THE INITIAL-VALUE PROBLEM FOR A STRETCHED STRING*

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1. Introduction. There have been many studies in the past thirty-five years of the motion of a stretched elastic string. In 1945 Carrier [1] developed an approximate equation for transverse motion for such a string which accounted, at least to second order, for the fact that such a motion was inherently nonlinear. These equations were reformulated by Narasimha [5] in 1967. In this paper, Narasimha introduced a term which accounted for external damping of the string.

The general theory of the Carrier-Narasimha equations has been partially worked out by Dickey and Nisida. Dickey [2, 3] and Nisida [6, 7] have investigated the initial-boundary value problem for these equations for a string of finite length and Dickey [4] has obtained partial results for the pure initial-value problem for these equations for a string of infinite length.

In [4], Dickey considered the initial value problem for the equation

$$U_{tt} = C^2(t)U_{xx}, \qquad -\infty < x < \infty \text{ and } t > 0 \tag{1.1}$$

where the speed of sound $C(\cdot)$ is related to the vertical displacement $U(\cdot, \cdot)$ by

$$C^2(t) = 1 + \varepsilon \int_{-\infty}^{\infty} U_\xi^2(\xi, t)\, d\xi, \qquad 0 < \varepsilon. \tag{1.2}$$

Through the use of certain differential inequalities Dickey was able to show that the initial-value problem for the system (1.1) and (1.2) was uniquely solvable on some interval $0 \le t < T(\varepsilon)$ where $T(\varepsilon) = 0(1/\varepsilon)$. He was also able to show, via direct substitution, that this system was capable of supporting traveling waves; that is, solutions of the form $U = f(x \pm C^*t)$ where C^* is a constant depending on f.

In this paper, we restrict our attention to the equations considered by Dickey in [4]. Our basic result is that if the parameter ε in Eq. (1.2) is sufficiently small and if the initial data for problem is sufficiently well-behaved, then the initial-value problem for (1.1) and (1.2) is uniquely solvable for all time t. Moreover, we establish the existence of functions $R_\pm(\cdot)$ and $L_\pm(\cdot)$ and a unique positive number C_* such that

$$\lim_{t\to\pm\infty} U(x + C_*t, t) = R_\pm(x), \qquad \lim_{t\to\pm\infty} U(x - C_*t, t) = L_\pm(x). \tag{1.3, 1.4}$$

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Here,

$$C_*^2 = \lim_{t\to\pm\infty}\left(1+\varepsilon\int_{-\infty}^{\infty} U_\xi^2(\xi, t)\, d\xi\right), \tag{1.5}$$

Our attack is similar to Dickey's. In Sec. 2, we Fourier-transform the original equations of motion and obtain integro-differential equations for the Fourier transform of the motion U. The form of these equations suggests a study of a certain linear problem. This is done in Sec. 3. The thrust of this section is to show that if a certain potential $q(\cdot)$, which is related to the sound speed $C(\cdot)$, has certain decay properties in $|t|$, then the Fourier transform U must itself have similar properties. Equipped with these results we are able to show (see Sec. 4) that it is possible to construct, through the identity (1.2), a well-defined mapping of functions $C(\cdot)$ into functions $C(\cdot)$; we also show that for $0 < \varepsilon \ll 1$ this map is a contraction. This fact yields the results claimed above.

2. Transformations. In this section we transform our original integro-differential equation into ordinary differential equations for the Fourier coefficients of U. Our original equation is

$$U_{tt} - C^2(t)U_{xx} = 0 \tag{2.1}$$

where U satisfies the initial conditions

$$U(x, 0) = f(x), \qquad U_t(x, 0) = g(x) \tag{IC}$$

and C^2 is given by $C^2(t) = 1 + \varepsilon \int_{-\infty}^{\infty} U_x^2(x, t)\, dx$.

We assume that f' and g are twice continuously differentiable and that $(1 + x^2)|f'(x)| < \infty$ and $(1 + x^2)|g(x)| < \infty$. If we let $C_0^2 = 1 + \varepsilon \int_{-\infty}^{\infty} U_x^2(x, 0)\, dx = 1 + \varepsilon \int_{-\infty}^{\infty} (f'(x))^2\, dx$ and $g(x) = C_0 g_1(x)$, the initial condition may be rewritten as

$$U(x, 0) = f(x), \qquad U_t(x, 0) = C_0 g_1(x). \tag{IC$'$}$$

We shall use this fact later.

Our first task is to write Eq. (2.1) as a system. We let $A(x, t) = U_t(x, t) - C(t)U_x(x, t)$ and $B(x, t) = U_t(x, t) + C(t)U_x(x, t)$. The fact that Eq. (2.1) may be rewritten as

$$\left(\frac{\partial}{\partial t} + C(t)\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - C(t)\frac{\partial}{\partial x}\right)U = -\dot{C}U_x,$$

$$\left(\frac{\partial}{\partial t} - C(t)\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + C(t)\frac{\partial}{\partial x}\right)U = \dot{C}U_x \tag{2.2}$$

yields the following equations for A and B:

$$A_t + C(t)A_x = \frac{-\dot{C}(t)}{2C(t)}(B - A), \qquad B_t - C(t)B_x = \frac{\dot{C}(t)}{2C(t)}(B - A). \tag{2.3}$$

The initial conditions for A and B are computable in terms of f' and g. They are

$$A(x, 0) = A_0(x) \overset{\text{def}}{=} C_0(g_1(x) - f'(x)),$$

$$B(x, 0) = B_0(x) \overset{\text{def}}{=} C_0(g_1(x) + f'(x)). \tag{IC$_1$}$$

The defining relation for C^2 becomes

$$C^2(t) = 1 + \frac{\varepsilon}{4C^2(t)} \int_{-\infty}^{\infty} (B - A)^2(x, t)\, dx. \tag{2.3a}$$

We now introduce the change of variable $\tau = \int_0^t C(s)\, ds$. Clearly, $d\tau/dt = C(t) > 1$ and thus τ is a strictly increasing function of t. We denote its inverse function by $t = T(\tau)$ and regard C as a function of τ, i.e. $C(\tau) = C(T(\tau))$. Then, $dC/dt = (dC/d\tau) \cdot (d\tau/dt) = C(\tau) \times (dC/d\tau)$.

If we apply the change of variable to Eq. (2.3), we obtain

$$A_\tau + A_x = -\frac{C_\tau}{2C}(B - A), \qquad B_\tau - B_x = \frac{C_\tau}{2C}(B - A). \tag{2.4}$$

We are now regarding A and B as function of x and τ. The initial condition and the defining relationship for C are the same as described in $(\mathrm{IC})_1$ and (2.3a).

We now Fourier-transform the system (2.4). We let

$$\mathscr{A}_*(\omega, \tau) = \int_{-\infty}^{\infty} e^{-i\omega x} A(x, \tau)\, dx, \qquad \mathscr{B}_*(\omega, \tau) = \int_{-\infty}^{\infty} e^{-i\omega x} B(x, \tau)\, dx.$$

The coefficients $\mathscr{A}_*$ and $\mathscr{B}_*$ satisfy

$$\mathscr{A}_{*\tau} + i\omega \mathscr{A}_* = -\frac{C_\tau}{2C}(\mathscr{B}_* - \mathscr{A}_*), \qquad \mathscr{B}_{*\tau} - i\omega \mathscr{B}_* = \frac{C_\tau}{2C}(\mathscr{B}_* - \mathscr{A}_*) \tag{2.5}$$

and C is connected to $\mathscr{A}_*$ and $\mathscr{B}_*$ by

$$C^2(\tau) = 1 + \frac{\varepsilon}{8\pi C^2(\tau)} \int_{-\infty}^{\infty} (\mathscr{B}_*(\omega, \tau) - \mathscr{A}_*(\omega, \tau)) \cdot (\mathscr{B}_*(-\omega, \tau) - \mathscr{A}_*(-\omega, \tau))\, d\omega.^1 \tag{2.3b}$$

The initial condition for $\mathscr{A}_*$ and $\mathscr{B}_*$ are

$$\mathscr{A}_*(\omega, 0) = C_0(\hat{g}_1 - \hat{f}_1)(\omega) \overset{\text{def}}{=} C_0^{\frac{1}{2}} \tilde{\mathscr{A}}(\omega), \qquad \mathscr{B}_*(\omega, 0) = C_0(\hat{g}_1 - \hat{f}_1)(\omega) \overset{\text{def}}{=} C_0^{\frac{1}{2}} \tilde{\mathscr{B}}(\omega), \tag{(IC)$_2$}$$

where $\hat{f}_1(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f'(x)\, dx$.

Finally, if we let

$$\mathscr{A}(\omega, \tau) = \frac{\mathscr{A}_*(\omega, \tau)}{C^{\frac{1}{2}}(\tau)} e^{i\omega\tau}, \qquad \mathscr{B}(\omega, \tau) = \frac{\mathscr{B}_*(\omega, \tau)}{C^{\frac{1}{2}}(\tau)} e^{-i\omega\tau},$$

then Eq. (2.5) transforms to

$$\mathscr{A}_\tau(\omega, \tau) = -\frac{C_\tau(\tau)}{2C(\tau)} e^{2i\omega\tau} \mathscr{B}(\omega, \tau), \qquad \mathscr{B}_\tau(\omega, \tau) = -\frac{C_\tau(\tau)}{2C(\tau)} e^{-2i\omega\tau} \mathscr{A}(\omega, \tau) \tag{2.6}$$

and $\mathscr{A}$ and $\mathscr{B}$ satisfy the initial condition

$$\mathscr{A}(\omega, 0) = \tilde{\mathscr{A}}(\omega), \qquad \mathscr{B}(\omega, 0) = \tilde{\mathscr{B}}(\omega). \tag{(IC)$_3$}$$

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[1] Throughout, we exploit the identities $\mathscr{A}_*(-\omega, \tau) = \overline{\mathscr{A}_*(\omega, \tau)}$ and $\mathscr{B}_*(-\omega, \tau) = \overline{\mathscr{B}_*(\omega, \tau)}$.

