ELASTIC-PLASTIC FIELD AT THE TIP OF A PROPAGATING SHEAR CRACK*

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Abstract. Asymptotic stress and strain fields are presented for cracks propagating dynamically in incompressible elastic-perfectly plastic materials. Perfect plasticity solutions for stationary cracks are recovered upon setting the velocity of propagation of the crack tip equal to zero. The stress field near the crack tip is found to consist of a combination of uniform and non-uniform stress regions.

1. Introduction. This paper presents an asymptotic analysis for cracks propagating dynamically in elastic-perfectly plastic materials under shear loading in the far field. The medium in which the crack advances is assumed to be homogeneous, isotropic and incompressible. The results are based on a small-strain formulation of elastic-perfect plasticity which employs the Prandtl–Reuss flow rule.

Results for dynamic crack tip fields were obtained by Slepyan [1] for cracks in plane strain tensile loading (Mode I) and anti-plane shear loading (Mode III). Similar analysis in the Mode III case was also carried out by Achenbach et al. [2]. The present paper extends the work of Slepyan [1] by carrying out an analysis for rapidly advancing cracks under far-field shear loading (Mode II). The analysis is two-dimensional and the conditions of plane strain are assumed.

For the stationary crack, the plasticity solution was obtained by Hutchinson [3] and the analyses for quasi-static growing cracks were carried out by Slepyan [4] and Rice et al. [5].

2. Governing equations for the dynamic propagating crack under plane strain shear loading. The equations of motion are given by

$$\partial\sigma_{\beta\alpha}/\partial x_\beta = \rho\ddot{u}_\alpha \tag{2.1}$$

where $\sigma_{\alpha\beta}$ is the symmetric stress tensor, ρ the density of the material and u_α the displacement components. The superimposed dot denotes material time derivative. In Eq. (2.1), the Greek indices have range 1, 2 and the summation convention is implied. The Cartesian axes are chosen such that $x_1 - x_2$ is in a plane normal to the crack front and x_1 is in the direction of crack propagation. The polar coordinates r, θ which will be used later are chosen to be in the $x_1 - x_2$ plane with the origin located instantaneously at the advancing crack tip; $\theta = 0$ is chosen to be the positive x-axis.

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For incompressible, isotropic elastic-perfectly plastic materials satisfying the Mises yield condition, the constitutive equations in plane strain take the form

$$\dot{\varepsilon}_{11} = 3/4E(\dot{\sigma}_{11} - \dot{\sigma}_{22}) + \dot{\Lambda}(\sigma_{11} - \sigma_{22}) = -\dot{\varepsilon}_{22}, \tag{2.2}$$

$$\dot{\varepsilon}_{12} = 3/2E\dot{\sigma}_{12} + \dot{\Lambda}\sigma_{12}, \tag{2.3}$$

where

$$\dot{\varepsilon}_{\alpha\beta} = \dot{\varepsilon}^{e}_{\alpha\beta} + \dot{\varepsilon}^{p}_{\alpha\beta} \equiv \tfrac{1}{2}(\partial v_{\alpha}/\partial x_{\beta} + \partial v_{\beta}/\partial x_{\alpha}) \tag{2.4}$$

is the strain-rate tensor $\dot{\varepsilon}_{\alpha\beta}$ written as the sum of the elastic and plastic strain rates; the components of the velocity field are denoted by v_{α}. The plastic dissipation $\dot{\Lambda}$ in (2.2) and (2.3) is assumed to be non-negative and E is Young's modulus. Because of incompressibility ($\dot{\varepsilon}_{\alpha\alpha} = 0$), the Poisson's ratio ν has been set equal to one-half in applying Hooke's law to the elastic strain rate in (2.2) and (2.3).

