ON TRANSVERSELY ISOTROPIC FUNCTIONS OF VECTORS, SYMMETRIC SECOND-ORDER TENSORS AND SKEW-SYMMETRIC SECOND-ORDER TENSORS*

By

G. F. SMITH

Lehigh University

1. Introduction. There are five groups $T_1, \ldots, T_5$ which define the symmetry properties of materials which are referred to as being transversely isotropic. We define these groups by listing the matrices which generate the groups:

$$\begin{aligned} &T_1: \mathbf{Q}(\theta), \\ &T_2: \mathbf{Q}(\theta), \mathbf{R}_1 = \operatorname{diag}(-1, 1, 1), \\ &T_3: \mathbf{Q}(\theta), \mathbf{R}_3 = \operatorname{diag}(1, 1, -1), \\ &T_4: \mathbf{Q}(\theta), \mathbf{R}_1 = \operatorname{diag}(-1, 1, 1), \mathbf{R}_3 = \operatorname{diag}(1, 1, -1), \\ &T_5: \mathbf{Q}(\theta), \mathbf{D}_2 = \operatorname{diag}(-1, 1, -1). \end{aligned} \tag{1.1}$$

In (1.1), $\mathbf{Q}(\theta)$ denotes the matrix

$$\mathbf{Q}(\theta) = \left\| \begin{matrix} \cos\theta, & \sin\theta, & 0 \\ -\sin\theta, & \cos\theta, & 0 \\ 0, & 0, & 1 \end{matrix} \right\| . \tag{1.2}$$

$\mathbf{Q}(\theta)$ corresponds to a rotation about the x_3 axis. $\mathbf{R}_1$ and $\mathbf{R}_3$ correspond to reflections in planes perpendicular to the x_1 axis and the x_3 axis respectively. $\mathbf{D}_2$ corresponds to a rotation through 180 degrees about the x_2 axis.

In this paper, we determine integrity bases for polynomial functions $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M, \mathbf{W}_1, \ldots, \mathbf{W}_P)$ of N three-dimensional second-order symmetric tensors $\mathbf{A}_p = \|A_{ij}^p\|$ $(p = 1, \ldots, N)$, M three-dimensional vectors $\mathbf{V}_q = V_i^q$ $(q = 1, \ldots, M)$ and P three-dimensional second-order skew-symmetric tensors $\mathbf{W}_r = \|W_{ij}^r\|$ $(r = 1, \ldots, P)$ which are invariant under any given group chosen from $T_1, \ldots, T_5$. Adkins [1, 2] has considered the problem of determining integrity bases for functions $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M)$ which are invariant under the group T_1 and for functions $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M)$ which are invariant under the group T_2. Long and McIntire [3] have considered the problem of determining an integrity basis for functions $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M, \mathbf{W}_1, \ldots, \mathbf{W}_P)$ which are invariant under the group T_4. The results obtained here for this case differ from those given in [3].

* Received November 6, 1980. This work was supported by a grant from the National Science Foundation to Lehigh University.

2. An integrity basis for functions invariant under T_1. Let us employ the notation

$$\begin{aligned} B^i_\alpha &= A^i_{3\alpha} \qquad (\alpha = 1, 2; i = 1, \ldots, N), \\ B^{N+j}_\alpha &= V^j_\alpha \qquad (\alpha = 1, 2; j = 1, \ldots, M), \\ B^{N+M+k}_\alpha &= W^k_{3\alpha} \qquad (\alpha = 1, 2; k = 1, \ldots, P). \end{aligned} \tag{2.1}$$

It is readily seen that the problem of determining the form of a polynomial function $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M, \mathbf{W}_1, \ldots, \mathbf{W}_P)$ which is invariant under the group T_1 is equivalent to that of determining the form of a polynomial function $G(A^i_{33}, \ldots, B^r_1 - \imath B^r_2)$ which is subject to the restrictions that

$$\begin{aligned} &G(A^i_{33}, A^i_{11} + A^i_{22}, A^i_{11} - A^i_{22} + 2\imath A^i_{12}, A^i_{11} - A^i_{22} - 2\imath A^i_{12}, V^p_3, \\ &W^m_{12}, B^r_1 + \imath B^r_2, B^r_1 - \imath B^r_2) = G(A^i_{33}, A^i_{11} + A^i_{22}, (A^i_{11} - A^i_{22} + 2\imath A^i_{12})e^{-2\imath\theta}, \\ &(A^i_{11} - A^i_{22} - 2\imath A^i_{12})e^{2\imath\theta}, V^p_3, W^m_{12}, (B^r_1 + \imath B^r_2)e^{-\imath\theta}, (B^r_1 - \imath B^r_2)e^{\imath\theta}) \end{aligned} \tag{2.2}$$

shall hold for $0 \le \theta \le 2\pi$. In (2.2), $\imath^2 = -1$ and $i = 1, \ldots, N$; $p = 1, \ldots, M$; $m = 1, \ldots, P$; $r = 1, \ldots, N + M + P$. It is immediately seen that $G(A^i_{33}, \ldots)$ is expressible as a polynomial in the quantities (2.3) listed below which then form an integrity basis for functions $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M, \mathbf{W}_1, \ldots, \mathbf{W}_P)$ which are invariant under T_1.