These are equivalent to the integral equations

$$\mathscr{A}(\omega, \tau) = \tilde{\mathscr{A}}(\omega) - \int_0^\tau e^{2i\omega s} \frac{C_s(s)}{2C(s)} \mathscr{B}(\omega, s)\, ds,$$

$$\mathscr{B}(\omega, \tau) = \tilde{\mathscr{B}}(\omega) - \int_0^\tau e^{-2i\omega s} \frac{C_s(s)}{2C(s)} \mathscr{A}(\omega, s)\, ds, \tag{2.7}$$

and C is connected to $\mathscr{A}$ and $\mathscr{B}$ by

$$C^2(\tau) = 1 + \frac{\varepsilon}{8\pi C(\tau)} \int_{-\infty}^{\infty} (\mathscr{B}(\omega, \tau) e^{i\omega\tau} - \mathscr{A}(\omega, \tau) e^{-i\omega\tau}) \times (\mathscr{B}(-\omega, \tau) e^{-i\omega\tau} - \mathscr{A}(-\omega, \tau) e^{i\omega\tau})\, d\omega. \tag{2.8}$$

Now our goal is to show that for $0 < \varepsilon \ll 1$ there exists a unique solution to (2.7) and (2.8). We shall also show there exists a function $(\mathscr{A}_{\pm\infty}, \mathscr{B}_{\pm\infty})(\omega)$ and a unique positive number $C_* > 0$ such that

$$\lim_{\tau \to \pm\infty} (\mathscr{A}(\omega, \tau), \mathscr{B}(\omega, \tau)) = (\mathscr{A}_{\pm\infty}, \mathscr{B}_{\pm\infty})(\omega)$$

and

$$\lim_{\tau \to \pm\infty} C(\tau) = C_* > 0.$$

These latter limit relations imply that the functions

$$A(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \mathscr{A}_*(\omega, \tau)\, d\omega, \qquad B(x, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \mathscr{B}_*(\omega, \tau)\, d\omega$$

satisfy

$$\lim_{\tau \to \pm\infty} A(x + \tau, \tau) = A_{\pm\infty}(x), \qquad \lim_{\tau \to \pm\infty} B(x - \tau, \tau) = B_{\pm\infty}(x)$$

where

$$A_{\pm\infty}(\xi) = \frac{C_*^{\frac{1}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\xi} \mathscr{A}_{\pm\infty}(\omega)\, d\omega, \qquad B_{\pm\infty}(\xi) = \frac{C_*^{\frac{1}{2}}}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\xi} \mathscr{B}_{\pm\infty}(\omega)\, d\omega.$$

That is, the system (2.3) and $(\mathrm{IC})_1$ has a well-defined scattering theory.

That such a result should have been anticipated follows from Dickey's observation [4] that the original equation $U_{tt} - C^2(t) U_{xx} = 0$ with $C^2(t) = 1 + \varepsilon \int_{-\infty}^{\infty} U_x^2(x, t)\, dx$ supports right- and left-facing traveling waves.

3. The linear problem. In this section we shall confine our attention to solutions of the integral equations:

$$\mathscr{A}(\omega, \tau) = \tilde{\mathscr{A}}(\omega) - \int_0^\tau e^{2i\omega s} q(s) \mathscr{B}(\omega, s)\, ds$$

$$\mathscr{B}(\omega, \tau) = \tilde{\mathscr{B}}(\omega) - \int_0^\tau e^{-2i\omega s} q(s) \mathscr{A}(\omega, s)\, ds \tag{3.1}$$

when $q(\cdot)$ is prescribed.

In fact it will suffice to examine carefully solutions of

$$\alpha(\omega, \tau) = 1 - \int_0^\tau e^{2i\omega s} q(s)\beta(\omega, s)\, ds, \qquad \beta(\omega, \tau) = -\int_0^\tau e^{-2i\omega s} q(s)\alpha(\omega, s)\, ds \tag{3.2}$$

because the solution of (3.1) may be expressed in terms of the functions α and β by

$$\begin{pmatrix} \mathscr{A} \\ \mathscr{B} \end{pmatrix}(\omega, \tau) = \begin{pmatrix} \alpha(\omega, \tau), & \beta(-\omega, \tau) \\ \beta(\omega, \tau), & \alpha(-\omega, \tau) \end{pmatrix} \begin{pmatrix} \tilde{\mathscr{A}}(\omega) \\ \tilde{\mathscr{B}}(\omega) \end{pmatrix}. \tag{3.3}$$

This last result is a consequence of the variation of constants formula for the system (3.1) and the fact that $W(\omega, \tau) \overset{\text{def}}{=} \alpha(\omega, \tau)\alpha(-\omega, \tau) - \beta(\omega, \tau)\beta(-\omega, \tau)$ satisfies $dW/d\tau = 0$.

3.1. *Boundedness and continuity lemmas.* In what follows we shall let

$$\|\cdot\| = \sup_{\omega, \tau} |\cdot|, \qquad \|\cdot\|_1 = \int_{-\infty}^{\infty} |\cdot|(s)\, ds \tag{3.4}$$

and shall employ the working hypothesis that *the function* $q(\cdot)$ in (3.1) *satisfies* $\|q\|_1 < 1$. This assumption guarantees that the functions α and β of (3.2) are well defined. When comparing solutions of (3.1) we shall let $(\mathscr{A}_i, \mathscr{B}_i)$, $i = 1$ and 2, denote the unique solutions of (3.1) corresponding to the potential $q_i(\cdot)$, $i = 1$ and 2. The following notation will be used throughout the remainder of this section:

$$M_{n,i}(\tau) = \max\left[\sup_\omega |\omega^n \mathscr{A}_i(\omega, \tau)|, \sup_\omega |\omega^n \mathscr{B}_i(\omega, \tau)|\right], \tag{3.5}$$

$$N_{n,i}(\tau) = \max\left[\int_{-\infty}^{\infty} |\omega^n \mathscr{A}_i(\omega, \tau)|\, d\omega, \int_{-\infty}^{\infty} |\omega^n \mathscr{B}_i(\omega, \tau)|\, d\omega\right], \tag{3.6}$$

$$M_n^*(\tau) = \max_{i=1,2} M_{n,i}, \qquad N_n^*(\tau) = \max_{i=1,2} N_{n,i}(\tau), \tag{3.7}$$

$$\Delta_n(\tau) = \max\left[\sup_\omega |\omega^n(\mathscr{A}_2 - \mathscr{A}_1)(\omega, \tau)|, \sup_\omega |\omega^n(\mathscr{B}_2 - \mathscr{B}_1)(\omega, \tau)|\right], \tag{3.8}$$

$$D_n(\tau) = \max\left[\int_{-\infty}^{\infty} |\omega^n(\mathscr{A}_2 - \mathscr{A}_1)(\omega, \tau)|\, d\omega, \int_{-\infty}^{\infty} |\omega^n(\mathscr{B}_2 - \mathscr{B}_1)(\omega, \tau)|\, d\omega\right]. \tag{3.9}$$

We shall now give some estimates which tie down the dependence of the functions (α_i, β_i) and $(\mathscr{A}_i, \mathscr{B}_i)$ on the potentials $q_i(\cdot)$.

LEMMA 3.1. (a) The functions α_i and β_i satisfy

$$\|\alpha_i\| \le \frac{1}{1 - \|q_i\|_1^2}, \qquad \|\beta_i\| \le \frac{\|q_i\|_1}{1 - \|q_i\|_1^2}; \tag{3.10}$$

(b) the functions $M_{n,i}$ and $N_{n,i}$ satisfy

$$M_{n,i}(\tau) \le \frac{M_{n,i}(0)}{1 - \|q_i\|_1} \qquad N_{n,i}(\tau) \le \frac{N_{n,i}(0)}{1 - \|q_i\|_1}; \tag{3.11}$$

(c) the functions Δ_n and D_n satisfy

$$\Delta_n(\tau) \leq \frac{M_n^*(0)\|q_2 - q_1\|_1}{\left(1 - \max_{m=1,2} \|q_m\|_1\right)^2}, \tag{3.12}$$

$$D_n(\tau) \leq \frac{N_n^*(0)\|q_2 - q_1\|_1}{\left(1 - \max_{m=1,2} \|q_m\|_1\right)^2}. \tag{3.13}$$

Proof. We establish (3.10) first. We discuss only the case where $\tau > 0$ and suppress the indicies i. Eq. (3.2) yields the identity

$$\alpha(\omega, \tau) = 1 + \int_0^\tau e^{2i\omega s} q(s) \int_0^s e^{-2i\omega y} q(y) \alpha(\omega, y)\, dy\, ds$$

and from this we obtain the inequalities:

$$|\alpha(\omega, \tau)| \leq 1 + \int_0^\tau |q(s)| \int_0^s |q(y)|\, |\alpha(\omega, y)|\, dy\, ds \leq 1 + \|q\|_1^2 \sup_{\substack{\omega \\ 0 \leq y \leq \tau}} |\alpha(\omega, y)|,$$

$$\|\alpha\| \leq 1 + \|q\|_1^2 \|\alpha\|, \qquad \|\alpha\| \leq \frac{1}{1 - \|q\|_1^2}.$$

We also have

$$\beta(\omega, \tau) = -\int_0^\tau e^{-2i\omega s} q(s)\, ds + \int_0^\tau e^{-2i\omega s} q(s) \int_0^s e^{2i\omega y} q(y) \beta(\omega, y)\, dy\, ds$$

which in turn yields

$$|\beta(\omega,\tau) \leq \int_0^\tau |q(s)|\, ds + \int_0^\tau |q(s)| \int_0^s |q(y)| \|\beta(\omega, y|\, dy\, ds$$

$$\leq \|q\|_1 + \|q\|_1^2 \sup_{\substack{\omega \\ 0 \leq y \leq \tau}} |\beta(\omega, y)|,$$

$$\|\beta\| \leq \|q\|_1 + \|q\|_1^2 \|\beta\|, \qquad \|\beta\| \leq \frac{\|q\|_1}{1 - \|q\|_1^2}.$$

The identity (3.3) connecting the solution of (3.1) to the solution of (3.2) and the results of (3.10) yield (3.11).

To obtain (3.12) and (3.13) we first note that

$$(\mathscr{A}_2 - \mathscr{A}_1)(\omega, \tau) = -\int_0^\tau (q_2 - q_1)(s) e^{2i\omega s} \mathscr{B}_1(\omega, s)\, ds$$

$$- \int_0^\tau (q_2(s) e^{2i\omega s} (\mathscr{B}_2 - \mathscr{B}_1)(\omega, s)\, ds$$

and

$$(\mathscr{B}_2 - \mathscr{B}_1)(\omega, \tau) = -\int_0^\tau (q_2 - q_1)(s)e^{-2i\omega s}\mathscr{A}_1(\omega, s)\,ds$$
$$-\int_0^\tau q_2(s)e^{-2i\omega s}(\mathscr{A}_2 - \mathscr{A}_1)(\omega, s)\,ds.$$

From the above we obtain

$$\Delta_n(\tau) \le \left(\int_{-|\tau|}^{|\tau|} |q_2 - q_1|(s)\,ds\right) \sup_{|s|\le|\tau|} M_n^*(s) + \max_{m=1,2}\left(\int_{-|\tau|}^{|\tau|} |q_m|(s)\,ds\right) \sup_{|s|\le|\tau|} D_n(s),$$
$$D_n(\tau) \le \left(\int_{-|\tau|}^{|\tau|} |q_2 - q_1|(s)\,ds\right) \sup_{|s|\le|\tau|} N_n^*(s) + \max_{m=1,2}\left(\int_{-|\tau|}^{|\tau|} |q_m|(s)\,ds\right) \sup_{|s|\le|\tau|} \Delta_n(s),$$

and these, when combined with $\max_{m=1,2}\|q_m\|_1 < 1$, yield

$$\Delta_n(\tau) \le \frac{\sup_{\tau\in(-\infty,\infty)} M_n^*(\tau)}{1 - \max_{m=1,2}\|q_m\|_1}\|q_2 - q_1\|_1, \qquad D_n(\tau) \le \frac{\sup_{\tau\in(-\infty,\infty)} N_n^*(\tau)}{1 - \max_{m=1,2}\|q_m\|_1}\|q_2 - q_1\|_1.$$

The desired result then follows from the above estimates and the inequalities

$$M_n^*(\tau) \le \frac{M_n^*(0)}{1 - \max_{m=1,2}\|q_m\|_1}, \qquad N_n^*(\tau) \le \frac{N_n^*(0)}{1 - \max_{m=1,2}\|q_m\|_1},$$

the latter being a direct consequence of (3.11).