Since we are concerned with an asymptotic analysis near the crack tip, we shall only be interested in the local behavior of the stress and deformation fields as $r \to 0$. Consequently, if we consider the material time derivative of the stress tensor $\sigma_{\alpha\beta}$, treating it as a function of the polar coordinates r, θ and time t,

$$\begin{aligned}\dot{\sigma}_{\alpha\beta}(r, \theta, t) &= (\partial\sigma_{\alpha\beta}/\partial r)\dot{r} + (\partial\sigma_{\alpha\beta}/\partial\theta)\dot{\theta} + (\partial\sigma_{\alpha\beta}/\partial t)\\ &= -(\partial\sigma_{\alpha\beta}/\partial r)\sin\theta\dot{a} + (\partial\sigma_{\alpha\beta}/\partial\theta)r^{-1}\cos\theta\dot{a} + \partial\sigma_{\alpha\beta}/\partial t,\end{aligned} \tag{2.5}$$

we will only retain those terms which dominate asymptotically near the crack-tip. In the context of ideal plasticity in which the stresses are constrained to be bounded, it is seen from (2.5) that as $r \to 0$,

$$\dot{\sigma}_{\alpha\beta} = r^{-1}\cos\theta\dot{a}(\partial\sigma_{\alpha\beta}/\partial\theta), \tag{2.6}$$

which is equivalent to assuming that the stress tensor $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(\theta)$. This property was used in the analyses of stationary cracks [3] and quasistatic growing cracks [4, 5] and it will also be used here at the outset to establish the asymptotic structure of the deformation field at the crack tip.

Consider now the material time derivative of the velocity field, say

$$\dot{v}_{\alpha} \equiv \partial v_{\alpha}/\partial t - \dot{a}\partial v_{\alpha}/\partial x_1 = -\dot{a}\partial v_{\alpha}/\partial x_1 \quad \text{as} \quad r \to 0. \tag{2.7}$$

The relation is exact for steady-state crack propagation but, as observed by Freund and Clifton [6], Rice and Sorensen [7] and Hui and Riedel [8], for non-steady crack growth, the result (2.7) still holds asymptotically as $r \to 0$.

With the asymptotic relation (2.7), the equations of motion become

$$\partial\sigma_{\beta\alpha}/\partial x_{\beta} = -\rho\dot{a}\partial v_{\alpha}/\partial x_1 \tag{2.8}$$

The stress-strain relations assume the form

$$(-\dot{a})\partial v_1/\partial x_1 = 3/4E(-\dot{a})\partial/\partial x_1(\sigma_{11} - \sigma_{22}) + \dot{\Lambda}(\sigma_{11} - \sigma_{22}), \tag{2.9}$$

$$(-\dot{a})(\partial v_1/\partial x_2 + \partial v_2/\partial x_1) = 3/E(-\dot{a})\partial\sigma_{12}/\partial x_1 + 2\dot{\Lambda}\sigma_{12}. \tag{2.10}$$

Finally, the traction-free boundary conditions for the crack are

$$\sigma_{22} = \sigma_{12} = 0 \quad \text{on} \quad \theta = \pi \qquad (x_2 = 0,\ x_1 < 0). \tag{2.11}$$

3. Asymptotic structure of the deformation field. With the assumption that $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(\theta)$ only, the equations of motion (2.1) imply that the stress gradient $\sigma_{\alpha\beta,\alpha}$ and hence the velocity gradients $\partial v_\beta/\partial x_1$ ($\beta = 1, 2$) must be of the order

$$\partial v_1/\partial x_1 = O(r^{-1}), \qquad \partial v_2/\partial x_1 = O(r^{-1}) \quad \text{as} \quad r \to 0. \tag{3.1}$$

Therefore, the most general form (asymptotically) the deformation gradient $\partial u_1/\partial x_1$ permitted by the asymptotic behavior of $\partial v_1/\partial x_1$ in (3.1) is

$$\partial u_1/\partial x_1 = p(\theta) + B \log r \tag{3.2}$$

where $p(\theta)$ is an arbitrary function of the polar angle θ and B is a constant. The coefficient multiplying $\log r$ cannot be a function of θ (and hence can at most be a constant) because $\partial v_1/\partial x_1$ will then be

$$\partial v_1/\partial x_1 = O(r^{-1} \log r), \tag{3.3}$$

which is inconsistent with (3.1) and the equations of motion (2.8). Similarly, from the other equation of motion, we conclude that

$$\partial u_2/\partial x_1 = \tilde{q}(\theta) + C \log r \tag{3.4}$$

where $\tilde{q}(\theta)$ is again a function of θ only and C a constant. Note, however, that $\partial u_2/\partial x_1$ may be written as a function of θ plus a function of x_2 only,

$$\begin{aligned} \partial u_2/\partial x_1 &= \tilde{q}(\theta) - C \log \sin \theta + C \log (r \sin \theta) \\ &= q(\theta) + C \log x_2. \end{aligned} \tag{3.5}$$

In the subsequent analysis, the constant C will be shown to vanish, but for now, without loss of generality, it will be taken as zero as long as we are dealing with the x_1-derivative of $\partial u_2/\partial x_1$.