$$\begin{aligned} &A^i_{33}, A^i_{11} + A^i_{22}, V^p_3, W^m_{12}, \\ &(A^i_{11} - A^i_{22})(A^j_{11} - A^j_{22}) + 4A^i_{12}A^j_{12} \qquad (i \le j), \\ &(A^i_{11} - A^i_{22})A^j_{12} - (A^j_{11} - A^j_{22})A^i_{12} \qquad (i < j), \\ &B^r_1 B^s_1 + B^r_2 B^s_2 \qquad (r \le s), \qquad B^r_1 B^s_2 - B^r_2 B^s_1 \qquad (r < s), \\ &(A^i_{11} - A^i_{22})(B^r_1 B^s_1 - B^r_2 B^s_2) + 2A^i_{12}(B^r_1 B^s_2 + B^r_2 B^s_1) \qquad (r \le s), \\ &(A^i_{11} - A^i_{22})(B^r_1 B^s_2 + B^r_2 B^s_1) - 2A^i_{12}(B^r_1 B^s_1 - B^r_2 B^s_2) \qquad (r \le s). \end{aligned} \tag{2.3}$$

In (2.3), $i, j = 1, \ldots, N$; $p = 1, \ldots, M$; $m = 1, \ldots, P$; $r, s = 1, \ldots, N + M + P$ subject to the restrictions indicated. We observe from (2.3) that the integrity basis for functions $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M)$ invariant under T_1 given by Adkins [1, 2] contains redundant terms.

3. An integrity basis for functions invariant under T_2. The group T_2 is generated by the matrices $\mathbf{Q}(\theta)$ and $\mathbf{R}_1$. We have seen in Sec. 2 that any polynomial function $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M, \mathbf{W}_1, \ldots, \mathbf{W}_P)$ which is invariant under the group T_1 generated by $\mathbf{Q}(\theta)$ is expressible as a polynomial in the quantities (2.3). In order to determine the general form of the function $F(\mathbf{A}_1, \ldots, \mathbf{W}_P)$ which is invariant under T_2, we need only determine the general form of a polynomial function of the quantities (2.3) which is invariant under $\mathbf{R}_1$. The elements of (2.3) either remain invariant under $\mathbf{R}_1$ or change sign under $\mathbf{R}_1$. Let

$I_1, \ldots, I_a$ and $J_1, \ldots, J_b$ denote the elements of (2.3) which remain invariant under $\mathbf{R}_1$ and which change sign under $\mathbf{R}_1$ respectively. With (2.3), we see that the $J_1, \ldots, J_b$ are given by

$$\begin{gathered} W_{12}^m,\ (A_{11}^i - A_{22}^i)A_{12}^j - (A_{11}^j - A_{22}^j)A_{12}^i \quad (i < j), \qquad B_1^r B_2^s - B_2^r B_1^s \quad (r < s), \\ (A_{11}^i - A_{22}^i)(B_1^r B_2^s + B_2^r B_1^s) - 2A_{12}^i(B_1^r B_1^s - B_2^r B_2^s) \quad (r \leq s) \end{gathered} \tag{3.1}$$

where $i, j = 1, \ldots, N$; $m = 1, \ldots, P$; $r, s = 1, \ldots, N + M + P$ subject to the restrictions indicated. The $I_1, \ldots, I_a$ are the elements of (2.3) not listed in (3.1). An integrity basis for functions $F(\mathbf{A}_1, \ldots, \mathbf{W}_P)$ which are invariant under T_2 is then given by $I_1, \ldots, I_a$ and $J_p J_q$ $(p, q = 1, \ldots, b; p \leq q)$.

After eliminating the redundant elements from the set $J_p J_q$, we obtain the result that an integrity basis for functions $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M, \mathbf{W}_1, \ldots, \mathbf{W}_P)$ which are invariant under T_2 is given by

$$\begin{gathered} A_{33}^i,\ A_{11}^i + A_{22}^i,\ V_3^p,\ (A_{11}^i - A_{22}^i)(A_{11}^j - A_{22}^j) + 4A_{12}^i A_{12}^j \quad (i \leq j), \\ B_1^r B_1^s + B_2^r B_2^s \quad (r \leq s), \\ (A_{11}^i - A_{22}^i)(B_1^r B_1^s - B_2^r B_2^s) + 2A_{12}^i(B_1^r B_2^s + B_2^r B_1^s) \quad (r \leq s), \\ W_{12}^m W_{12}^n \quad (m \leq n), \qquad W_{12}^m(A_{11}^i - A_{22}^i)A_{12}^j - W_{12}^m(A_{11}^j - A_{22}^j)A_{12}^i \quad (i < j), \\ W_{12}^m(B_1^r B_2^s - B_2^r B_1^s) \quad (r < s), \\ W_{12}^m(A_{11}^i - A_{22}^i)(B_1^r B_2^s + B_2^r B_1^s) - 2W_{12}^m A_{12}^i(B_1^r B_1^s - B_2^r B_2^s) \quad (r \leq s), \\ ((A_{11}^i - A_{22}^i)A_{12}^j - (A_{11}^j - A_{22}^j)A_{12}^i)(B_1^r B_2^s - B_2^r B_1^s) \quad (i < j, r < s) \end{gathered} \tag{3.2}$$

