AN INEQUALITY FOR THE COEFFICIENT σ OF THE FREE BOUNDARY $s(t) = 2\sigma\sqrt{t}$ OF THE NEUMANN SOLUTION FOR THE TWO-PHASE STEFAN PROBLEM*

By

DOMINGO ALBERTO TARZIA

Universidad Nacional de Rosario, Argentina

Abstract. We consider a semi-infinite body (e.g. ice), represented by $(0, +\infty)$, with an initial temperature $-c < 0$ having a heat flux $h(t) = -h_0/\sqrt{t}$ $(h_0 > 0)$ in the fixed face $x = 0$. If $h_0 > ck_1/\sqrt{\pi}\, a_1$ there exists a solution, of Neumann type, for the resulting two-phase Stefan problem. If we connect it with the Neumann problem (on $x = 0$ the body has a temperature $b > 0$) we obtain the inequality $\operatorname{erf}(\sigma/a_2) < (k_2 b a_1/k_1 c a_2)$ for the coefficient σ of the free boundary $s(t) = 2\sigma\sqrt{t}$, where k_i and a_i^2 are respectively the thermal conductivity and thermal diffusivity coefficients of the corresponding i phase ($i = 1$: solid phase, $i = 2$: liquid phase). If $h_0 < ck_1/\sqrt{\pi}\, a_1$ there is no solution of the initial problem and if $h_0 = ck_1/\sqrt{\pi}\, a_1$ the problem has no physical meaning and corresponds to the case where the latent heat of fusion L tends to infinity.

Notation.

$\Omega = (0, +\infty)$	semi-infinite body
x	space coordinate variable in Ω
t	time
$s(t)$	position of the solid-liquid interface (free boundary) at time $t > 0$
$\theta(x, t)$	temperature defined for $x > 0$, $t > 0$
$\theta_2(x, t)$	water temperature defined for $0 < x < s(t)$, $t > 0$
$\theta_1(x, t)$	ice temperature defined for $x > s(t)$, $t > 0$
c_2	specific heat of water
c_1	specific heat of ice
l	latent heat of fusion
ρ	mass density
k_2	thermal conductivity of water
k_1	thermal conductivity of ice
$-c < 0$	initial temperature
$b > 0$	temperature in the fixed face $x = 0$
$h(t)$	heat flux in the fixed face $x = 0$

* Received October 1, 1980.

$C_2 = \rho c_2$, $C_1 = \rho c_1$, $L = \rho l$
$a_2 = (k_2/C_2)^{1/2}$ $\quad a_1 = (k_1/C_1)^{1/2}$
a_2^2 thermal diffusivity of water
a_1^2 thermal diffusivity of ice.

I. The problem. We shall consider the two-phase Stefan problem for a semi-infinite body, represented by $\Omega = (0, +\infty)$ with null change phase temperature (case: water-ice). That is, we shall find the functions $s = s(t) > 0$ (free boundary), defined for $t > 0$ with $s(0) = 0$, and

$$\begin{aligned} \theta(x, t) &= \theta_2(x, t) > 0 \quad \text{if} \quad 0 < x < s(t) \\ &= 0 \qquad\qquad\quad\ \ \text{if} \quad x = s(t) \\ &= \theta_1(x, t) < 0 \quad \text{if} \quad s(t) < x, \end{aligned} \tag{1}$$

defined for $x > 0$ and $t > 0$, such that they satisfy the following conditions:

$$C_1 \frac{\partial \theta_1}{\partial t} - k_1 \frac{\partial^2 \theta_1}{\partial x^2} = 0, \qquad \text{in} \quad s(t) < x,\ t > 0, \tag{2}$$

$$C_2 \frac{\partial \theta_2}{\partial t} - k_2 \frac{\partial^2 \theta_2}{\partial x^2} = 0, \qquad \text{in} \quad 0 < x < s(t),\ t > 0, \tag{3}$$

$$\theta_1(s(t), t) = 0, \qquad \forall t > 0, \tag{4}$$

$$\theta_2(s(t), t) = 0, \qquad \forall t > 0, \tag{5}$$

$$k_1 \frac{\partial \theta_1}{\partial x}(s(t), t) - k_2 \frac{\partial \theta_2}{\partial x}(s(t), t) = Ls'(t), \qquad \forall t > 0, \tag{6}$$

$$\theta_1(x, 0) = -c < 0, \qquad \forall x > 0, \tag{7}$$

$$k_2 \frac{\partial \theta_2}{\partial x}(0, t) = h(t), \qquad \forall t > 0. \tag{8}$$

The function $h(t)$ represents the heat flux that the material Ω receives in its fixed face $x = 0$.