For shear deformations, symmetry demands that directly ahead of the crack, $x_1 > 0$, $x_2 = 0$ ($\theta = 0$)

$$\partial u_1/\partial x_1 = 0. \tag{3.6}$$

This implies that the constant B in (3.2) is identically zero. Further, from incompressibility and (3.2), we have

$$\partial u_2/\partial x_2 = m(\theta) = -p(\theta). \tag{3.7}$$

4. Yield condition. Under conditions of plane strain, the Mises yield condition is reduced to

$$(\sigma_{11} - \sigma_{22})^2/4 + \sigma_{12}^2 = \tau_0^2 \tag{4.1}$$

where τ_0 is the yield stress in shear. The relation is exact for rigid-ideally plastic materials and follows immediately from the first of the constitutive equations (2.2) and (2.3), in which the elastic strains are neglected. However, the condition (4.1) remains *exact* for elastic-perfectly plastic solids if the material is assumed to be incompressible ($\nu = 1/2$). In quasistatic crack growth [5], Eq. (4.1) holds only asymptotically in singular plastic sectors in which at least one of the in-plane plastic strain components approaches infinity at the crack tip.

5. Analysis for the shear crack. The governing equations (2.8)–(2.10) together with the observation in (2.6), the asymptotic structure of the deformation gradients (3.2), (3.6) and (3.8), the yield condition (4.1) and the boundary conditions (2.11), constitute the problem for the stress and deformation fields near the tip of a shear crack propagating dynamically.

For a solution to this problem, first note that, based on the remark after (2.6), the stress tensor can be taken as $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(\theta)$. Further, the yield condition (4.1) is identically satisfied if one assumes without loss of generality that

$$\sigma_{11} = \sigma - \tau_0 \cos(\omega - 2\theta), \qquad \sigma_{22} = \sigma + \tau_0 \cos(\omega - 2\theta),$$
$$\sigma_{12} = -\tau_0 \sin(\omega - 2\theta). \tag{5.1}$$

where $\sigma = \sigma(\theta)$, $\omega = \omega(\theta)$ are for now arbitrary functions of θ. With the use of (5.1), the stress gradients $\sigma_{\alpha\beta,\alpha}$ and the velocity gradients $v_{\beta,1}$ can be calculated and they are then substitutded into the equations of motion (2.8) to give, for $r \to 0$,

$$-\sigma' \sin\theta - \tau_0(\omega' - 2)\cos(\omega - 3\theta) = -\rho\dot{a}^2 p' \sin\theta, \tag{5.2}$$

$$\sigma' \cos\theta - \tau_0(\omega' - 2)\sin(\omega - 3\theta) = -\rho\dot{a}^2 q' \sin\theta, \tag{5.3}$$

where primes denote differentiation with respect to the polar angle θ. The elastic-plastic constitutive equations (2.9) and (2.10) when expressed in terms of $p(\theta)$, $q(\theta)$, $\omega(\theta)$ $\lambda \equiv \lim_{r\to 0} r\dot{\Lambda}$ become

$$p' \cos\theta - q' \sin\theta = 3/E\tau_0(\omega' - 2)\cos(\omega - 2\theta)\sin\theta + 2\lambda\dot{a}^{-1}\tau_0 \sin(\omega - 2\theta), \tag{5.4}$$

$$-p' \sin\theta = -3 - 2E\tau_0(\omega' - 2)\sin(\omega - 2\theta)\sin\theta + \lambda\dot{a}^{-1}\tau_0 \cos(\omega - 2\theta). \tag{5.5}$$

The dissipation amplitude λ can first be solved in terms of ω, p, q and the resulting expression substituted in (2.9). The quantity σ' is then eliminated from (5.4) and (5.5) and after some algebra, the following equation for $\omega(\theta)$ is obtained:

$$(\omega' - 2)[\cos^2(\omega - 4\theta) - \alpha^2 \sin^2\theta] = 0, \tag{5.6}$$

where $\alpha^2 \equiv 3\rho\dot{a}/E$. The same equation was obtained previously by Slepyan [1] in his analysis of Mode I cracks.