where $i, j = 1, \ldots, N$; $p = 1, \ldots, M$; $m, n = 1, \ldots, P$; $r, s = 1, \ldots, N + M + P$ subject to the restrictions indicated. The quantities B_α^r $(\alpha = 1, 2; r = 1, \ldots, N + M + P)$ are defined by (2.1). If we set the W_{12}^m, W_{31}^m, W_{23}^m $(m = 1, \ldots, P)$ appearing in (3.2) equal to zero, we obtain an integrity basis for functions $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M)$ which are invariant under T_2. The integrity basis so obtained contains fewer terms than that given by Adkins [2], which would indicate the presence of redundant terms in the basis listed in [2].

4. An integrity basis for functions invariant under T_3. The group T_3 is generated by the matrices $\mathbf{Q}(\theta)$ and $\mathbf{R}_3$. We have seen in Sec. 2 that any polynomial function $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M, \mathbf{W}_1, \ldots, \mathbf{W}_P)$ which is invariant under the group T_1 generated by $Q(\theta)$ is expressible as a polynomial in the quantities (2.3). In order to determine the general form of the function $F(\mathbf{A}_1, \ldots, \mathbf{W}_P)$ which is invariant under T_3, we need only determine the general form of a polynomial function of the quantities (2.3) which is invariant under $\mathbf{R}_3$. The elements of (2.3) either remain invariant under $\mathbf{R}_3$ or change sign under $\mathbf{R}_3$. Let $K_1, \ldots, K_c$ and $L_1, \ldots, L_d$ denote the elements of (2.3) which remain invariant under $\mathbf{R}_3$ and which change sign under $\mathbf{R}_3$ respectively. Let

$$C_\alpha^i = A_{3\alpha}^i \quad (\alpha = 1, 2; i = 1, \ldots, N), \qquad C_\alpha^{N+j} = W_{3\alpha}^j \quad (\alpha = 1, 2; j = 1, \ldots, P). \tag{4.1}$$

With (2.3) and (4.1), we see that the elements $K_1, \ldots, K_c$ of (2.3) which remain invariant under $\mathbf{R}_3$ are given by

$$\begin{aligned}
&A^i_{33},\ A^i_{11} + A^i_{22},\ W^m_{12},\ (A^i_{11} - A^i_{22})(A^j_{11} - A^j_{22}) + 4A^i_{12}A^j_{12} \qquad (i \le j),\\
&(A^i_{11} - A^i_{22})A^j_{12} - (A^j_{11} - A^j_{22})A^i_{12} \qquad (i < j), \qquad C^r_1 C^s_1 + C^r_2 C^s_2 \qquad (r \le s),\\
&C^r_1 C^s_2 - C^r_2 C^s_1 \qquad (r < s), \qquad V^p_1 V^q_1 + V^p_2 V^q_2 \qquad (p \le q),\\
&V^p_1 V^q_2 - V^p_2 V^q_1 \qquad (p < q),\\
&(A^i_{11} - A^i_{22})(C^r_1 C^s_1 - C^r_2 C^s_2) + 2A^i_{12}(C^r_1 C^s_2 + C^r_2 C^s_1) \qquad (r \le s),\\
&(A^i_{11} - A^i_{22})(V^p_1 V^q_1 - V^p_2 V^q_2) + 2A^i_{12}(V^p_1 V^q_2 + V^p_2 V^q_1) \qquad (p \le q),\\
&(A^i_{11} - A^i_{22})(C^r_1 C^s_2 + C^r_2 C^s_1) - 2A^i_{12}(C^r_1 C^s_1 - C^r_2 C^s_2) \qquad (r \le s),\\
&(A^i_{11} - A^i_{22})(V^p_1 V^q_2 + V^p_2 V^q_1) - 2A^i_{12}(V^p_1 V^q_1 - V^p_2 V^q_2) \qquad (p \le q)
\end{aligned} \tag{4.2}$$

where $i, j = 1, \ldots, N$; $p, q = 1, \ldots, M$; $m = 1, \ldots, P$; $r, s = 1, \ldots, N + P$ subject to the restrictions indicated. With (2.3) and (4.1), we see that the elements $L_1, \ldots, L_d$ of (2.3) which change sign under $\mathbf{R}_3$ are given by