3.2. *Decay estimates for a single potential* $q(\cdot)$. Our task now is to obtain decay estimates for the solution of (3.1) for a given potential $q(\cdot)$. Recalling that $\mathscr{A}$ and $\mathscr{B}$ satisfy:

$$\mathscr{A}_\tau(\omega, \tau) = -q(\tau)e^{2i\omega\tau}\mathscr{B}(\omega, \tau), \qquad \mathscr{B}_\tau(\omega, \tau) = -q(\tau)e^{-2i\omega\tau}\mathscr{A}(\omega, \tau),$$

we see that

$$\frac{\partial}{\partial\tau}(\mathscr{A}\bar{\mathscr{A}} + \mathscr{B}\bar{\mathscr{B}})(\omega, \tau) = -2q(\tau)(e^{-2i\omega\tau}(\mathscr{A}\bar{\mathscr{B}})(\omega, \tau) + e^{2i\omega\tau}(\bar{\mathscr{A}}\mathscr{B})(\omega, \tau)),$$
$$\frac{\partial}{\partial\tau}(\mathscr{A}\bar{\mathscr{B}})(\omega, \tau) = -q(\tau)e^{2i\omega\tau}(\mathscr{A}\bar{\mathscr{A}} + \mathscr{B}\bar{\mathscr{B}})(\omega, \tau),$$

or equivalently that

$$(\mathscr{A}\bar{\mathscr{A}} + \mathscr{B}\bar{\mathscr{B}})(\omega, \tau)$$
$$= (\tilde{\mathscr{A}}\bar{\tilde{\mathscr{A}}} + \tilde{\mathscr{B}}\bar{\tilde{\mathscr{B}}})(\omega) - 2\int_0^\tau q(s)(e^{-2i\omega s}(\mathscr{A}\bar{\mathscr{B}})(\omega, s) + e^{2i\omega s}(\bar{\mathscr{A}}\mathscr{B})(\omega, s))\,ds \quad (3.14)$$

$$(\mathscr{A}\bar{\mathscr{B}})(\omega, \tau) = (\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega) - \int_0^\tau q(s)e^{2i\omega s}(\mathscr{A}\bar{\mathscr{A}} + \mathscr{B}\bar{\mathscr{B}})(\omega, s)\,ds. \quad (3.15)$$

Our first task is to obtain a decay estimate for

$$J(\tau) \overset{\text{def}}{=} \int_{-\infty}^{\infty} \omega e^{-2i\omega\tau}(\mathscr{A}\bar{\mathscr{B}})(\omega, \tau)\, d\omega. \tag{3.16}$$

Eq. (3.15) implies that $J(\tau) = J^1(\tau) - J^2(\tau)$, where

$$J^1(\tau) = \int_{-\infty}^{\infty} \omega e^{-2i\omega\tau}(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\, d\omega \tag{3.17}$$

and

$$J^2(\tau) = \int_0^{\tau} q(s) \int_{-\infty}^{\infty} \omega e^{-2i\omega(\tau - s)}(\mathscr{A}\bar{\mathscr{A}} + \mathscr{B}\bar{\mathscr{B}})(\omega, s)\, d\omega\, ds. \tag{3.18}$$

Our first result is an estimate for $J^1(\tau)$. This estimate depends exclusively on the data $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{B}}$ and is independent of the potential $q(\cdot)$. In what follows we adopt the notation

$$\|f\|^k \overset{\text{def}}{=} \sup_{\tau} (1 + \tau^{2k})|f(\tau)|. \tag{3.19}$$

Lemma 3.2. If

$$\lim_{|\omega| \to \infty} \frac{d^r}{d\omega^r}(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega) = 0 \text{ for } 0 \le r \le 2k, \tag{3.20}$$

then

$$(1 + \tau^{2k})J^1(\tau) = \int_{-\infty}^{\infty} e^{-2i\omega\tau}\left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}}\right)(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\, d\omega, \tag{3.21}$$

and

$$\|J^1\|^k \le \int_{-\infty}^{\infty} \left|\left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}}\right)(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\right| d\omega. \tag{3.22}$$

Moreover, these latter terms are finite if the initial data $\tilde{\mathscr{A}}(\omega)$ and $\tilde{\mathscr{B}}(\omega)$ are sufficiently well-behaved.

Proof. Repeated integration by parts and repeated application of the limit relations (3.20) yield the identity

$$\tau^{2k} J^1(\tau) = \int_{-\infty}^{\infty} e^{-2i\omega\tau}\left(\frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}}(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\right) d\omega,$$

and from this we obtain

$$(1 + \tau^{2k})J^1(\tau) = \int_{-\infty}^{\infty} e^{-2i\omega\tau}\left(1 + \frac{1}{(2i)^{2k}} \cdot \frac{d^{2k}}{d\omega^{2k}}\right)(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\, d\omega.$$

The inequality (3.22) is an immediate consequence of (3.21).

We now seek a similar estimate for $J^2(\tau)$ (see Eq. (3.18)). We first prove the following preliminary lemma.

LEMMA 3.3. Let $k \geq 1$ and $H(\cdot, \cdot)$ and $q(\cdot)$ be two functions satisfying

$$(1 + \tau^{2k})|H(\tau, s)| \leq \bar{H}_k < \infty \text{ for all } \tau \text{ and } s, \tag{3.23}$$

$$\|q\|^k < \infty. \tag{3.24}$$

Then the following inequality holds:

$$(1 + \tau^{2k})\left|\int_0^\tau q(s)H(\tau - s, s)\, ds\right| \leq 2^{2k+1}\|q\|^k \bar{H}_k \int_0^\infty \frac{ds}{1 + s^{2k}}. \tag{3.25}$$

Proof. We shall consider only the case where $\tau > 0$. The case $\tau < 0$ is similar. For $\tau > 0$

$$\begin{aligned}\left|\int_0^\tau q(s)H(\tau - s, s)\, ds\right| &\leq \int_0^\tau |q(s)|\,|H(\tau - s, s)|ds \\ &= \int_0^\tau \frac{(1 + s^{2k})|q(s)|}{1 + s^{2k}} \frac{(1 + (\tau - s)^{2k})|H(\tau - s, s)|}{(1 + (\tau - s)^{2k})}\, ds \\ &\leq \|q\|^k \bar{H}_k \int_0^\tau \frac{ds}{(1 + s^{2k})(1 + (\tau - s)^{2k})} = 2\|q\|^k \bar{H}_k \int_0^{\tau/2} \frac{ds}{(1 + s^{2k})(1 + (\tau - s)^{2k}} \\ &\leq \frac{2\|q\|^k \bar{H}_k}{1 + \left(\frac{\tau}{2}\right)^{2k}} \int_0^{\tau/2} \frac{ds}{1 + s^{2k}} \leq \frac{2^{2k+1}\|q\|^k \bar{H}_k}{1 + \tau^{2k}} \int_0^\infty \frac{ds}{1 + s^{2k}}\end{aligned}$$

and this is the desired result.

By virtue of the preceding lemma it suffices to show that (3.23) holds for

$$H(\tau, s) \overset{\text{def}}{=} \int_{-\infty}^\infty e^{-2i\omega\tau}\omega(\mathscr{A}\bar{\mathscr{A}} + \mathscr{B}\bar{\mathscr{B}})(\omega, s)\, d\omega. \tag{3.26}$$

It is easily checked that H is odd in its first argument. Thus it suffices to establish (3.23) when $\tau > 0$. We shall limit our discussion to the situation $\tau > 0$ and $s > 0$. The case $\tau > 0$ and $s < 0$ is similar. To analyze H we shall make use of the following identity:

$$\begin{aligned}H(\tau, s) = &\int_{-\infty}^\infty e^{-2i\omega\tau}\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{A}}} + \tilde{\mathscr{B}}\bar{\tilde{\mathscr{B}}})(\omega)\, d\omega \\ &-2\int_0^s q(\eta)\int_{-\infty}^\infty (e^{-2i\omega(\tau+\eta)}(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega) + e^{-2i\omega(\tau-\eta)}(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\, d\omega\, d\eta \\ &+2\int_0^s q(\eta)\int_0^\eta q(\zeta)(H(\tau + \eta - \zeta, \zeta) + H(\tau - \eta + \zeta, \zeta))\, d\zeta\, d\eta.\end{aligned} \tag{3.27}$$

This is a direct consequence of (3.14) and (3.15).

LEMMA 3.4. Let $\tau \geq 0$, $s \geq 0$, $0 \leq \eta \leq s$, and $k \geq 1$, and suppose that

$$\lim_{|\omega|\to\infty} \frac{d^r}{d\omega^r}(\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{A}}} + \tilde{\mathscr{B}}\bar{\tilde{\mathscr{B}}})(\omega) = 0 \tag{3.28}$$

and

$$\lim_{|\omega|\to\infty} \frac{d^r}{d\omega^r}(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega) = 0 \text{ for } 0 \le r \le 2k. \tag{3.29}$$

Then the following inequalities hold:

$$\left|(1+\tau^{2k})\int_{-\infty}^{\infty} e^{-2i\omega\tau}\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{A}}} + \tilde{\mathscr{B}}\bar{\tilde{\mathscr{B}}})(\omega)\, d\omega\right|$$
$$\le \int_{-\infty}^{\infty}\left|\left(1+\frac{1}{(2i)^{2k}}\cdot\frac{d^{2k}}{d\omega^{2k}}\right)(\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{A}}} + \tilde{\mathscr{B}}\bar{\tilde{\mathscr{B}}})(\omega))\right| d\omega, \tag{3.30}$$

$$\left|(1+\tau^{2k})\int_0^s q(\eta)\int_{-\infty}^{\infty} e^{-2i\omega(\tau+\eta)}\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}}(\omega)\, d\omega\, d\eta\right|$$
$$\le \|q\|^k\int_0^{\infty}\frac{d\eta}{1+\eta^{2k}}\int_{-\infty}^{\infty}\left|\left(1+\frac{1}{(2i)^{2k}}\frac{d^{2k}}{d\omega^{2k}}\right)(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}}(\omega)\right| d\omega, \tag{3.31}$$

$$\left|(1+\tau^{2k})\int_0^s q(\eta)\int_{-\infty}^{\infty} e^{-2i\omega(\tau-\eta)}(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\, d\omega\, d\eta\right|$$
$$\le \|q\|^k(1+2^{2k+1})\int_0^{\infty}\frac{d\eta}{1+\eta^{2k}}\int_{-\infty}^{\infty}\left|\left(1+\frac{1}{(2i)^{2k}}\frac{d^{2k}}{d\omega^{2k}}\right)(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\right| d\omega. \tag{3.32}$$