Satisfaction of Eq. (5.6) demands that the stress field should be one or more of the following:

Uniform stress field: $\omega' = 2$ and hence, from (5.1),

$$\sigma_{\alpha\beta} = \text{constant} \tag{5.7}$$

Nonuniform stress field:

$$\cos(\omega - 4\theta) = \pm\alpha \sin\theta \tag{5.8}$$

from which follows

$$\sin(\omega - 4\theta) = \pm(1 - \alpha^2 \sin^2\theta)^{1/2}. \tag{5.9}$$

It is seen from (5.8) that there is an ambiguity in sign for $\cos(\omega - 4\theta)$ and similarly for $\sin(\omega - 4\theta)$.

It can be shown from (5.1)–(5.4) and (5.8), (5.9) that the dissipation amplitude λ is

$$\lambda = \frac{(\omega' - 2)}{\rho\dot{a} \sin\theta} \sin(\omega - 4\theta)\cos(\omega - 4\theta). \tag{5.10}$$

Thus it follows immediately that in order for λ to be non-negative, like signs must be chosen in (5.8) and (5.9) if $\omega' \geq 2$. Note also that for uniform stress regions, λ vanishes identically.

From Eq. (5.2), (5.3), we have the following expressions for q', σ':

$$q' = \frac{\tau_0(\omega' - 2)}{\rho \dot{a}^2} \cos(\omega - 4\theta), \tag{5.11}$$

$$\sigma' = \tau_0(\omega' - 2)\sin(\omega - 4\theta), \tag{5.12}$$

and from incompressibility and compatibility of assumptions (3.5), (3.7), there follows

$$m' = -p' = -q' \cos\theta - C/\sin^2\theta. \tag{5.13}$$

6. Stress fields. Guided by the results for the stationary cracks [3], we assume that near the crack tip the stress field consists of the following regions:[1]

$$\begin{aligned}
&\text{(i) nonuniform stress region DOE for } 0 \leq \theta \leq \theta_1;\\
&\text{(ii) uniform stress region COD for } \theta_1 \leq \theta \leq \theta_2;\\
&\text{(iii) nonuniform stress region BOC for } \theta_2 \leq \theta \leq \theta_3;\\
&\text{(iv) uniform stress region AOB for } \theta_3 \leq \theta \leq \pi,
\end{aligned} \tag{6.1}$$

where the angles θ_1, θ_2, θ_3 are to be determined. A solution to the problem is considered found if the stress deformation field constructed according to (6.1) satisfies all the conditions mentioned in the beginning of Sec. 5 and traction continuity conditions across θ_1, θ_2, and θ_3.

The choice of signs in (5.8) and (5.9) is dictated by the requirement of non-negative plastic work, by the satisfaction of the boundary conditions and by the plausible assumption that the results for the stationary crack should be recovered by setting the velocity of propagation $\dot{a}$ equal to zero. For reference, the stationary crack results [3] are given by (6.1) with

$$\theta_1 = \pi/8 + 1/4, \qquad \theta_2 = \theta_1 + \pi/2, \qquad \theta_3 = 3\pi/4. \tag{6.2}$$

We start by focusing on the region AOB in which the stress field is assumed to be *uniform*. Because of the traction-free boundary conditions (2.11), we have from (5.1) and (5.7) that

$$\begin{aligned}
&\sigma(\theta) = -\tau_0,\\
&\omega(\theta) = 2\theta \quad \text{for} \quad \theta_3 \leq \theta \leq \pi \quad \text{in} \quad \text{AOB}.
\end{aligned} \tag{6.3}$$

This gives $\sigma_{11} = -2\tau_0, \sigma_{22} = \sigma_{12} = 0$ in AOB.