$$\begin{aligned}
&V^p_3,\ C^r_1 V^p_1 + C^r_2 V^p_2,\ C^r_1 V^p_2 - C^r_2 V^p_1,\\
&(A^i_{11} - A^i_{22})(C^r_1 V^p_1 - C^r_2 V^p_2) + 2A^i_{12}(C^r_1 V^p_2 + C^r_2 V^p_1),\\
&(A^i_{11} - A^i_{22})(C^r_1 V^p_2 + C^r_2 V^p_1) - 2A^i_{12}(C^r_1 V^p_1 - C^r_2 V^p_2)
\end{aligned} \tag{4.3}$$

where $i = 1, \ldots, N; p = 1, \ldots, M; r = 1, \ldots, N + P$. An integrity basis for functions $F(\mathbf{A}_1, \ldots, \mathbf{W}_P)$ which are invariant under T_3 is then given by the quantities $K_1, \ldots, K_c$ and $L_p L_q$ $(p, q = 1, \ldots, d;\ p \le q)$. After eliminating the redundant terms from the set of invariants $L_p L_q$, we obtain the result that an integrity basis for functions $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M, \mathbf{W}_1, \ldots, \mathbf{W}_P)$ which are invariant under T_3 is given by

$$\begin{aligned}
&A^i_{33},\ A^i_{11} + A^i_{22},\ W^m_{12},\ (A^i_{11} - A^i_{22})(A^j_{11} - A^j_{22}) + 4A^i_{12}A^j_{12} \qquad (i \le j),\\
&(A^i_{11} - A^i_{22})A^j_{12} - (A^j_{11} - A^j_{22})A^i_{12} \qquad (i < j),\\
&C^r_1 C^s_1 + C^r_2 C^s_2 \qquad (r \le s),\\
&C^r_1 C^s_2 - C^r_2 C^s_1 \qquad (r < s), \qquad V^p_1 V^q_1 + V^p_2 V^q_2 \qquad (p \le q),\\
&V^p_1 V^q_2 - V^p_2 V^q_1 \qquad (p < q),\\
&V^p_3 V^q_3 \qquad (p \le q), \qquad (A^i_{11} - A^i_{22})(C^r_1 C^s_1 - C^r_2 C^s_2)\\
&\quad + 2A^i_{12}(C^r_1 C^s_2 + C^r_2 C^s_1) \qquad (r \le s),\\
&(A^i_{11} - A^i_{22})(C^r_1 C^s_2 + C^r_2 C^s_1) - 2A^i_{12}(C^r_1 C^s_1 - C^r_2 C^s_2) \qquad (r \le s),\\
&(A^i_{11} - A^i_{22})(V^p_1 V^q_1 - V^p_2 V^q_2) + 2A^i_{12}(V^p_1 V^q_2 + V^p_2 V^q_1) \qquad (p \le q),\\
&(A^i_{11} - A^i_{22})(V^p_1 V^q_2 + V^p_2 V^q_1) - 2A^i_{12}(V^p_1 V^q_1 - V^p_2 V^q_2) \qquad (p \le q),\\
&V^p_3(C^r_1 V^q_1 + C^r_2 V^q_2), \qquad V^p_3(C^r_1 V^q_2 - C^r_2 V^q_1),\\
&V^p_3(A^i_{11} - A^i_{22})(C^r_1 V^q_1 - C^r_2 V^q_2) + 2V^p_3 A^i_{12}(C^r_1 V^q_2 + C^r_2 V^q_1),
\end{aligned} \tag{4.4}$$

$$V_3^p(A_{11}^i - A_{22}^i)(C_1^r V_2^q + C_2^r V_1^q) - 2V_3^p A_{12}^i(C_1^r V_1^q - C_2^r V_2^q),$$
$$(C_1^r C_1^s - C_2^r C_2^s)(V_1^p V_1^q - V_2^p V_2^q) + (C_1^r C_2^s + C_2^r C_1^s)(V_1^p V_2^q + V_2^p V_1^q) \qquad (r \le s,\ p \le q),$$
$$(C_1^r C_1^s - C_2^r C_2^s)(V_1^p V_2^q + V_2^p V_1^q) - (C_1^r C_2^s + C_2^r C_1^s)(V_1^p V_1^q - V_2^p V_2^q) \qquad (r \le s,\ p \le q)$$

where $i, j = 1, \ldots, N$; $p, q = 1, \ldots, M$; $m = 1, \ldots, P$; $r, s = 1, \ldots, N + P$ subject to the restrictions indicated. The quantities C_α^r ($\alpha = 1, 2; r = 1, \ldots, N + P$) are defined by (4.1).