Proof. First we prove (3.30). The proof mimics the proof of Lemma 3.2. Repeated integration by parts and application of the limit relation (3.28) yields

$$\tau^{2k}\int_{-\infty}^{\infty} e^{-2i\omega\tau}\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{A}}} + \tilde{\mathscr{B}}\bar{\tilde{\mathscr{B}}})(\omega)\, d\omega = \int_{-\infty}^{\infty} e^{-2i\omega\tau}\left(\frac{1}{(2i)^{2k}}\cdot\frac{d^{2k}}{d\omega^{2k}}\right)(\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{A}}} + \tilde{\mathscr{B}}\bar{\tilde{\mathscr{B}}})(\omega))\, d\omega,$$

and thus we have

$$\left|(1+\tau^{2k})\int_{-\infty}^{\infty} e^{-2i\omega\tau}\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{A}}} + \tilde{\mathscr{B}}\bar{\tilde{\mathscr{B}}})(\omega)\, d\omega\right|$$
$$= \left|\int_{-\infty}^{\infty} e^{-2i\omega\tau}\left(1+\frac{1}{(2i)^{2k}}\cdot\frac{d^{2k}}{d\omega^{2k}}\right)(\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{A}}} + \tilde{\mathscr{B}}\bar{\tilde{\mathscr{B}}})(\omega))\, d\omega\right|$$
$$\le \int_{-\infty}^{\infty}\left|\left(1+\frac{1}{(2i)^{2k}}\cdot\frac{d^{2k}}{d\omega^{2k}}\right)(\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{A}}} + \tilde{\mathscr{B}}\bar{\tilde{\mathscr{B}}})(\omega))\right| d\omega,$$

as claimed.

If we apply the arguments used in Lemma 3.2 to the integral appearing in (3.31), we obtain

$$(1+\tau^{2k})\int_0^s q(\eta)\int_{-\infty}^{\infty} e^{-2i\omega(\tau+\eta)}(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\, d\omega\, d\eta$$
$$= (1+\tau^{2k})\int_0^s \frac{q(\eta)(1+(\tau+\eta)^{2k})}{1+(\tau+\eta)^{2k}}\int_{-\infty}^{\infty} e^{-2i\omega(\tau+\eta)}(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\, d\omega\, d\eta$$
$$= (1+\tau^{2k})\int_0^s \frac{q(\eta)}{1+(\tau+\eta)^{2k}}\int_{-\infty}^{\infty} e^{-2i\omega(\tau+\eta)}\left(1+\frac{1}{(2i)^{2k}}\cdot\frac{d^{2k}}{d\omega^{2k}}\right)(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\, d\omega\, d\eta,$$

and this implies

$$\left|(1+\tau^{2k})\int_0^s q(\eta)\int_{-\infty}^{\infty} e^{-2i\omega(\tau+\eta)}\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\,d\omega\,d\eta\right|$$
$$\le (1+\tau^{2k})\int_0^s \frac{(1+\eta^{2k})|q(\eta)|}{(1+\eta^{2k})(1+(\tau+\eta)^{2k})}\int_{-\infty}^{\infty}\left|\left(1+\frac{1}{(2i)^{2k}}\cdot\frac{1}{d\omega^{2k}}\right)(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\right|d\omega\,d\eta$$
$$\le \|q\|^k\int_0^{\infty}\frac{d\eta}{1+\eta^{2k}}\int_{-\infty}^{\infty}\left|\left(1+\frac{1}{(2i)^{2k}}\cdot\frac{d^{2k}}{d\omega^{2k}}\right)(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\right|d\omega,$$

as claimed.

Finally, we turn to (3.32). We have

$$(1+\tau^{2k})\int_0^s q(\eta)\int_{-\infty}^{\infty} e^{-2i\omega(\tau-\eta)}(\omega\bar{\tilde{\mathscr{A}}}\tilde{\mathscr{B}})(\omega)\,d\omega\,d\eta$$
$$=(1+\tau^{2k})\int_0^s \frac{q(\eta)(1+(\tau-\eta)^{2k})}{1+(\tau-\eta)^{2k}}\int_{-\infty}^{\infty} e^{-2i\omega(\tau-\eta)}(\omega\bar{\tilde{\mathscr{A}}}\tilde{\mathscr{B}})(\omega)\,d\omega\,d\eta$$
$$=(1+\tau^{2k})\int_0^s \frac{q(\eta)}{1+(\tau-\eta)^{2k}}\int_{-\infty}^{\infty} e^{-2i\omega(\tau-\eta)}\left(1+\frac{1}{(2i)^{2k}}\cdot\frac{d^{2k}}{d\omega^{2k}}\right)(\omega\bar{\tilde{\mathscr{A}}}\tilde{\mathscr{B}})(\omega)\,d\omega\,d\eta,$$

and thus the inequality

$$\left|(1+\tau^{2k})\int_0^s q(\eta)\int_{-\infty}^{\infty} e^{-2i\omega(\tau-\eta)}(\omega\bar{\tilde{\mathscr{A}}}\tilde{\mathscr{B}})(\omega)\,d\omega\,d\eta\right|$$
$$\le (1+\tau^{2k})\int_0^s \frac{(1+\eta^{2k})|q(\eta)|}{(1+\eta^{2k})(1+(\tau-\eta)^{2k})}\int_{-\infty}^{\infty}\left|\left(1+\frac{1}{(2i)^{2k}}\cdot\frac{d^{2k}}{d\omega^{2k}}\right)(\omega\bar{\tilde{\mathscr{A}}}\tilde{\mathscr{B}})(\omega)\right|d\omega\,d\eta$$
$$\le \|q\|^k\int_{-\infty}^{\infty}\left|\left(1+\frac{1}{(2i)^{2k}}\frac{d^{2k}}{d\omega^{2k}}\right)(\omega\bar{\tilde{\mathscr{A}}}\tilde{\mathscr{B}})(\omega)\right|d\omega\int_0^s\frac{(1+\tau^{2k})\,d\eta}{(1+\eta^{2k})(1+(\tau-\eta)^{2k})}$$

The desired result, (3.32), now follows from the inequality above and the fact that

$$\int_0^s \frac{(1+\tau^{2k})\,d\eta}{(1+\eta^{2k})(1+(\tau-\eta)^{2k})}\le(1+2^{2k+1})\int_0^{\infty}\frac{d\eta}{1+\eta^{2k}}. \tag{3.33}$$

Our final task is to show that the function H of (3.26) decays.

LEMMA 3.5. Let $k\ge 1$ and suppose there exist constants $0<\delta_k<\infty$ and $0<Q_k<\infty$ such that

$$\|q\|^k\le\delta_k, \tag{3.34}$$

$$\int_{-\infty}^{\infty}\left|\left(1+\frac{1}{(2i)^{2k}}\cdot\frac{d^{2k}}{d\omega^{2k}}\right)(\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{A}}}+\tilde{\mathscr{B}}\bar{\tilde{\mathscr{B}}})(\omega))\right|d\omega\le 2Q_k, \tag{3.35}$$

$$\int_{-\infty}^{\infty}\left|\left(1+\frac{1}{(2i)^{2k}}\cdot\frac{d^{2k}}{d\omega^{2k}}\right)(\omega\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)\right|d\omega\le Q_k, \tag{3.36}$$

and

$$\int_{-\infty}^{\infty}\left|\left(1+\frac{1}{(2i)^{2k}}\cdot\frac{d^{2k}}{d\omega^{2k}}\right)(\omega\bar{\tilde{\mathscr{A}}}\tilde{\mathscr{B}})(\omega)\right|d\omega\le Q_k. \tag{3.37}$$

Then the function $H(\tau, s)$ defined in (3.26) satisfies the estimate

$$((1+\tau^{2k})|H(\tau, s)|) \leq \frac{Q_k\left(2 + 4\delta_k(1+2^{2k})\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)}{1 - 4(1+2^{2k+1})\delta_k^2\left(\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)^2} \stackrel{\text{def}}{=} \bar{H}_k \tag{3.38}$$

provided

$$\delta_k < \frac{1}{2(1+2^{2k+1})^{\frac{1}{2}}\int_0^\infty \frac{d\eta}{1+\eta^{2k}}}. \tag{3.39}$$

Proof. By recalling that H is odd in its first argument it suffices to confine our attention to the case $\tau > 0$. We shall further limit our discussion to the case $s > 0$. The case $s < 0$ is similar. The results of Lemma 3.4 and the identity (3.27) yield

$$\begin{aligned}(1+\tau^{2k})|H(\tau, s)| &\leq 2\left(1 + 2(1+2^{2k})\delta_k\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)Q_k \\ &+ 2(1+\tau^{2k})\int_0^s |q(\eta)|\int_0^\eta |q(\zeta)|\,|H(\tau+\eta-\zeta, \zeta) + H(\tau-\eta+\zeta, \zeta)|\,d\zeta\, d\eta. \end{aligned} \tag{3.40}$$

We now let

$$h_k(\tau, s) \stackrel{\text{def}}{=} (1+\tau^{2k})|H(\tau, s)|. \tag{3.41}$$

Clearly h_k is nonnegative. The fact that H is odd in τ implies that h_k is even in τ. Moreover, (3.40) implies that h_k satisfies

$$\begin{aligned}h_k(\tau, s) &\leq 2\left(1 + 2(1+2^{2k})\delta_k\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)Q_k \\ &+ 2\delta_k^2(1+\tau^{2k})\int_0^s \frac{1}{1+\eta^{2k}}\int_0^\eta \frac{1}{1+\zeta^{2k}}\left(\frac{h_k(\tau+\eta-\zeta, \zeta)}{1+(\tau+\eta-\zeta)^{2k}} + \frac{h_k(\tau-\eta+\zeta, \zeta)}{1+(\tau-\eta+\zeta)^{2k}}\right) d\zeta\, d\eta.\end{aligned} \tag{3.42}$$

We now let

$$h_k^*(T) \stackrel{\text{def}}{=} \sup_{\substack{0 \leq s,\ 0 \leq \tau \\ s+\tau \leq T}} h_k(\tau, s). \tag{3.43}$$