In the case

$$h(t) = -(h_0/\sqrt{t}) \qquad (h_0 > 0), \tag{9}$$

we prove that there is not always a solution of Neumann type [1, 2, 3, 4, 7] for the problem (2) to (9). Moreover, the explicit solution exists if the constant h_0 satisfies a certain inequality (19). This idea was suggested in [5], where simple exact solutions are given for the steady-state two-phase Stefan problem in which the heat flux satisfies an inequality on a given portion of the body's boundary and the temperature has a constant sign (for example, positive) on the remaining body's boundary.

II. Solution of problem (2)—(9). Following the idea of Neumann for the two-phase Stefan problem [1, 2, 3, 4, 7], we propose:

$$\begin{aligned} \theta_1(x, t) &= A_1 + B_1 f(x/2a_1\sqrt{t}), \\ \theta_2(x, t) &= A_2 + B_2 f(x/2a_2\sqrt{t}), \\ s(t) &= 2\omega\sqrt{t}, \qquad \omega > 0, \end{aligned} \tag{10}$$

where

$$f(y) = \frac{2}{\sqrt{\pi}} \int_0^y \exp(-u^2)\, du, \qquad a_1 = \sqrt{k_1/C_1}, \qquad a_2 = \sqrt{k_2/C_2}. \tag{11}$$

The four conditions (4), (5), (7), (8) give rise to the two systems of equations

$$A_1 + B_1 = -c, \qquad A_1 + f(\omega/a_1)B_1 = 0, \tag{12}$$

$$B_2 = -\sqrt{\pi}\,\frac{a_2}{k_2}\,h_0, \qquad A_2 + f(\omega/a_2)B_2 = 0. \tag{13}$$

Solving (12) and (13) as functions of ω, we obtain:

$$A_1(\omega) = c\,\frac{f\left(\dfrac{\omega}{a_1}\right)}{1 - f\left(\dfrac{\omega}{a_1}\right)}, \qquad B_1(\omega) = \frac{-c}{1 - f\left(\dfrac{\omega}{a_1}\right)},$$

$$A_2(\omega) = \sqrt{\pi}\,\frac{a_2 h_0}{k_2} f\left(\frac{\omega}{a_2}\right), \qquad B_2(\omega) = -\sqrt{\pi}\,\frac{a_2 h_0}{k_2}. \tag{14}$$

The condition (6) is satisfied if ω is a solution of the equation

$$F_0(\omega) = \omega, \qquad \omega > 0 \tag{15}$$

where

$$F_0(\omega) = \frac{h_0}{L}\exp\left(-\frac{\omega^2}{a_2^2}\right) - \frac{ck_1}{L\sqrt{\pi}\,a_1} F_1\left(\frac{\omega}{a_1}\right), \qquad F_1(x) = \frac{\exp(-x^2)}{1 - f(x)}. \tag{16}$$

Taking into account the following properties of the function F_1:

$$F_1(0^+) = 1, \qquad F_1(+\infty) = +\infty,$$
$$\frac{dF_1}{dx}(x) > 0, \qquad \forall x > 0 \qquad \text{(cf. Appendix)}, \tag{17}$$

we deduce for the function F_0

$$F_0(0^+) = \frac{1}{L}\left(h_0 - \frac{ck_1}{\sqrt{\pi}\,a_1}\right),$$
$$F_0(+\infty) = -\infty, \qquad \frac{dF_0}{d\omega}(\omega) < 0, \qquad \forall \omega > 0, \tag{18}$$

and therefore we obtain

LEMMA 1. There exists a solution (10) of the problem (2)—(9) iff $F_0(0^+) > 0$ and iff

$$h_0 > ck_1/\sqrt{\pi}\,a_1 \tag{19}$$

Proof. Using the properties (18), Eq. (15) has a unique solution iff $F_0(0^+) > 0$.

Remark 1. If $h_0 < ck_1/\sqrt{\pi}\, a_1$, there is no solution of the problem (2)—(9) of the type (10). The limit case $h_0 = ck_1/\sqrt{\pi}\, a_1$ has no physical meaning and corresponds to the case $L \to +\infty$ (cf. Sec. IV).