5. An integrity basis for functions invariant under T_4. The group T_4 is generated by the matrices $\mathbf{Q}(\theta)$, $\mathbf{R}_1$ and $\mathbf{R}_3$. We have seen in Sec. 3 that any polynomial function $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M, \mathbf{W}_1, \ldots, \mathbf{W}_P)$ which is invariant under the group T_2 generated by $Q(\theta)$ and $\mathbf{R}_1$ is expressible as a polynomial in the quantities (3.2). In order to determine the general form of a function $F(\mathbf{A}_1, \ldots, \mathbf{W}_P)$ which is invariant under T_4, we need only determine the general form of a polynomial function of the quantities (3.2) which is invariant under $\mathbf{R}_3$. We observe that the elements of (3.2) either remain invariant under $\mathbf{R}_3$ or change sign under $\mathbf{R}_3$. Let

$$C_\alpha^i = A_{3\alpha}^i \quad (\alpha = 1, 2;\ i = 1, \ldots, N), \qquad C_\alpha^{N+j} = W_{3\alpha}^j \quad (\alpha = 1, 2;\ j = 1, \ldots, P). \tag{5.1}$$

With (3.2) and (5.1), we see that the elements $M_1, \ldots, M_e$ of (3.2) which remain invariant under R_3 are given by

$$\begin{aligned}
&A_{33}^i,\ A_{11}^i + A_{22}^i,\ (A_{11}^i - A_{22}^i)(A_{11}^j - A_{22}^j) + 4A_{12}^i A_{12}^j \qquad (i \le j),\\
&C_1^r C_1^s + C_2^r C_2^s \qquad (r \le s), \qquad V_1^p V_1^q + V_2^p V_2^q \qquad (p \le q),\\
&W_{12}^m W_{12}^n \qquad (m \le n),\\
&(A_{11}^i - A_{22}^i)(C_1^r C_1^s - C_2^r C_2^s) + 2A_{12}^i(C_1^r C_2^s + C_2^r C_1^s) \qquad (r \le s),\\
&((A_{11}^i - A_{22}^i)A_{12}^j - (A_{11}^j - A_{22}^j)A_{12}^i)(C_1^r C_2^s - C_2^r C_1^s) \qquad (i < j,\ r < s),\\
&(A_{11}^i - A_{22}^i)(V_1^p V_1^q - V_2^p V_2^q) + 2A_{12}^i(V_1^p V_2^q + V_2^p V_1^q) \qquad (p \le q),\\
&((A_{11}^i - A_{22}^i)A_{12}^j - (A_{11}^j - A_{22}^j)A_{12}^i)(V_1^p V_2^q - V_2^p V_1^q) \qquad (i < j,\ p < q),\\
&W_{12}^m(A_{11}^i - A_{22}^i)A_{12}^j - W_{12}^m(A_{11}^j - A_{22}^j)A_{12}^i \qquad (i < j),\\
&W_{12}^m(C_1^r C_2^s - C_2^r C_1^s) \qquad (r < s),\\
&W_{12}^m(V_1^p V_2^q - V_2^p V_1^q) \qquad (p < q),\\
&W_{12}^m(A_{11}^i - A_{22}^i)(C_1^r C_2^s + C_2^r C_1^s) - 2W_{12}^m A_{12}^i(C_1^r C_1^s - C_2^r C_2^s) \qquad (r \le s),\\
&W_{12}^m(A_{11}^i - A_{22}^i)(V_1^p V_2^q + V_2^p V_1^q) - 2W_{12}^m A_{12}^i(V_1^p V_1^q - V_2^p V_2^q) \qquad (p \le q)
\end{aligned} \tag{5.2}$$

where $i, j = 1, \ldots, N$; $p, q = 1, \ldots, M$; $m, n = 1, \ldots, P$; $r, s = 1, \ldots, N + P$ subject to the restrictions indicated. With (3.2) and (5.1), we see that the elements $N_1, \ldots, N_f$ of (3.2) which change sign under $\mathbf{R}_3$ are given by

$$
\begin{aligned}
& V_3^p,\ C_1^r V_1^p + C_2^r V_2^p,\ W_{12}^m(C_1^r V_2^p - C_2^r V_1^p), \\
& (A_{11}^i - A_{22}^i)(C_1^r V_1^p - C_2^r V_2^p) + 2A_{12}^i(C_1^r V_2^p + C_2^r V_1^p), \\
& W_{12}^m(A_{11}^i - A_{22}^i)(C_1^r V_2^p + C_2^r V_1^p) - 2W_{12}^m A_{12}^i(C_1^r V_1^p - C_2^r V_2^p), \\
& ((A_{11}^i - A_{22}^i)A_{12}^j - (A_{11}^j - A_{22}^j)A_{12}^i)(C_1^r V_2^p - C_2^r V_1^p) \qquad (i < j)
\end{aligned}
\tag{5.3}
$$

where $i, j = 1, \ldots, N;\ p = 1, \ldots, M;\ m = 1, \ldots, P;\ r = 1, \ldots, N + P$ subject to the restrictions indicated. An integrity basis for functions $F(\mathbf{A}_1, \ldots, \mathbf{W}_P)$ which are invariant under T_4 is then given by $M_1, \ldots, M_e$ and $N_p N_q$ $(p, q = 1, \ldots, f;\ p \leq q)$. After eliminating the redundant elements from the set of invariants $N_p N_q$, we obtain the result that an integrity basis for functions $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M, \mathbf{W}_1, \ldots, \mathbf{W}_P)$ which are invariant under T_4 is given by