Our goal is to show that h_k^* obeys the upper bound (3.38). For $0 \leq \tau$, $0 \leq \zeta \leq \eta \leq s$ and $s + \tau \leq T$ the following inequalities are valid:

$$0 \leq \tau \leq \tau + \eta - \zeta \leq \tau + \eta \leq \tau + s \leq T,$$

$$0 \leq |\tau - \eta + \zeta| + \zeta \leq \max[\tau + \zeta, \eta - \tau] \leq \tau + s \leq T.$$

The above relations, together with the fact that $h_k(\cdot, \cdot)$ is even in its first argument, then yield

$$h_k(\tau+\eta-\zeta, \zeta) \leq h_k^*(T), \qquad h_k(\tau-\eta+\zeta, \zeta) \leq h_k^*(T),$$

and this in turn implies that the second integral on the right-hand side of (3.42) is bounded from above by

$$2\delta_k^2 h_k^*(T)\left[\left(\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)^2 + (1+\tau^{2k})\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\int_0^\eta \frac{d\zeta}{(1+\zeta^{2k})(1+(\tau-\eta+\zeta)^{2k})}d\eta\right].$$

Moreover, it is a relatively trivial matter to show that

$$(1+\tau^{2k})\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\int_0^\eta \frac{1}{(1+\zeta^{2k})(1+(\tau-\eta+\zeta)^{2k})}\,d\zeta\,d\eta \le (1+2^{2k+2})\left(\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)^2.$$

Combining the last two facts with the inequality (3.42), we see that

$$h_k(\tau, s) \le 2\left(1 + 2(1+2^k)\delta_k\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)Q_k + 4(1+2^{2k+1})\delta_k^2\left(\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)^2 h_k^*(T) \quad (3.44)$$

for all $\tau \ge 0$ and $s \ge 0$ such that $s + \tau \le T$. The last inequality yields the result that $h_k^*(T)$ obeys the upper bound (3.38) and, since this result is independent of T, we have the result claimed for $H(\cdot, \cdot)$.

From the last lemma, we know that for δ_k sufficiently small, the function H of (3.26) satisfies the hypotheses of Lemma 3.3 with $\bar{H}_k$ given by the right-hand side of (3.38). This in turn implies

LEMMA 3.6. If $\|q\|^k \le \delta_k$ and δ_k is sufficiently small and if the hypotheses of Lemmas 3.2, 3.4, and 3.5 hold, then the function $J^2(\cdot)$ of (3.18) satisfies

$$\|J^2\|^k \le 2^{2k+1}\delta_k \bar{H}_k \int_0^\infty \frac{d\eta}{1+\eta^{2k}}. \quad (3.45)$$

3.3 *Additional continuity and decay estimates.* Now suppose that $(\mathscr{A}_1, \mathscr{B}_1)$ and $(\mathscr{A}_2, \mathscr{B}_2)$ are the unique solutions of (3.1) corresponding to the same data $(\tilde{\mathscr{A}}(\omega), \tilde{\mathscr{B}}(\omega))$. For $m = 1$ and 2 we let

$$J_m(\tau) = \int_{-\infty}^{\infty} \omega e^{-2i\omega\tau}(\mathscr{A}_m \bar{\mathscr{B}}_m)(\omega, \tau)\,d\omega, \quad (3.46)$$

$$H_m(\tau, s) = \int_{-\infty}^{\infty} \omega e^{-2i\omega\tau}(\mathscr{A}_m \bar{\mathscr{A}}_m + \mathscr{B}_m \bar{\mathscr{B}}_m)(\omega, s)\,d\omega, \quad (3.47)$$

$$K(\tau, s) = H_2(\tau, s) - H_1(\tau, s). \quad (3.48)$$

Arguments similar to those previously employed may now be used to obtain

LEMMA 3.7. If the data $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{B}}$ and the potentials q_m, $m = 1$ and 2, satisfy the conditions of Lemmas 3.2, 3.4, and 3.5, then K satisfies

$$(1+\tau^{2k})|K(\tau, s)| \le \bar{K}_k \|q_2 - q_1\|^k \quad \text{for all } \tau \text{ and } s, \quad (3.49)$$

where

$$\bar{K}_k = \frac{4(1+2^{2k})Q_k\int_0^\infty \frac{d\eta}{1+\eta^{2k}} + 8(1+2^{2k+1})\bar{H}_k\delta_k\left(\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)^2}{1 - 4(1+2^{2k+1})\delta_k^2\left(\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)^2} \quad (3.50)$$

and δ_k, Q_k, and $\bar{H}_k$ are the constants defined in Lemma 3.5. Moreover, for $m = 1$ and 2 and all τ,

$$(1+\tau^{2k})\left|\int_0^\tau q_m(s)K(\tau-s,s)\,ds\right| \le 2^{2k+1}\delta_k\bar{K}_k\|q_2-q_1\|^k\int_0^\infty \frac{d\eta}{1+\eta^{2k}}. \tag{3.51}$$

4. The nonlinear problem. In this section we turn to the nonlinear problem derived in Sec. 2:

$$\mathscr{A}_\tau = -\frac{C_\tau}{2C}e^{2i\omega\tau}\mathscr{B}, \qquad \mathscr{B}_\tau = -\frac{C_\tau}{2C}e^{-2i\omega\tau}\mathscr{A} \tag{4.1}$$

and

$$(\mathscr{A},\mathscr{B})(\omega,0) = (\tilde{\mathscr{A}},\tilde{\mathscr{B}})(\omega) = C_0^{\frac{1}{2}}(\hat{g}_1-\hat{f}_1,\hat{g}_1+\hat{f}_1)(\omega), \tag{4.2}$$

where

$$(\hat{g}_1,\hat{f}_1)(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}(g_1,f')(x)\,dx, \tag{4.3}$$

$$C_0^2 = 1+\frac{\varepsilon}{2\pi}\int_{-\infty}^{\infty}\hat{f}_1(\omega)\hat{f}_1(-\omega)\,d\omega, \tag{4.4}$$

$$C^2(\tau) = 1+\frac{\varepsilon}{8\pi C(\tau)}\int_{-\infty}^{\infty}(\mathscr{B}(\omega,\tau)e^{i\omega\tau}-\mathscr{A}(\omega,\tau)e^{-i\omega\tau}$$
$$\times(\mathscr{B}(-\omega,\tau)e^{-i\omega\tau}-\mathscr{A}(-\omega,\tau)e^{i\omega\tau})\,d\omega. \tag{4.5}$$

It is easily checked that if $(\mathscr{A},\mathscr{B},C)$ is a solution of (4.1)-(4.5), then

$$(\bar{\mathscr{A}},\bar{\mathscr{B}})(\omega,\tau) = (\mathscr{A},\mathscr{B})(-\omega,\tau) \quad\text{and}\quad C^2(0)=C_0^2. \tag{4.6}$$

For any integer $k \ge 1$, we introduce the class of functions

$$\mathscr{C}_k = \left\{C(\cdot)\,|\,C(\cdot)\in BC^1(-\infty,\infty)\text{ and }\sup_\tau(1+\tau^{2k})|\dot{C}(\tau)|<\infty\right\}.^2 \tag{4.7}$$

For functions $C(\cdot)$ in $\mathscr{C}_k$, we let

$$|||C|||^k \overset{\text{def}}{=} \max\left[\sup_\tau|C(\tau)|,\ \sup_\tau(1+\tau^{2k})|\dot{C}(\tau)|\right]. \tag{4.8}$$

It is easily checked that $\mathscr{C}_k$ with the norm $|||\cdot|||$ is a Banach space. For functions $C(\cdot)$ in $\mathscr{C}_k$ with $k \ge 1$, we also have the inequality

$$\|\dot{C}\|_1 \overset{\text{def}}{=} \int_{-\infty}^{\infty}|\dot{C}(\tau)|\,d\tau \le 2\int_0^\infty\frac{d\eta}{1+\eta^{2k}}\,|||C|||^k. \tag{4.9}$$

For positive numbers δ we let

$$\mathscr{C}_{k,\delta} = \left\{C(\cdot)\in\mathscr{C}_k\,|\,C(0)=C_0\ge1,\ C(\tau)\ge1,\text{ and }\sup_\tau(1+\tau^{2k})|\dot{C}(\tau)|\le\delta\right\}. \tag{4.10}$$

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[2] We employ the notation $\dot{f}(\tau) = (df/d\tau)(\tau)$.

The number $C_0 \geq 1$ is the positive square root of the right side of Eq. (4.4). It is easily checked that $\mathscr{C}_{k,\delta}$ is a closed convex subset of $\mathscr{C}_k$ under the norm $|||\cdot|||^k$.

For functions $C(\cdot)$ in $\mathscr{C}_{k,\delta}$ we let

$$q_C(\tau) = \dot{C}(\tau)/2C(\tau), \tag{4.11}$$

and we let $(\mathscr{A}_C, \mathscr{B}_C)(\omega, \tau)$ be the unique solution of

$$\mathscr{A}_\tau = -q_C(\tau)e^{2i\omega\tau}\mathscr{B}, \qquad \mathscr{B}_\tau = -q_C(\tau)e^{-2i\omega\tau}\mathscr{A} \tag{4.1}$$

satisfying

$$(\mathscr{A}_C, \mathscr{B}_C)(\omega, 0) = C_0^{\frac{1}{2}}(\hat{g}_1 - \hat{f}_1, \hat{g}_1 + \hat{f}_1)(\omega). \tag{4.2}$$

We define

$$\Gamma_C^2(\tau) = 1 + \frac{\varepsilon}{8\pi C(\tau)}\int_{-\infty}^{\infty}(\mathscr{B}_C(\omega, \tau)e^{i\omega\tau} - \mathscr{A}_C(\omega, \tau)e^{-i\omega\tau})$$
$$\times (\mathscr{B}_C(-\omega, \tau)e^{-i\omega\tau} - \mathscr{A}_C(-\omega, \tau)e^{i\omega\tau})\, d\omega, \tag{4.12}$$

and Γ_C as the positive square root of the right side of (4.12).

The estimates of Lemma 3.1 and the inequality

$$\|q_C\|_1 = \frac{1}{2}\int_{-\infty}^{\infty}\left|\frac{\dot{C}}{C}\right|(\tau)\, d\tau \leq \delta\int_0^{\infty}\frac{d\eta}{1+\eta^{2k}} \tag{4.13}$$

for all $C(\cdot) \in \mathscr{C}_{k,\delta}$ insure that $\mathscr{A}_C$, $\mathscr{B}_C$, and Γ_C are well-defined for all

$$\delta < 1\bigg/\int_0^{\infty}\frac{d\eta}{1+\eta^{2k}},$$

provided the initial data $(\tilde{\mathscr{A}}, \tilde{\mathscr{B}})(\omega)$ is well enough behaved.