In the adjoining region of *non-uniform* stress field BOC, $\omega(\theta)$ satisfies (5.8). But for continuity at the interface θ_3, $\omega(\theta_3) = 2\theta_3$. Thus (5.8) implies

$$\cos 2\theta_3 = \pm\alpha \sin\theta_3. \tag{6.4}$$

It can easily be shown that Eq. (6.4) admits four roots:

$$\theta_3^{(1),(2)} = \sin^{-1}[\pm\alpha/4 + 1/4(\alpha^2 + 8)^{1/2}]$$

where
$$\pi/6 \le \theta_3^{(1)} \le \pi/4, \qquad \pi/4 \le \theta_3^{(2)} \le \pi/2, \tag{6.5}$$
$$\theta_3^{(3),(4)} = \pi - \theta_3^{(1),(2)}, \tag{6.6}$$
which implies $\pi/2 \le \theta_3^{(4)} \le 3\pi/4$, $3\pi/4 \le \theta_3^{(3)} \le 5\pi/6$. From consideration of the stationary crack limit ($\dot{a} = 0$), it is concluded that the first two roots $\theta_3^{(1),(2)}$ should be discarded.

In the region DOE directly ahead of the crack at $\theta = 0$, we have
$$\sigma_{22} = 0, \qquad \sigma_{12} = \tau_0 . \tag{6.7}$$
It follows from (5.1) then that
$$\sigma(0) = 0, \qquad \omega(0) = -\pi/2 \tag{6.8}$$
with ω satisfying (5.8), (5.9). For consistency with the second of (6.8), the negative sign in (5.8) and hence in (5.9) must be chosen for the region DOE, resulting in
$$\begin{aligned} \cos(\omega - 4\theta) &= -\alpha \sin \theta, \\ \sin(\omega - 4\theta) &= -(1-\alpha^2 \sin^2 \theta)^{1/2} \qquad 0 \le \theta \le \theta_1 \text{ in DOE.} \end{aligned} \tag{6.9}$$
In the constant-stress region COD, the stresses must take on the value in (5.1) on $\theta = \theta_1$. Therefore, in the non-uniform stress state region BOC, continuity of traction across $\theta = \theta_2$ gives
$$\sigma(\theta_1) = \sigma(\theta_2), \qquad \cos(\omega(\theta_1) - 2\theta_1) = \cos(\omega(\theta_2) - 2\theta_2). \tag{6.10}$$
Although conditions (6.7), (6.8) enable us to choose the sign in (6.9) in the region DOE, the sign for cos $(\omega - 4\theta)$ in region BOC is as yet undetermined. Consideration of (5.8) reveals that the positive sign in (5.8) must be chosen in order for the stationary cracks results to be recovered as a special case when $\dot{a} = 0$, i.e. in BOC
$$\begin{aligned} \cos(\omega - 4\theta) &= \alpha \sin \theta, \\ \sin(\omega - 4\theta) &= (1 - \alpha^2 \sin^2 \theta)^{1/2} \quad \text{for} \quad \theta_2 \le \theta \le \theta_3 . \end{aligned} \tag{6.11}$$
Because of the continuity condition (6.4), this requires that $\cos 2\theta_3 > 0$. Thus, for consistency, we must choose
$$\theta_3 = \theta_3^{(3)} = \pi - \theta_3^{(1)} = \pi - \sin^{-1}(-\alpha/4 + 1/4(\alpha^2 + 8)^{1/2}) \tag{6.12}$$
so that $\theta_3^{(3)} \ge 3\pi/4$ for $\alpha \ge 0$.

From the expression for σ' in (5.12), integration from $\theta = 0$ in the region DOE, after (6.9) and the initial conditions (6.8) have been used, gives σ in DOE:
$$\sigma(\theta) = +\tau_0 \alpha \sin \theta - 2\tau_0 \int_0^\theta (1 - \alpha^2 \sin^2 \phi)^{1/2}\, d\phi \quad \text{for} \quad 0 \le \theta \le \theta_1 . \tag{6.13}$$
In the region BOC, integration from $\theta = \pi$ gives
$$\sigma(\theta) = -\tau_0 - \tau_0 \alpha(\sin \theta - \sin \theta_3) + 2\tau_0 \int_{\theta_3}^\theta (1 - \alpha^2 \sin^2 \phi)^{1/2} d\phi \quad \text{for} \quad \theta_2 \le \theta \le \theta_3. \tag{6.14}$$
where the uniform stress property in AOB has been used.