$$
\begin{aligned}
& A_{33}^i,\ A_{11}^i + A_{22}^i,\ (A_{11}^i - A_{22}^i)(A_{11}^j - A_{22}^j) + 4A_{12}^i A_{12}^j \qquad (i \leq j), \\
& C_1^r C_1^s + C_2^r C_2^s \qquad (r \leq s), \qquad V_1^p V_1^q + V_2^p V_2^q \qquad (p \leq q), \\
& V_3^p V_3^q \qquad (p \leq q), \qquad W_{12}^m W_{12}^n \qquad (m \leq n), \\
& (A_{11}^i - A_{22}^i)(C_1^r C_1^s - C_2^r C_2^s) + 2A_{12}^i(C_1^r C_2^s + C_2^r C_1^s) \qquad (r \leq s), \\
& ((A_{11}^i - A_{22}^i)A_{12}^j - (A_{11}^j - A_{22}^j)A_{12}^i)(C_1^r C_2^s - C_2^r C_1^s) \qquad (i < j,\ r < s), \\
& (A_{11}^i - A_{22}^i)(V_1^p V_1^q - V_2^p V_2^q) + 2A_{12}^i(V_1^p V_2^q + V_2^p V_1^q) \qquad (p \leq q), \\
& ((A_{11}^i - A_{22}^i)A_{12}^j - (A_{11}^j - A_{22}^j)A_{12}^i)(V_1^p V_2^q - V_2^p V_1^q) \qquad (i < j,\ p < q), \\
& W_{12}^m(A_{11}^i - A_{22}^i)A_{12}^j - W_{12}^m(A_{11}^j - A_{22}^j)A_{12}^i \qquad (i < j), \\
& (C_1^r C_1^s - C_2^r C_2^s)(V_1^p V_1^q - V_2^p V_2^q) \\
& \quad + (C_1^r C_2^s + C_2^r C_1^s)(V_1^p V_2^q + V_2^p V_1^q) \qquad (r \leq s,\ p \leq q), \\
& (C_1^r C_2^s - C_2^r C_1^s)(V_1^p V_2^q - V_2^p V_1^q) \qquad (r < s,\ p < q), \\
& V_3^p(C_1^r V_1^q + C_2^r V_2^q),\ W_{12}^m(C_1^r C_2^s - C_2^r C_1^s) \qquad (r < s), \\
& W_{12}^m(V_1^p V_2^q - V_2^p V_1^q) \qquad (p < q), \\
& ((A_{11}^i - A_{22}^i)(C_1^r C_2^s + C_2^r C_1^s) \\
& \quad - 2A_{12}^i(C_1^r C_1^s - C_2^r C_2^s))(V_1^p V_2^q - V_2^p V_1^q) \qquad (r \leq s,\ p < q), \\
& ((A_{11}^i - A_{22}^i)(V_1^p V_2^q + V_2^p V_1^q) \\
& \quad - 2A_{12}^i(V_1^p V_1^q - V_2^p V_2^q))(C_1^r C_2^s - C_2^r C_1^s) \qquad (p \leq q,\ r < s), \\
& W_{12}^m(A_{11}^i - A_{22}^i)(C_1^r C_2^s + C_2^r C_1^s) - 2W_{12}^m A_{12}^i(C_1^r C_1^s - C_2^r C_2^s) \qquad (r \leq s), \\
& W_{12}^m(A_{11}^i - A_{22}^i)(V_1^p V_2^q + V_2^p V_1^q) - 2W_{12}^m A_{12}^i(V_1^p V_1^q - V_2^p V_2^q) \qquad (p \leq q),
\end{aligned}
\tag{5.4}
$$

$$W^m_{12}(C^r_1 C^s_1 - C^r_2 C^s_2)(V^p_1 V^q_2 + V^p_2 V^q_1)$$
$$- W^m_{12}(C^r_1 C^s_2 + C^r_2 C^s_1)(V^p_1 V^q_1 - V^p_2 V^q_2) \qquad (r \le s, p \le q),$$
$$V^p_3(A^i_{11} - A^i_{22})(C^r_1 V^q_1 - C^r_2 V^q_2) + 2V^p_3 A^i_{12}(C^r_1 V^q_2 + C^r_2 V^q_1),$$
$$V^p_3((A^i_{11} - A^i_{22})A^j_{12} - (A^j_{11} - A^j_{22})A^i_{12})(C^r_1 V^q_2 - C^r_2 V^q_1) \qquad (i < j),$$
$$V^p_3 W^m_{12}(C^r_1 V^q_2 - C^r_2 V^q_1),$$
$$V^p_3 W^m_{12}(A^i_{11} - A^i_{22})(C^r_1 V^q_2 + C^r_2 V^q_1) - 2V^p_3 W^m_{12} A^i_{12}(C^r_1 V^q_1 - C^r_2 V^q_2)$$

where $i, j = 1, \ldots, N$; $p, q = 1, \ldots, M$; $m, n = 1, \ldots, P$; $r, s = 1, \ldots, N + P$ subject to the restrictions indicated. The quantities C^r_α ($\alpha = 1, 2; r = 1, \ldots, N + P$) are defined by (5.1).