We also observe that if for some $k \geq 1$ and δ sufficiently small, the mapping $C \to \Gamma_C$ has a fixed point in $\mathscr{C}_{k,\delta}$, then the problem (4.1)-(4.5) has a globally defined solution for all time τ. Moreover, the estimate (4.13), the integral identities

$$\mathscr{A}(\omega, \tau) = \tilde{\mathscr{A}}(\omega) - \frac{1}{2}\int_0^{\tau}\frac{\dot{C}}{C}(s)e^{2i\omega s}\mathscr{B}(\omega, s)\, ds,$$

$$\mathscr{B}(\omega, \tau) = \tilde{\mathscr{B}}(\omega) - \frac{1}{2}\int_0^{\tau}\frac{\dot{C}}{C}(s)e^{-2i\omega s}\mathscr{A}(\omega, s)\, ds \tag{4.14}$$

and the results of Lemma 3.1 will guarantee the existence of functions $(\mathscr{A}_{\pm\infty}, \mathscr{B}_{\pm\infty})(\omega)$ such that

$$\lim_{\tau\to\pm\infty}(\mathscr{A}, \mathscr{B})(\omega, \tau) = (\mathscr{A}_{\pm\infty}, \mathscr{B}_{\pm\infty})(\omega). \tag{4.15}$$

The fact that $C(\cdot) \in \mathscr{C}_{k,\delta}$ will guarantee the existence of numbers $C_{\pm\infty}$ such that

$$\lim_{\tau\to\pm\infty} C(\tau) = C_{\pm\infty}. \tag{4.16}$$

That these two numbers are equal will follow from (4.5), the conservation of energy identity which states that any solution of (4.1)-(4.5) satisfies

$$C^4(\tau) + \frac{\varepsilon}{4\pi} C(\tau) \int_{-\infty}^{\infty} (\mathscr{B}(\omega, \tau)e^{i\omega\tau} + \mathscr{A}(\omega, \tau)e^{-i\omega\tau}) \cdot (\mathscr{B}(-\omega, \tau)e^{-i\omega\tau} + \mathscr{A}(-\omega, \tau)e^{i\omega\tau})\, d\omega = \text{constant}, \quad (4.17)$$

and the fact that such solutions will satisfy

$$\lim_{\tau \to \pm\infty} \int_{-\infty}^{\infty} \mathscr{A}(-\omega, \tau)\mathscr{B}(\omega, \tau)e^{2i\omega\tau}\, d\omega = 0. \quad (4.18)$$

By virtue of the last remarks, we shall confine our attention to showing that for some $k \geq 1$ and δ sufficiently small the mapping $C \to \Gamma_C$ has a fixed point in $\mathscr{C}_{k,\delta}$. For definiteness, we fix $k \geq 1$ and assume that the initial data

$$(\tilde{\mathscr{A}}, \tilde{\mathscr{B}})(\omega) = C_0^{\frac{1}{2}}(\hat{g}_1 - \hat{f}_1, \hat{g}_1 + \hat{f}_1)(\omega) \quad (4.2)$$

satisfies

$$M_n(0) = \max \left[\sup_{\omega} |\omega^n \tilde{\mathscr{A}}(\omega)|, \sup_{\omega} |\omega^n \tilde{\mathscr{B}}(\omega)| \right] < \infty, \qquad n = 0, 1, \quad (4.19)$$

$$N_n(0) = \max \left[\int_{-\infty}^{\infty} |\omega^n \tilde{\mathscr{A}}(\omega)|\, d\omega, \int_{-\infty}^{\infty} |\omega^n \tilde{\mathscr{B}}(\omega)|\, d\omega \right] < \infty, \qquad n = 0, 1, \quad (4.20)$$

and

$$Q_n(0) = \max \left[\frac{1}{2} \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2n}} \frac{d^{2n}}{d\omega^{2n}}\right) (\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{A}}} + \tilde{\mathscr{B}}\bar{\tilde{\mathscr{B}}})(\omega)) \right| d\omega, \right.$$
$$\int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2n}} \frac{d^{2n}}{d\omega^{2n}}\right) (\omega(\tilde{\mathscr{A}}\bar{\tilde{\mathscr{B}}})(\omega)) \right| d\omega,$$
$$\left. \int_{-\infty}^{\infty} \left| \left(1 + \frac{1}{(2i)^{2n}} \frac{d^{2n}}{d\omega^{2n}}\right) (\omega(\bar{\tilde{\mathscr{A}}}\tilde{\mathscr{B}})(\omega)) \right| d\omega \right] < \infty, \qquad n = 0, 1, \ldots, k. \quad (4.21)$$

Our first task is to show that for fixed $k \geq 1$ and $0 < \delta$ sufficiently small, there is an $\tilde{\varepsilon}(k, \delta)$ such that for all $C(\cdot) \in \mathscr{C}_{k,\delta}$ and $0 < \varepsilon \leq \tilde{\varepsilon}(k, \delta)$ the function $\Gamma_C(\cdot)$ is in $\mathscr{C}_{k,\delta}$. Before starting to prove this assertion, we simplify our notation. We let $\mathscr{A}_C^\omega(\tau) = \mathscr{A}_C(\omega, \tau)$, $\mathscr{B}_C^\omega(\tau) = \mathscr{B}_C(\omega, \tau)$, $\bar{\mathscr{A}}_C^\omega(\tau) = \mathscr{A}_C(-\omega, \tau)$ and $\bar{\mathscr{B}}_C^\omega(\tau) = \bar{\mathscr{B}}_C(\omega, \tau) = \mathscr{B}_C(-\omega, \tau)$.

THEOREM 4.1. For $k \geq 1$ fixed,

$$0 < \delta < \frac{1}{(1 + 2^{2k+1})^{\frac{1}{2}} \int_0^{\infty} \frac{d\eta}{1 + \eta^{2k}}},$$

and

$$0 < \varepsilon \leq \tilde{\varepsilon}(k, \delta) \overset{\text{def}}{=} \frac{4\pi\delta}{Q_k(0) + 2^{2k}\delta \bar{H}_k \int_0^{\infty} \frac{d\eta}{1 + \eta^{2k}}}, \quad (4.22)$$

the mapping $C \to \Gamma_C$ of (4.12) takes $\mathscr{C}_{k,\delta}$ into $\mathscr{C}_{k,\delta}$.

Proof. From the definition of $\Gamma_C(\cdot)$ and Lemma 3.1 it is easy to check that $\Gamma_C(0) = C_0$, $\Gamma_C(\cdot) \in C^1(-\infty, \infty)$, and

$$1 \leq \Gamma_C^2(\tau) \leq 1 + \frac{\varepsilon(M_0 N_0)(0)}{2\pi\left(1 - \delta\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)^2}.$$

We need only show that, for $0 < \varepsilon \leq \tilde{\varepsilon}(k, \delta)$, $\|\dot{\Gamma}_C\|^k \leq \delta$.

The defining relationship (4.12) implies that

$$\begin{aligned}
2\Gamma_C\dot{\Gamma}_C = &-\frac{\varepsilon\dot{C}}{8\pi C^2}\int_{-\infty}^{\infty}(\mathscr{A}_C^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_C^\omega(\tau)e^{i\omega\tau})(\bar{\mathscr{A}}_C^\omega(\tau)e^{i\omega\tau} - \bar{\mathscr{B}}_C^\omega(\tau)e^{-i\omega\tau})\,d\omega\\
&+\frac{\varepsilon}{8\pi C}\int_{-\infty}^{\infty}(\mathscr{A}_{C,\tau}^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_{C,\tau}^\omega(\tau)e^{i\omega\tau})(\bar{\mathscr{A}}_C^\omega(\tau)e^{i\omega\tau} - \bar{\mathscr{B}}_C^\omega(\tau)e^{-i\omega\tau})\,d\omega\\
&+\frac{\varepsilon}{8\pi C}\int_{-\infty}^{\infty}(\mathscr{A}_C^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_C^\omega(\tau)e^{i\omega\tau})(\bar{\mathscr{A}}_{C,\tau}^\omega(\tau)e^{i\omega\tau} - \bar{\mathscr{B}}_{C,\tau}^\omega(\tau)e^{-i\omega\tau})\,d\omega\\
&+\frac{\varepsilon}{8\pi C}\int_{-\infty}^{\infty}(-i\omega)(\mathscr{A}_C^\omega(\tau)e^{-i\omega\tau} + \mathscr{B}_C^\omega(\tau)e^{i\omega\tau})(\bar{\mathscr{A}}_C^\omega(\tau)e^{i\omega\tau} - \bar{\mathscr{B}}_C^\omega(\tau)e^{-i\omega\tau})\,d\omega\\
&+\frac{\varepsilon}{8\pi C}\int_{-\infty}^{\infty}(i\omega)(\mathscr{A}_C^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_C^\omega(\tau)e^{i\omega\tau})(\bar{\mathscr{A}}_C^\omega(\tau)e^{i\omega\tau} + \bar{\mathscr{B}}_C^\omega(\tau)e^{-i\omega\tau})\,d\omega
\end{aligned}$$

and this when combined with Eq. (4.1) yields:

$$\Gamma_C\dot{\Gamma}_C = \frac{\varepsilon}{8\pi C}\int_{-\infty}^{\infty} i\omega(e^{-2i\omega\tau}(\mathscr{A}_C^\omega\bar{\mathscr{B}}_C^\omega)(\tau) - e^{2i\omega\tau}(\bar{\mathscr{A}}_C^\omega\mathscr{B}_C^\omega)(\tau))\,d\omega.$$

Moreover, the fact that

$$-\int_{-\infty}^{\infty} i\omega e^{2i\omega\tau}(\bar{\mathscr{A}}_C^\omega\mathscr{B}_C^\omega)(\tau)\,d\omega = \int_{-\infty}^{\infty} i\omega e^{-2i\omega\tau}(\mathscr{A}_C^\omega\bar{\mathscr{B}}_C^\omega)(\tau)\,d\omega$$

reduces the last equation to

$$\dot{\Gamma}_C = \frac{\varepsilon}{4\pi(C\Gamma_C)(\tau)}\int_{-\infty}^{\infty} i\omega e^{-2i\omega\tau}(\mathscr{A}_C^\omega\bar{\mathscr{B}}_C^\omega)(\tau)\,d\omega \overset{\text{def}}{=} \frac{i\varepsilon}{4\pi\Gamma_C C}J(\tau), \tag{4.23}$$

where $J(\tau)$ is the function defined in (3.16). The fact that Γ_C and C are both greater than or equal to unity implies

$$|\dot{\Gamma}_C(\tau)| \leq \frac{\varepsilon}{4\pi}|J(\tau)| \tag{4.24}$$

and

$$(1+\tau^{2k})|\dot{\Gamma}_C(\tau)| \leq \frac{\varepsilon}{4\pi}(1+\tau^{2k})|J(\tau)|. \tag{4.25}$$

The results of Lemmas 3.2, 3.5 and 3.6 and (4.25) above imply that

$$(1+\tau^{2k})|\dot{\Gamma}_C(\tau)| \le \frac{\varepsilon}{4\pi}(1+\tau^{2k})|J(\tau)| \le \frac{\varepsilon}{4\pi}\left(Q_k(0) + 2^{2k}\delta \bar{H}_k \int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right) \le \delta \tag{4.26}$$

provided $0 < \varepsilon \le \tilde{\varepsilon}(k, \delta)$. In the last inequality

$$\bar{H}_k = \frac{\left(2 + 2(1+2^{2k})\delta \int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right) Q_k(0)}{1 - (1+2^{2k+1})\delta^2 \left(\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)^2} < \infty. \tag{4.27}$$

THEOREM 4.2. Let $k \ge 1$,

$$0 < \delta < \frac{1}{(1+2^{2k+1})^{\frac{1}{2}} \int_0^\infty \frac{d\eta}{1+\eta^{2k}}},$$

and ε satisfy (4.22). Then there exists a constant D which may be chosen independently of ε such that

$$|||\Gamma_{C_2} - \Gamma_{C_1}|||^k \le \varepsilon D\, |||C_2 - C_1|||^k.$$

It is now an immediate consequence of the contraction mapping principle that if the hypotheses of Theorems 4.1 and 4.2 hold then the map $C \to \Gamma_C$ has a unique fixed point in $C_{k,\delta}$ provided ε is small enough. This is the desired result.

