0$. Hence $\phi \equiv 0$ in D, or $\phi_1 \equiv \phi_2$ in D.

In the corresponding three-dimensional beach problem we seek a velocity potential of the form $\exp\{i(\sigma t + kz)\}\phi(x, y)$ with $\phi(x, y)$ satisfying:

$$\nabla^2\phi - k^2\phi = 0 \quad \text{in } D = \{(r,\theta): r > 0, 0 < \theta < \omega\}, \tag{5.13}$$

$$-\phi_\theta - \phi = 0 \quad \text{on } \theta = 0, \tag{5.14}$$

$$\phi_\theta = 0 \quad \text{on } \theta = \omega, \tag{5.15}$$

$$\phi = \bar{\phi}\log r + \bar{\bar{\phi}} \quad \text{for } r \ll 1, \tag{5.16}$$

$$\phi = O(r^\alpha), \qquad \phi = O(r^\alpha) \quad \text{with } 0 < \alpha < 1, \tag{5.17}$$

where $0 \le k \le 1$ and we have adopted the same set of polar coordinates as we have just used. We note that the only difference between the two-dimensional case and the three-dimensional case is that between (5.1) and (5.13): one extra term $-k^2\phi$ appears in (5.13). We can prove uniqueness of solutions for (5.13)–(5.17) by following exactly the same procedure as was used in the two-dimensional proof, only now in (5.6) there is an extra term $-k^2\Phi$, i.e. in place of $L\Phi$ we have $(L - k^2)\Phi$. If we use the same w_k in (5.11) then in place of (5.12) we have $(L - k^2)w_k = -k^2w_k \le 0$ in D_k, since $w_k > 0$. The condition (2.2) in the P-L principle is still satisfied. Therefore, we have proved the following theorem.

THEOREM. The solution of the boundary-value problem consisting of (5.13)–(5.17) is unique.

It should be noted that our approach, in fact, allows much weaker conditions at infinity for beaches of small sloping angle ω. Since

$$w_k = r^n \operatorname{sech} m(\omega-\theta)[n\cos n\theta + m\sin n\theta\tanh m(\omega-\theta)], \qquad n = 1, 2, 3, \ldots$$

is positive in D_k and $Lw_k = 0$ in D_k, it satisfies (5.11) and (5.12) or (2.1) and (2.2) when $0 < \theta < \omega < \pi/2n$. In this way the uniqueness theorem is still valid by only requiring in (5.5) that

$$\phi = O(r^\beta) \qquad \text{and} \qquad \phi_\theta = O(r^\beta) \qquad \text{as } r \to \infty,$$

with $0 < \beta < n$. We should also point out that the dock problems can be handled in a similar way by constructing suitable w_k functions. As a final remark we mention that explicit solutions for the cliff and beach problems can be found in Stoker [2].

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