6. An integrity basis for functions invariant under T_5. The group T_5 is generated by the matrices $\mathbf{Q}(\theta)$ and $\mathbf{D}_2$. We have seen in Sec. 2 that any polynomial function $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M, \mathbf{W}_1, \ldots, \mathbf{W}_P)$ which is invariant under the group T_1 generated by $\mathbf{Q}(\theta)$ is expressible as a polynomial in the quantities (2.3). In order to determine the general form of the function $F(\mathbf{A}_1, \ldots, \mathbf{W}_P)$ which is invariant under T_5, we need only determine the general form of a polynomial function of the quantities (2.3) which is invariant under $\mathbf{D}_2$. The elements of (2.3) either remain invariant under $\mathbf{D}_2$ or change sign under $\mathbf{D}_2$. Let

$$C^i_\alpha = A^i_{3\alpha} \quad (\alpha = 1, 2; i = 1, \ldots, N), \qquad C^{N+j}_\alpha = W^j_{3\alpha} \quad (\alpha = 1, 2; j = 1, \ldots, P). \tag{6.1}$$

With (2.3) and (6.1), we see that the elements $P_1, \ldots, P_g$ of (2.3) which remain invariant under $\mathbf{D}_2$ are given by

$$\begin{gathered} A^i_{33}, A^i_{11} + A^i_{22}, (A^i_{11} - A^i_{22})(A^j_{11} - A^j_{22}) + 4A^i_{12} A^j_{12} \qquad (i \le j), \\ C^s_1 C^t_1 + C^s_2 C^t_2 \quad (s \le t), \qquad V^p_1 V^q_1 + V^p_2 V^q_2 \quad (p \le q), \qquad C^s_1 V^p_2 - C^s_2 V^p_1, \\ (A^i_{11} - A^i_{22})(C^s_1 C^t_1 - C^s_2 C^t_2) + 2A^i_{12}(C^s_1 C^t_2 + C^s_2 C^t_1) \qquad (s \le t), \\ (A^i_{11} - A^i_{22})(V^p_1 V^q_1 - V^p_2 V^q_2) + 2A^i_{12}(V^p_1 V^q_2 + V^p_2 V^q_1) \qquad (p \le q), \\ (A^i_{11} - A^i_{22})(C^s_1 V^p_2 + C^s_2 V^p_1) - 2A^i_{12}(C^s_1 V^p_1 - C^s_2 V^p_2) \end{gathered} \tag{6.2}$$

where $i, j = 1, \ldots, N$; $p, q = 1, \ldots, M$; $s, t = 1, \ldots, N + P$ subject to the restrictions indicated. With (2.3) and (6.1), we see that the elements $Q_1, \ldots, Q_h$ of (2.3) which change sign under $\mathbf{D}_2$ are given by

$$\begin{gathered} V^p_3, W^m_{12}, (A^i_{11} - A^i_{22})A^j_{12} - (A^j_{11} - A^j_{22})A^i_{12} \qquad (i < j), \\ C^s_1 V^p_1 + C^s_2 V^p_2, C^s_1 C^t_2 - C^s_2 C^t_1 \quad (s < t), \qquad V^p_1 V^q_2 - V^p_2 V^q_1 \quad (p < q), \\ (A^i_{11} - A^i_{22})(C^s_1 V^p_1 - C^s_2 V^p_2) + 2A^i_{12}(C^s_1 V^p_2 + C^s_2 V^p_1), \\ (A^i_{11} - A^i_{22})(C^s_1 C^t_2 + C^s_2 C^t_1) - 2A^i_{12}(C^s_1 C^t_1 - C^s_2 C^t_2) \qquad (s \le t), \\ (A^i_{11} - A^i_{22})(V^p_1 V^q_2 + V^p_2 V^q_1) - 2A^i_{12}(V^p_1 V^q_1 - V^p_2 V^q_2) \qquad (p \le q) \end{gathered} \tag{6.3}$$

where $i, j = 1, \ldots, N$; $p, q = 1, \ldots, M$; $m = 1, \ldots, P$; $s, t = 1, \ldots, N + P$ subject to the restrictions indicated. An integrity basis for functions $F(\mathbf{A}_1, \ldots, \mathbf{W}_p)$ which are invariant under T_5 is then given by $P_1, \ldots, P_g$ and $Q_r Q_s$ ($r, s = 1, \ldots, h; r \le s$). After eliminating the redundant elements from the set of invariants $Q_r Q_s$, we obtain the result that an integrity basis for functions $F(\mathbf{A}_1, \ldots, \mathbf{A}_N, \mathbf{V}_1, \ldots, \mathbf{V}_M, \mathbf{W}_1, \ldots, \mathbf{W}_P)$ which are invariant under T_5 is given by