Proof of Theorem 4.2. We shall employ the simplified notation:

$$\Gamma_m = \Gamma_{C_m}, \qquad q_m = q_{C_m} = \dot{C}_m/2C_m, \qquad \mathscr{A}_m^\omega(\tau) = \mathscr{A}_{C_m}^\omega(\tau),$$
$$\mathscr{B}_m^\omega(\tau) = \mathscr{B}_{C_m}^\omega(\tau), \qquad m = 1 \text{ and } 2.$$

The functions $\mathscr{A}_m^\omega$ and $\mathscr{B}_m^\omega$ satisfy the identities

$$\mathscr{A}_m^\omega(\tau) = \tilde{\mathscr{A}}(\omega) - \int_0^\tau \mathscr{A}_m(s) e^{2i\omega s} \mathscr{B}_m^\omega(s)\, ds$$

$$\mathscr{B}_m^\omega(\tau) = \tilde{\mathscr{B}}(\omega) - \int_0^\tau q_m(s) e^{-2i\omega s} \mathscr{A}_m^\omega(s)\, ds \qquad \text{for } m = 1, 2.$$

The results of Lemma 3.1 yield

$$\sup_{\omega,\tau} |\omega^n \mathscr{A}_m^\omega(\tau)| \le \frac{M_n(0)}{1 - \|q_m\|_1}, \tag{4.31}$$

$$\sup_{\omega,\tau} |\omega^n \mathscr{B}_m^\omega(\tau)| \le \frac{M_n(0)}{1 - \|q_m\|_1}, \tag{4.32}$$

$$\sup_{\tau} \int_{-\infty}^{\infty} |\omega^n \mathscr{A}_m^\omega(\tau)|\, d\omega \le \frac{N_n(0)}{1 - \|q_m\|_1}, \tag{4.33}$$

$$\sup_\tau \int_{-\infty}^{\infty} |\omega^n \mathscr{B}_m^\omega(\tau)|\, d\omega \le \frac{N_n(0)}{1 - \|q_m\|_1}, \tag{4.34}$$

$$\sup_\tau \Delta_n(\tau) \le \frac{M_n(0)}{\left(1 - \max_{m=1,2} \|q_m\|_1\right)^2} \|q_2 - q_1\|_1, \tag{4.35}$$

$$\sup_\tau D_n(\tau) \le \frac{N_n(0)}{\left(1 - \max_{m=1,2} \|q_m\|_1\right)^2} \|q_2 - q_1\|_1 \tag{4.36}$$

where $\Delta_n(\tau)$ and $D_n(\tau)$ are defined in Eqs. (3.8) and (3.9) and $M_n(0)$ and $N_n(0)$ in Eqs. (4.19) and (4.20). We shall state without proof a crucial relation among the norms $\|\cdot\|_1$, $\|\cdot\|^k$, and $|||\cdot|||^k$.

LEMMA 4.1. For $C_1, C_2 \in \mathscr{C}_{k,\delta}$, we have

$$\|q_2 - q_1\|_1 \le (1 + \delta)\int_0^\infty \frac{d\eta}{1 + \eta^{2k}} |||C_2 - C_1|||^k,$$

$$\|q_2 - q_1\|^k \le \tfrac{1}{2}(1 + \delta)|||C_2 - C_1|||^k.$$

From the definition of $\Gamma_m(\cdot)$, we have

$$\Gamma_m^2(\tau) = 1 + \frac{\varepsilon}{8\pi C_m(\tau)} \int_{-\infty}^{\infty} (\mathscr{A}_m^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_m^\omega(\tau)e^{i\omega\tau})(\bar{\mathscr{A}}_m^\omega(\tau)e^{i\omega\tau} - \bar{\mathscr{B}}_m^\omega(\tau)e^{-i\omega\tau})\, d\omega$$

$$= 1 + \frac{\varepsilon}{8\pi C_m(\tau)} \int_{-\infty}^{\infty} |\mathscr{A}_m^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_m^\omega(\tau)e^{i\omega\tau}|^2 d\omega. \tag{4.37}$$

The above identity and the fact that $\Gamma_m(\tau) \ge 1$ yield

$$\begin{aligned}
|\Gamma_2 - \Gamma_1|(\tau) &= \left|\frac{\Gamma_2^2 - \Gamma_1^2}{\Gamma_2 + \Gamma_1}\right|(\tau) \\
&\le \frac{1}{2} |\Gamma_2^2 - \Gamma_1^2|(\tau) \\
&\le \frac{1}{2}\left| \frac{\varepsilon}{8\pi C_2(\tau)} \int_{-\infty}^{\infty} |\mathscr{A}_2^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_2^\omega(\tau)e^{i\omega\tau}|^2\, d\omega \right. \\
&\quad \left. - \frac{\varepsilon}{8\pi C_1(\tau)} \int_{-\infty}^{\infty} |\mathscr{A}_1^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_1^\omega(\tau)e^{i\omega\tau}|^2\, d\omega \right| \\
&\le \frac{\varepsilon |C_2 - C_1|}{16\pi C_2 C_1}(\tau) \int_{-\infty}^{\infty} |\mathscr{A}_2^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_2^\omega(\tau)e^{i\omega\tau}|^2\, d\omega \\
&\quad + \frac{\varepsilon}{16\pi C_1(\tau)} \left| \int_{-\infty}^{\infty} (|\mathscr{A}_2^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_2^\omega(\tau)e^{i\omega\tau}|^2 \right. \\
&\quad \left. - |\mathscr{A}_1^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_1^\omega(\tau)e^{i\omega\tau}|^2)\, d\omega \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\varepsilon|C_2 - C_1|(\tau)}{16\pi}\int_{-\infty}^{\infty} |\mathscr{A}_2^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_2^\omega(\tau)e^{i\omega\tau}|^2\, d\omega \\
&+ \frac{\varepsilon}{16\pi}\left|\int_{-\infty}^{\infty} ((\mathscr{A}_2^\omega\bar{\mathscr{A}}_2^\omega - \mathscr{A}_1^\omega\bar{\mathscr{A}}_1^\omega)(\tau) + (\mathscr{B}_2^\omega\bar{\mathscr{B}}_2^\omega - \mathscr{B}_1^\omega\bar{\mathscr{B}}_1^\omega)(\tau))\, d\omega\right| \\
&+ \frac{\varepsilon}{16\pi}\left|\int_{-\infty}^{\infty} (e^{-2i\omega\tau}(\mathscr{A}_2^\omega\bar{\mathscr{B}}_2^\omega - \mathscr{A}_1^\omega\bar{\mathscr{B}}_1^\omega)(\tau)\right. \\
&\left. + e^{2i\omega\tau}(\bar{\mathscr{A}}_2^\omega\mathscr{B}_2^\omega - \bar{\mathscr{A}}_1^\omega\mathscr{B}_1^\omega)(\tau))\, d\omega\right|. \qquad (4.38)
\end{aligned}
$$

We shall estimate the terms in (4.38) separately. For the first term we have

$$
\begin{aligned}
&\frac{\varepsilon}{16\pi}|C_2 - C_1|(\tau)\int_{-\infty}^{\infty} |\mathscr{A}_2^\omega(\tau)e^{-i\omega\tau} - \mathscr{B}_2^\omega(\tau)e^{i\omega\tau}|^2\, d\omega \\
&\leq \frac{\varepsilon}{16\pi}|C_2 - C_1|(\tau)\int_{-\infty}^{\infty} (|\mathscr{A}_2^\omega\bar{\mathscr{A}}_2^\omega| + |\mathscr{B}_2^\omega\bar{\mathscr{B}}_2^\omega| + |\mathscr{A}_2^\omega\bar{\mathscr{B}}_2^\omega| + |\bar{\mathscr{A}}_2^\omega\mathscr{B}_2^\omega|)(\tau)\, d\omega \\
&\leq \frac{\varepsilon}{16\pi}|C_2 - C_1|(\tau)\cdot\frac{4(M_0N_0)(0)}{(1 - \|q_2\|_1)^2} \leq \frac{\varepsilon(M_0N_0)(0)}{4\pi(1 - \|q_2\|_1)^2}|C_2 - C_1|(\tau).
\end{aligned}
$$

The second term becomes

$$
\begin{aligned}
&\frac{\varepsilon}{16\pi}\left|\int_{-\infty}^{\infty} ((\mathscr{A}_2^\omega\bar{\mathscr{A}}_2^\omega - \mathscr{A}_1^\omega\bar{\mathscr{A}}_1^\omega)(\tau) + (\mathscr{B}_2^\omega\bar{\mathscr{B}}_2^\omega - \mathscr{B}_1^\omega\bar{\mathscr{B}}_1^\omega)(\tau))\, d\omega\right| \\
&\leq \frac{\varepsilon}{16\pi}\left|\int_{-\infty}^{\infty} (\mathscr{A}_2^\omega(\bar{\mathscr{A}}_2^\omega - \bar{\mathscr{A}}_1^\omega) + \bar{\mathscr{A}}_1^\omega(\mathscr{A}_2^\omega - \mathscr{A}_1^\omega) + \mathscr{B}_2^\omega(\bar{\mathscr{B}}_2^\omega - \bar{\mathscr{B}}_1^\omega)\right. \\
&\left. + \bar{\mathscr{B}}_1^\omega(\mathscr{B}_2^\omega - \mathscr{B}_1^\omega))(\tau)\, d\omega\right| \\
&\leq \frac{\varepsilon}{16\pi}\int_{-\infty}^{\infty} (|\mathscr{A}_2^\omega|\,|\bar{\mathscr{A}}_2^\omega - \bar{\mathscr{A}}_1^\omega| + |\bar{\mathscr{A}}_1^\omega|\,|\mathscr{A}_2^\omega - \mathscr{A}_1^\omega| + |\mathscr{B}_2^\omega|\,|\bar{\mathscr{B}}_2^\omega - \bar{\mathscr{B}}_1^\omega| \\
&+ |\bar{\mathscr{B}}_1^\omega|\,|\mathscr{B}_2^\omega - \mathscr{B}_1^\omega|)(\tau)\, d\omega \\
&\leq \frac{\varepsilon}{16\pi}\Delta_0(\tau)\int_{-\infty}^{\infty} (|\mathscr{A}_2^\omega| + |\bar{\mathscr{A}}_1^\omega| + |\mathscr{B}_2^\omega| + |\bar{\mathscr{B}}_1^\omega|)(\tau)\, d\omega \\
&\leq \frac{\varepsilon}{16\pi}\cdot\frac{M_0(0)\|q_2 - q_1\|_1}{\left(1 - \max_{m=1,2}\|q_m\|_1\right)^2}\cdot\frac{4N_0(0)}{1 - \max_{m=1,2}\|q_m\|_1} \\
&\leq \frac{\varepsilon(M_0N_0)(0)}{4\pi\left(1 - \max_{m=1,2}\|q_m\|_1\right)^3}\|q_2 - q_1\|_1.
\end{aligned}
$$