Combining (6.14) and (6.13), one gets from the first of the continuity conditions (6.10)

$$1 + \alpha(\sin\theta_2 - \sin\theta_3 + \sin\theta_1) + 2\int_{\theta_2}^{\theta_3} (1 - \alpha^2 \sin^2\phi)^{1/2}\,\phi$$
$$- 2\int_0^{\theta_1} (1 - \alpha^2 \sin^2\phi)^{1/2} d\phi = 0. \qquad (6.14)$$

Use of (6.9), (6.11) in the second continuity condition in (6.10) results in

$$\alpha \sin\theta_2 \cos 2(\theta_2 - \theta_1) - (1 - \alpha^2 \sin^2\theta_2)^{1/2} \sin 2(\theta_2 - \theta_1) = -\alpha \sin\theta_1 \qquad (6.15)$$

Eq. (6.14) and (6.15) were solved numerically for θ_1 and θ_2 over a range of α and the results are shown in Fig. 1. The stress distributions are displayed in Fig. 2 and Fig. 3. Remarkably, the stress field near the crack tip is almost completely unaffected by inertia, as can be seen from the figures. Observe also from (6.14) and (6.15) that when $\alpha = 0\,(\dot{a} = 0)$, the stationary crack results (6.2) are indeed obtained.

To summarize, the stress field around the crack tip consists of:

(i) non-uniform stress region DOE in which the stresses are

$$\sigma_{rr} = 2\tau_0\left(\alpha \sin\theta - \int_0^{\theta}(1 - \alpha^2 \sin^2\phi)^{1/2}\, d\phi\right),$$

$$\sigma_{\theta\theta} = -2\tau_0\int_0^{\theta}(1 - \alpha^2 \sin^2\phi)^{1/2}\, d\phi,$$

$$\sigma_{r\theta} = \tau_0\,(1 - \alpha^2 \sin^2\theta)^{1/2} \quad \text{for} \quad 0 \le \theta \le \theta_1,$$

where the integral can be expressed in terms of elliptic integrals of the second kind [9],

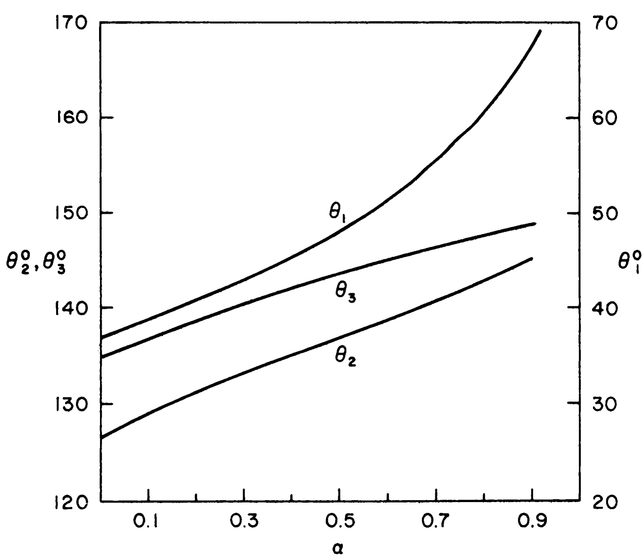

FIG. 1. Variation of the angles that separate the uniform and nonuniform regions with the dimensionless velocity.