III. Relationship with the Neumann solution. The temperature in the fixed face $x = 0$ is given by

$$b_0 = \theta_2(0, t) = A_2(\omega) = \sqrt{\pi}\, \frac{a_2 h_0}{k_2} f\left(\frac{\omega}{a_2}\right). \tag{20}$$

Since $b_0 > 0$, we can consider the two-phase Stefan problem consisting in finding the functions $s(t)$, $\theta_1(x, t)$, $\theta_2(x, t)$ solutions of (2)—(7) and (8 bis), where:

$$\theta_2(0, t) = b \quad \text{with} \quad b > 0. \tag{8 bis}$$

The solution of problem (2)—(7) and (8 bis), which is known as Neumann solution [1, 2, 3, 4, 7], is given by:

$$\begin{aligned} \theta_1(x, t) &= \alpha_1 + \beta_1 f\left(\frac{x}{2a_1\sqrt{t}}\right), \\ \theta_2(x, t) &= \alpha_2 + \beta_2 f\left(\frac{x}{2a_2\sqrt{t}}\right), \\ s(t) &= 2\sigma\sqrt{t}, \end{aligned} \tag{21}$$

with

$$\begin{aligned} \alpha_1(\sigma) &= c\left\{f\left(\frac{\sigma}{a_1}\right)\Big/\left[1 - f\left(\frac{\sigma}{a_1}\right)\right]\right\}, \\ \beta_1(\sigma) &= -c\Big/\left[1 - f\left(\frac{\sigma}{a_1}\right)\right], \\ \alpha_2(\sigma) &= b, \\ \beta_2(\sigma) &= -b\Big/f\left(\frac{\sigma}{a_2}\right). \end{aligned} \tag{22}$$

Here σ is the unique solution of the equation

$$F(\sigma) = \sigma, \qquad \sigma > 0 \tag{23}$$

with

$$F(\sigma) = \frac{k_1}{La_1\sqrt{\pi}}\, \beta_1(\sigma) \exp\left(-\frac{\sigma^2}{a_1^2}\right) - \frac{k_2}{La_2\sqrt{\pi}}\, \beta_2(\sigma) \exp\left(-\frac{\sigma^2}{a_2^2}\right) \tag{24}$$

which satisfies:

$$F(0^+) = +\infty, \qquad F(+\infty) = -\infty, \qquad \frac{dF}{d\sigma}(\sigma) < 0, \qquad \forall \sigma > 0. \tag{25}$$

Remark 2. Notice that

$$A_1(x) = \alpha_1(x), \qquad \forall x > 0, \qquad B_1(x) = \beta_1(x), \qquad \forall x > 0. \tag{26}$$

Lemma 2. If the condition (19) is valid and we take $b = b_0 > 0$, then

$$\sigma = \omega. \tag{27}$$

Proof. Since $\beta_2(\omega) = -\sqrt{\pi}(a_2 h_0/k_2)$, we have:

$$\begin{aligned} F(\omega) &= \frac{-ck_1}{La_1\sqrt{\pi}} \frac{\exp(-\omega^2/a_1^2)}{1-f(\omega/a_1)} + \frac{h_0}{L}\exp(-\omega^2/a_2^2) \\ &= \frac{h_0}{L}\exp(-\omega^2/a_2^2) - \frac{ck_1}{L\sqrt{\pi}\,a_1} F_1(\omega) = F_0(\omega) = \omega, \end{aligned} \tag{28}$$

and from the uniqueness of σ in Eq. (23), we deduce (27).

Remark 3. With the hypothesis of Lemma 2, we have:

$$A_2(x) = \alpha_2(x), \qquad \forall x > 0, \qquad B_2(x) = \beta_2(x), \qquad \forall x > 0. \tag{29}$$

Remark 4. With the hypothesis of Lemma 2, we deduce the following equivalence: Problem (2)—(9) $\Leftrightarrow$ Problem (2)—(7) and (8 bis); this implies the inequality:

$$k_2 b/\sqrt{\pi}\, a_2 f(\sigma/a_2) > ck_1/\sqrt{\pi}\, a_1$$

or

$$f(\sigma/a_2) < k_2 a_1 b/k_1 a_2 c = \frac{b}{c}\sqrt{k_2 C_2/k_1 C_1}. \tag{30}$$

Notice that this inequality takes into account the temperatures b, c and the coefficients k_1, a_1 and k_2, a_2 corresponding to the solid and liquid phase, but not the latent heat of fusion L.