The last term of (4.38) is similar to the second term; the result is

$$\frac{\varepsilon}{16\pi}\left|\int_{-\infty}^{\infty}(e^{-2i\omega\tau}(\mathscr{A}_2^\omega\bar{\mathscr{B}}_2^\omega - \mathscr{A}_1^\omega\bar{\mathscr{B}}_1^\omega)(\tau) - e^{2i\omega\tau}(\bar{\mathscr{A}}_2^\omega\mathscr{B}_2^\omega - \bar{\mathscr{A}}_1^\omega\mathscr{B}_1^\omega)(\tau))\,d\omega\right|$$

$$\leq \frac{\varepsilon(M_0N_0)(0)}{4\pi\left(1 - \max_{m=1,2}\|q_m\|_1\right)^3}\|q_2 - q_1\|_1.$$

The last three inequalities imply that

$$|\Gamma_2 - \Gamma_1|(\tau) \leq \frac{\varepsilon(M_0N_0)(0)}{4\pi(1 - \|q_2\|_1)^2}|C_2 - C_1|(\tau) + \frac{\varepsilon(M_0N_0)(0)}{2\pi\left(1 - \max_{m=1,2}\|q_m\|_1\right)^3}\|q_2 - q_1\|_1,$$

and since $\|q_m\|_1 \leq \delta\int_0^\infty (d\eta/(1 + \eta^{2k}))$, for $m = 1, 2$, we can majorize the last inequality by

$$|\Gamma_2 - \Gamma_1|(\tau) \leq \frac{\varepsilon(M_0N_0)(0)}{4\pi\left(1 - \delta\int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)^2}|C_2 - C_1|(\tau)$$

$$+ \frac{\varepsilon(M_0N_0)(0)}{2\pi\left(1 - \delta\int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)^3}\|q_2 - q_1\|_1.$$

From the definition of $|||\cdot|||^k$ and Lemma 4.1 we have $|C_2 - C_1|(\tau) \leq |||C_2 - C_1|||^k$ and

$$\|q_2 - q_1\|_1 \leq (1 + \delta)\int_0^\infty \frac{d\eta}{1 + \eta^{2k}}|||C_2 - C_1|||^k,$$

and these combine to give

$$|\Gamma_2 - \Gamma_1|(\tau) \leq \frac{\varepsilon(M_0N_0)(0)}{4\pi\left(1 - \delta\int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)^2}|||C_2 - C_1|||^k$$

$$+ \frac{\varepsilon(1 + \delta)\left(\int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)(M_0N_0)(0)}{2\pi\left(1 - \delta\int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)^3}|||C_2 - C_1|||^k$$

$$\leq \frac{\varepsilon\left(1 + (2 + \delta)\int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)(M_0N_0)(0)}{4\pi\left(1 - \delta\int_0^\infty \frac{d\eta}{1 + \eta^{2k}}\right)^3}|||C_2 - C_1|||^k. \tag{4.39}$$

We now turn to the derivatives. We have

$$|\dot{\Gamma}_2 - \dot{\Gamma}_1| = \left|\frac{\Gamma_2\dot{\Gamma}_2}{\Gamma_2} - \frac{\Gamma_2\dot{\Gamma}_1}{\Gamma_2}\right| \leq \frac{|\Gamma_2\dot{\Gamma}_2 - \Gamma_1\dot{\Gamma}_1|}{\Gamma_2} + \frac{|\dot{\Gamma}_1||\Gamma_2 - \Gamma_1|}{\Gamma_2}$$

and therefore if we exploit $1 \le \Gamma_2$ and $\sup_\tau(1+\tau^{2k})|\dot{\Gamma}_1| \le \delta$ we have

$$\sup_\tau(1+\tau^{2k})|\dot{\Gamma}_2 - \dot{\Gamma}_1|(\tau) \le \sup_\tau(1+\tau^{2k})|\Gamma_2\dot{\Gamma}_2 - \Gamma_1\dot{\Gamma}_1|(\tau) + \delta \sup_\tau|\Gamma_2 - \Gamma_1|(\tau). \tag{4.40}$$

The inequality (4.39) yields

$$\sup_\tau(1+\tau^{2k})|\dot{\Gamma}_2 - \dot{\Gamma}_1(\tau)| \le \sup_\tau(1+\tau^{2k})|\Gamma_2\dot{\Gamma}_2 - \Gamma_1\dot{\Gamma}_1|(\tau) + \varepsilon\delta D_1 \, |||C_2 - C_1|||^k, \tag{4.41}$$

where

$$D_1 = \frac{\left(1 + (2+\delta)\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)(M_0 N_0)(0)}{4\pi\left(1 - \delta\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\right)^3},$$

and therefore it suffices to estimate the first term on the right side of (4.41).

Recalling the relation

$$\Gamma_m\dot{\Gamma}_m = \frac{\varepsilon i}{4\pi C_m}\int_{-\infty}^{\infty} \omega(\mathscr{A}_m^\omega \mathscr{B}_m^\omega)(\tau)e^{-2i\omega\tau}\,d\omega$$

and the identities (3.46) and (3.47), we have

$$\begin{aligned} |\Gamma_2\dot{\Gamma}_2 - \Gamma_1\dot{\Gamma}_1|(\tau) &= \frac{\varepsilon}{4\pi}\left|\frac{J_2}{C_2} - \frac{J_1}{C_1}\right|(\tau) \\ &\le \frac{\varepsilon}{4\pi}\left|\frac{J_2 - J_1}{C_2} - \frac{J_1(C_2 - C_1)}{C_1 C_2}\right|(\tau) \\ &\le \frac{\varepsilon}{4\pi}\frac{|J_2 - J_1|}{C_2}(\tau) + \frac{\varepsilon}{4\pi}\cdot\frac{|J_1|\,|C_2 - C_1|}{C_1 C_2}(\tau). \end{aligned} \tag{4.42}$$

From the fact that $C_m \ge 1$, for $m = 1$ and 2, we are able to replace (4.42) by

$$\begin{aligned} &(1+\tau^{2k})|\Gamma_2\dot{\Gamma}_2 - \Gamma_1\dot{\Gamma}_1|(\tau) \\ &\quad\le \frac{\varepsilon}{4\pi}(1+\tau^{2k})|J_2 - J_1|(\tau) + \frac{\varepsilon}{4\pi}(1+\tau^{2k})|J_1|(\tau)\cdot|C_2 - C_1|(\tau) \\ &\quad\le \frac{\varepsilon}{4\pi}(1+\tau^{2k})\left|\int_0^\tau (q_2(s)H_2(\tau - s, s) - q_1(s)H_1(\tau - s, s))\,ds\right| \\ &\qquad + \frac{\varepsilon}{4\pi}(1+\tau^{2k})|J_1|(\tau)|C_2 - C_2|(\tau) \\ &\quad\le \frac{\varepsilon}{4\pi}(1+\tau^{2k})\int_0^\tau |q_2 - q_1|(s)|H_2(\tau - s, s)|\,ds \\ &\qquad + \frac{\varepsilon}{4\pi}(1+\tau^{2k})\int_0^\tau |q_1(s)|\,|(H_2 - H_1)(\tau - s, s)|\,ds \\ &\qquad + \frac{\varepsilon}{4\pi}(1+\tau^{2k})|J_1|(\tau)|C_2 - C_1|(\tau), \end{aligned} \tag{4.43}$$

where H_1 and H_2 are defined in Eq. (3.47).

The results of Lemmas 3.2-3.7 together with (4.26) and (4.27) yield

$$\begin{aligned}(1+\tau^{2k})|\Gamma_2\dot{\Gamma}_2-\Gamma_1\dot{\Gamma}_1|(\tau) \leq{}& \frac{\varepsilon}{4\pi}2^{2k+1}\bar{H}_k\|q_2-q_1\|^k \\ &+\frac{\varepsilon}{4\pi}2^{2k}\delta K_k\int_0^\infty \frac{d\eta}{1+\eta^{2k}}\|q_2-q_1\|^k \\ &+\frac{\varepsilon}{4\pi}\left(Q_k(0)+2^{2k}\delta\bar{H}_k\int_0^\infty\frac{d\eta}{1+\eta^{2k}}\right)|C_2-C_1|(\tau)\end{aligned} \tag{4.44}$$

where

$$K_k=\frac{4(1+2^{2k})Q_k(0)\displaystyle\int_0^\infty\frac{d\eta}{1+\eta^{2k}}+4(1+2^{2k+1})\bar{H}_k\delta\left(\int_0^\infty\frac{d\eta}{1+\eta^{2k}}\right)^2}{1-(2^{2k+1}+1)\delta^2\left(\displaystyle\int_0^\infty\frac{d\eta}{1+\eta^{2k}}\right)^2}$$

and $\bar{H}_k$ is defined as in (4.27).

The theorem now follows from (4.39), (4.41), (4.44) and Lemma 4.1.

REFERENCES

[1] G. F. Carrier, *On the nonlinear vibration problem of the elastic string*, Q. Appl. Math. **3**, 157 (1945)

[2] R. W. Dickey, *Infinite systems of nonlinear oscillation equations related to the string*, Proc. Amer. Math. Soc. **23**, 459–468 (1969)

[3] R. W. Dickey, *Infinite systems of nonlinear oscillation equations with linear damping*, SIAM J. Appl. Math. **19**, 208–214 (1970)

[4] R. W. Dickey, *The initial value problem for a nonlinear semi-infinite string*, University of Texas, Austin, Workshop, March 1977

[5] R. Narasimha, *Nonlinear vibration of an elastic string*, J. Sound Vib. **8**, 134–146 (1968)

[6] T. Nisida, *A note on the nonlinear vibration of an elastic string*, Mem. Fac. Eng., Kyoto University **34** (1971)

[7] T. Nisida, *Nonlinear vibration of an elastic string II*, preprint.