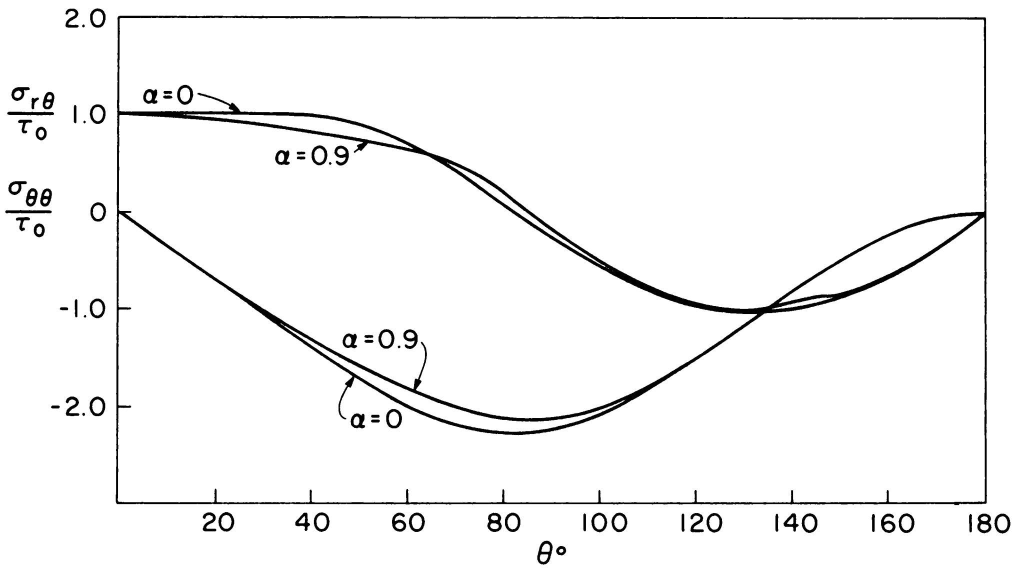

FIG. 2. Distribution of the stresses $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ for different velocities.

(ii) uniform stress region COD where

$$\sigma_{rr} = \sigma(\theta_1) + \alpha\tau_0 \sin\theta_1 \cos 2(\theta - \theta_1) + \tau_0(1 - \alpha^2 \sin^2\theta_1)^{1/2} \sin 2(\theta - \theta_1),$$

$$\sigma_{\theta\theta} = \sigma(\theta_1) - \alpha\tau_0 \sin\theta_1 \cos 2(\theta - \theta_1) - \tau_0(1 - \alpha^2 \sin^2\theta_1)^{1/2} \sin 2(\theta - \theta_1),$$

$$\sigma_{r\theta} = -\alpha\tau_0 \sin\theta_1 \sin 2(\theta - \theta_1) + \tau_0(1 - \alpha^2 \sin^2\theta_1)^{1/2} \cos 2(\theta - \theta_1)$$

$$\text{for} \quad \theta_1 \leq \theta \leq \theta_2\,;$$

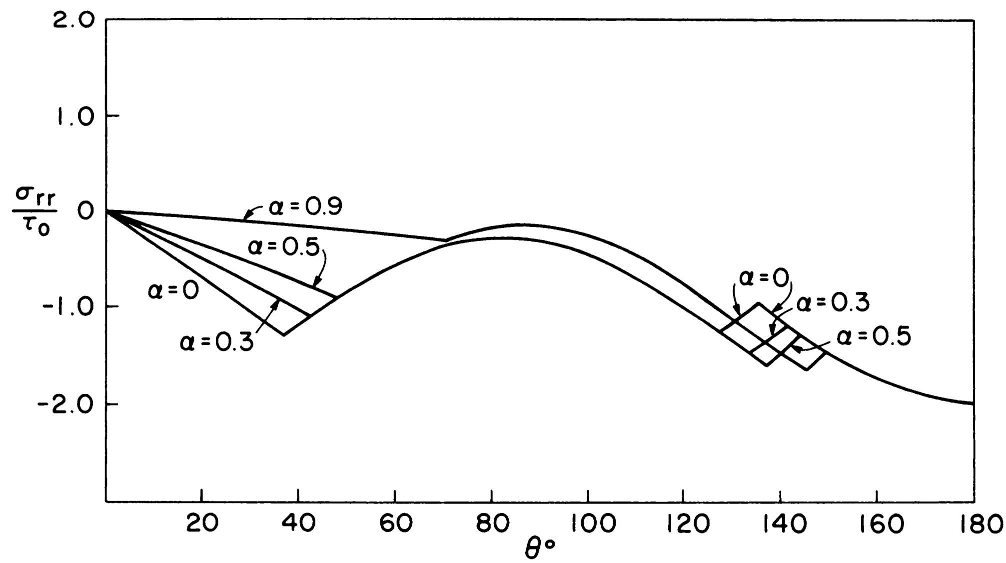

FIG. 3. Distribution of the stress σ_{rr} for different velocities.