$$
\begin{aligned}
&A^i_{33},\ A^i_{11} + A^i_{22},\ (A^i_{11} - A^i_{22})(A^j_{11} - A^j_{22}) + 4A^i_{12}A^j_{12} \qquad (i \le j),\\
&C^s_1 C^t_1 + C^s_2 C^t_2 \qquad (s \le t), \qquad C^s_1 V^p_2 - C^s_2 V^p_1,\ V^p_1 V^q_1 + V^p_2 V^q_2 \qquad (p \le q),\\
&(A^i_{11} - A^i_{22})(C^s_1 C^t_1 - C^s_2 C^t_2) + 2A^i_{12}(C^s_1 C^t_2 + C^s_2 C^t_1) \qquad (s \le t),\\
&(A^i_{11} - A^i_{22})(C^s_1 V^p_2 + C^s_2 V^p_1) - 2A^i_{12}(C^s_1 V^p_1 - C^s_2 V^p_2),\\
&(A^i_{11} - A^i_{22})(V^p_1 V^q_1 - V^p_2 V^q_2) + 2A^i_{12}(V^p_1 V^q_2 + V^p_2 V^q_1) \qquad (p \le q),\\
&((A^i_{11} - A^i_{22})A^j_{12} - (A^j_{11} - A^j_{22})A^i_{12})(C^s_1 C^t_2 - C^s_2 C^t_1) \qquad (i < j,\ s < t),\\
&((A^i_{11} - A^i_{22})A^j_{12} - (A^j_{11} - A^j_{22})A^i_{12})(C^s_1 V^p_1 + C^s_2 V^p_2) \qquad (i < j),\\
&((A^i_{11} - A^i_{22})A^j_{12} - (A^j_{11} - A^j_{22})A^i_{12})(V^p_1 V^q_2 - V^p_2 V^q_1) \qquad (i < j,\ p < q),\\
&W^m_{12} W^n_{12} \qquad (m \le n), \qquad W^m_{12}(A^i_{11} - A^i_{22})A^j_{12} - W^m_{12}(A^j_{11} - A^j_{22})A^i_{12} \qquad (i < j),\\
&W^m_{12}(C^s_1 C^t_2 - C^s_2 C^t_1) \qquad (s < t), \qquad W^m_{12}(C^s_1 V^p_1 + C^s_2 V^p_2),\\
&W^m_{12}(V^p_1 V^q_2 - V^p_2 V^q_1) \qquad (p < q), \qquad\qquad (6.4)\\
&W^m_{12}(A^i_{11} - A^i_{22})(C^s_1 C^t_2 + C^s_2 C^t_1) - 2W^m_{12} A^i_{12}(C^s_1 C^t_1 - C^s_2 C^t_2) \qquad (s \le t),\\
&W^m_{12}(A^i_{11} - A^i_{22})(C^s_1 V^p_1 - C^s_2 V^p_2) + 2W^m_{12} A^i_{12}(C^s_1 V^p_2 + C^s_2 V^p_1),\\
&W^m_{12}(A^i_{11} - A^i_{22})(V^p_1 V^q_2 + V^p_2 V^q_1) - 2W^m_{12} A^i_{12}(V^p_1 V^q_1 - V^p_2 V^q_2) \qquad (p \le q),\\
&W^m_{12} V^p_3,\ V^p_3 V^q_3 \qquad (p \le q),\\
&V^p_3(A^i_{11} - A^i_{22})A^j_{12} - V^p_3(A^j_{11} - A^j_{22})A^i_{12} \qquad (i < j),\\
&V^p_3(C^s_1 C^t_2 - C^s_2 C^t_1) \qquad (s < t), \qquad V^p_3(C^s_1 V^q_1 + C^s_2 V^q_2),\\
&V^p_3(V^q_1 V^r_2 - V^q_2 V^r_1) \qquad (q < r),\\
&V^p_3(A^i_{11} - A^i_{22})(C^s_1 V^q_1 - C^s_2 V^q_2) + 2V^p_3 A^i_{12}(C^s_1 V^q_2 + C^s_2 V^q_1),\\
&V^p_3(A^i_{11} - A^i_{22})(C^s_1 C^t_2 + C^s_2 C^t_1) - 2V^p_3 A^i_{12}(C^s_1 C^t_1 - C^s_2 C^t_2) \qquad (s \le t),\\
&V^p_3(A^i_{11} - A^i_{22})(V^q_1 V^r_2 + V^q_2 V^r_1) - 2V^p_3 A^i_{12}(V^q_1 V^r_1 - V^q_2 V^r_2) \qquad (q \le r)
\end{aligned}
$$

where $i, j = 1, \ldots, N$; $p, q, r = 1, \ldots, M$; $m, n = 1, \ldots, P$; $s, t = 1, \ldots, N + P$ subject to the restrictions indicated. The quantities C^s_α ($\alpha = 1, 2$; $s = 1, \ldots, N + P$) are defined by (6.1).

References

[1] J. E. Adkins, Arch. Rat. Mech. Anal. **4**, 193 (1960)
[2] J. E. Adkins, Arch. Rat. Mech. Anal. **5**, 263 (1960)
[3] R. L. Long and L. V. McIntire, private communication (1973)