IV. Limit cases. Using a method analogous to that in [6], we have

LEMMA 3.

$$F_0(0^+) = 0 \Leftrightarrow h_0 = \frac{ck_1}{a_1\sqrt{\pi}} \Leftrightarrow \omega = 0 \Leftrightarrow L = +\infty. \tag{31}$$

Let $\theta_L(x, t)$ be the function defined by (10), (14) and (15) for each $L > 0$. Then:

Remark 5. If $h_0 > ck_1/a_1\sqrt{\pi}$, the limit of $\theta_L(x, t)$ as $L \to +\infty$ is given by:

$$\begin{aligned} \theta_{L=\infty}(x, t) = \lim_{L\to+\infty} \theta_L(x, t) &= 0 && \text{if} \quad x = 0, \\ &= -cf\left(\frac{x}{2a_1\sqrt{t}}\right) < 0 && \text{if} \quad x \neq 0. \end{aligned} \tag{32}$$

Moreover, $\theta_{L=\infty}$ is a continuous function, but its heat flux in $x = 0$ is given by:

$$k_1 \frac{\partial \theta_{L=\infty}}{\partial x}(0, t) = \frac{-ck_1}{a_1\sqrt{\pi t}} \neq \frac{-h_0}{\sqrt{t}} = h(t). \tag{33}$$

This implies that the limit $L \to +\infty$ has no physical meaning, as was remarked in [6].

Remark 6. If $h_0 > ck_1/a_1\sqrt{\pi}$ the limit of $\theta_L(x, t)$ as $L \to 0$ is given by:

$$\theta_{L=0}(x, t) = \lim_{L\to 0} \theta_L(x, t) = \frac{\sqrt{\pi}\, a_2 h_0}{k_2}\left(1 - f\left(\frac{x}{2a_2\sqrt{t}}\right)\right). \tag{34}$$

The function $\theta_{L=0}$ is continuous and its heat flux in $x = 0$ is:

$$k_2(\partial\theta_{L=0}/\partial x)(0, t) = -h_0/\sqrt{t} = h(t). \tag{35}$$

Appendix. Let

$$F_1(x) = \frac{\exp(-x^2)}{1 - f(x)}, \qquad f(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-u^2)\,du = \operatorname{erf}(x),$$
$$H(x) = \frac{f'(x)}{1 - f(x)} = \frac{2}{\sqrt{\pi}} F_1(x), \qquad G(x) = H(x) - 2x. \tag{1}$$

We prove the following properties:

LEMMA.

(i) $H(0) = 2/\sqrt{\pi}, \qquad H(+\infty) = +\infty, \qquad H(x) > 0, \qquad \forall x > 0,$

(ii) $G(0) = 2/\sqrt{\pi}, \qquad G(+\infty) = 0,$

(iii) $H'(x) = G(x)\cdot H(x), \qquad G'(x) = H'(x) - 2,$ (2)

(iv) $G(x) > 0, \qquad \forall x > 0,$

(v) $F_1'(x) > 0, \qquad \forall x > 0,$

Proof. (i), (ii) and (iii) are evident by definition or by application of L'Hopital's rule. (iv) we suppose that there exists $x_0 > 0/G(x_0) = 0$. It follows that

$$H(x_0) = 2x_0, \qquad H'(x_0) = 0, \tag{3}$$

$$G(x_0) = 0, \qquad G'(x_0) = -2 < 0 \tag{4}$$

The conditions (4) implies that there exists $x_1 > x_0$,

$$G'(x_1) = 0, \qquad G(x_1) < 0. \tag{5}$$

Therefore

$$H'(x_1) = G(x_1)H(x_1) < 0.$$

Then

$$0 = G'(x_1) = H'(x_1) - 2 < -2,$$

which is a contradiction. (v) is evident using (iii) and (iv).

REFERENCES

[1] M. Brillouin, *Sur quelques problèmes non résolus de physique mathématique classique : propagation de la fusion*, Annales de l'Inst. H. Poincaré, **1**, 285–308 (1930/31)

[2] H. S. Carslaw and J. C. Jaeger, *Conduction of heat in solids*, Clarendon Press, Oxford (1959)

[3] J. Crank, *The mathematics of diffusion*, Clarendon Press, Oxford (1956)
[4] L. I. Rubinstein, *The Stefan problem*, Trans. Math. Monographs, **27**, Amer. Math. Soc., Providence (1971)
[5] D. A. Tarzia, *Sobre el caso estacionario del problema de Stefan a dos fases.* Mathematicae Notae, Año 28, 73–89 (1980/81)
[6] D. A. Tarzia, *La chaleur latente de fusion tend vers l'infini n'a pas de sense physique pour le problème de Stefan*, submitted to Int. J. Heat Mass Transfer
[7] H. Weber, *Die partiellen Differential-gleinchungen der mathematischen Physik, nach Riemanns Vorlesungen*, t. II, Braunwschweig (1901), 118–122