(iii) non-uniform stress region BOC where

$$\sigma_{rr} = \tau_0 \left[-\alpha(2 \sin \theta - \sin \theta_3) - 1 - 2 \int_\theta^{\theta_3} (1 - \alpha^2 \sin^2 \phi)^{1/2} \, d\phi \right],$$

$$\sigma_{\theta\theta} = \tau_0 \left[\alpha \sin \theta_3 - 1 - 2 \int_\theta^{\theta_3} (1 - \alpha^2 \sin^2 \phi)^{1/2} \, d\phi \right],$$

$$\sigma_{r\theta} = -\tau_0 (1 - \alpha^2 \sin^2 \theta)^{1/2} \quad \text{for} \quad \theta_2 \le \theta \le \theta_3 ;$$

(iv) uniform stress region AOB where

$$\sigma_{rr} = -\tau_0 (1 + \cos 2\theta), \qquad \sigma_{\theta\theta} = -\tau_0 (1 - \cos 2\theta),$$

$$\sigma_{r\theta} = \tau_0 \sin 2\theta \quad \text{for} \quad \theta_3 \le \theta \le \pi.$$

7. Strain field. Given the expression for $\omega(\theta)$ in (5.6) and p, q, m in terms of ω in (5.11) and (4.12), we can compute the strains near the crack tip. For example, in the region DOE, from (3.2), (3.4) and (3.7)

$$\varepsilon_{11} = -\varepsilon_{22} = -m(\theta) = \int_0^\theta q'(\phi) \cot \phi \, d\phi + C \int_0^\theta d\phi/\sin^2 \phi, \tag{7.1}$$

$$\varepsilon_{12} = -\frac{1}{2} \int^\theta q'(\phi) \cos 2\phi/\sin^2 \phi d \phi + C \log x_2 + f(x_2) \tag{7.2}$$

in which the function $f(x_2)$ will be chosen to make the shear strain component ε_{12} bounded at a fixed radial distance ($r > 0$) from the crack tip. Because of (3.6), it is seen that the constant C in (7.1) must vanish. Substituting (5.11), (6.9) for q' into (7.1) and (7.2) and integrating, we determine that, in the region DOE ahead of the crack, for example,

$$\varepsilon_{11} = -\varepsilon_{22} = 3/(\alpha E)\{-2 \sin \theta + \frac{1}{\alpha} \tilde{E}(\theta, \alpha) - (1 - \alpha^2)^{1/2}/\alpha \, F(\theta, \alpha)\}\tau_0, \tag{7.3}$$

$$\begin{aligned} \varepsilon_{12} = {} & 3\tau_0/(2\alpha E)\{4 \cos \theta + 2 \log \tan \theta/2 - 2/\alpha \, (1 - \alpha^2 \sin^2 \theta)^{1/2} \\ & + (\alpha/2)\log[(1 + (1 - \alpha^2 \sin^2 \theta)^{1/2})/(1 - (1 - \alpha^2 \sin^2 \theta)^{1/2})]\} + f(x_2) \\ & \qquad \text{for} \quad 0 \le \theta \le \theta_1, \end{aligned} \tag{7.4}$$

where in (7.3), F and $\tilde{E}$ are incomplete elliptic integrals of the first and second kind respectively [9]. In (7.4), based on the remark after (7.2), the function $f(x_2)$ must be of the form

$$f(x_2) = 3\tau_0(2 - \alpha)/(2\alpha E)\log(R/r \sin \theta) \tag{7.5}$$

where R is a length parameter undetermined from the asymptotic analysis. Hence, in contrast to the results in the Mode I analysis [1] in which all the strain components are finite for a *fixed* crack speed, here the shear strain in (7.4) becomes unbounded like log r near the crack tip. As pointed out by Slepyan [4], the fact that there is no elastic unloading region in the asymptotic solution does not necessarily mean that no unloading takes place near the crack tip. Rather, it may be that the region in which the asymptotic results hold may be too small to permit any unloading. This may explain why Slepyan's solution for a rapidly advancing crack in Mode I does not reduce to the quasistatic growing crack results [5]. Despite the fact that the stress field near the crack tip in this dynamic problem for *elastic*-perfectly plastic solids is similar to that of the stationary crack in *rigid*-ideally plastic materials, the crack propagation problem, in contrast to the results for the stationary crack, requires consideration of both the elastic and plastic strains.

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