APPLICATIONS AND IMPROVEMENTS OF THE SUMMED PROGRESSING WAVE FORMALISM*

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1. Introduction. This paper is concerned with initial-boundary-value problems associated with a general system of equations which covers in an unified way several classes of wave propagation phenomena in continuous media. This system encompasses, as special cases, integro-differential equations for electromagnetic waves in dispersive media [1], stress waves in viscoelastic solids, magnetoelastic waves in conducting solids [2], and symmetric hyperbolic systems governing wave propagation in elastic solids, for instance [3]; it notably includes the numerous one-dimensional theories that describe various wave propagation effects in rods (coupling of extensional, torsional, flexural waves in beams or helical springs [4], water-hammer effect in cylindrical viscoelastic ducts [5]) as well as asymmetric motion of shells of revolution.

The usual discretization methods (such as finite differences or finite elements) give a solution whose accuracy rapidly decreases as the length of the time interval increases. Wave-front expansion methods provide series solutions whose convergence is fast close to a wave-front, but soon becomes very slow at some distance from it. Note that convergence can be accelerated by the approach of Turchetti and Mainardi [6] who employed a Padé approximant technique.

These shortcomings prompt the proposal of a new integral-like representation of the solution which is termed the "summed progressing wave." It still constitutes a formal theory, but the excellent numerical results it provides already render its development rewarding.

In addition to the computational advantages, this formalism provides a better insight into and understanding of wave propagation phenomena, as it shows individually the importance of coupling, attenuation and boundary conditions.

2. Hyperbolic integro-differential system. The above-mentioned system is of the form:

$$\tilde{L}\tilde{U} = \frac{\partial}{\partial t}\tilde{U} + A^x\frac{\partial}{\partial x}\tilde{U} + A^y\frac{\partial}{\partial y}\tilde{U} + A^z\frac{\partial}{\partial z}\tilde{U} + \mathscr{D}_t\tilde{U} = 0,$$

where $\tilde{U}$ is the column vector of the n unknown functions of the independent variables (x, y, z, t) such that (x, y, z) are the space coordinates and t is the time. A^x, A^y, A^z are n

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by n constant coefficient matrices and $\mathscr{D}_t$ is an operator, integro-differential with respect to the variable t, defined by:

$$\mathscr{D}_t \tilde{U}(x, y, z, t) = \frac{\partial}{\partial t} \int_0^t D(t-\tau)\tilde{U}(x, y, z, \tau)\, d\tau = D(t) \circledast \tilde{U}(x, y, z, t)$$

where D is a n by n matrix whose coefficients are functions of time, several of which depend on the constitutive equations of the medium considered. For the sake of definiteness, we will confine ourselves to the situation where D is of the form:

$$D(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} D_\nu$$

where the matrices D_ν have constant coefficients, which is sufficient to encompass most usual cases.

(a) *Characteristic surfaces.* It can be verified that the system $\tilde{L}\tilde{U} = 0$ has the same characteristic surfaces as the partial differential system obtained by letting $D = 0$ [7]. The corresponding characteristic surfaces $\Phi(x, y, z, t) = 0$ meet the characteristic equation $Q(\Phi) = 0$ with $Q(\Phi) = \det A$, where the n by n characteristic matrix A is defined by:

$$A = \Phi_t I + \Phi_x A^x + \Phi_y A^y + \Phi_z A^z$$

where I is the n by n identity matrix [8].

(b) *Bicharacteristics.* Let $x_i = x_i(s)$; $i = 0, 1, 2, 3$ $(x_0 = t,\ x_1 = x,\ x_2 = y,\ x_3 = z)$ be the parametric equations of the bicharacteristic curves defined by: $(d/ds)x_i = \lambda(\partial Q/\partial \Phi_i)$, $(d/ds)\Phi_i = -\lambda(\partial Q/\partial x_i)$, where $\Phi_i = (\partial Q/\partial x_i)$, s is the curvilinear abscissa and λ is a parameter [8]. We will be interested in the bicharacteristics contained in the characteristic plane: $\Phi(z, t) \equiv z - ct + \phi = 0$ where ϕ is a given quantity, and such that $c \neq 0$ is a (possibly multiple) root of the equation $\det(A^z - cI) = 0$.

It can be verified that these curves are the straight lines

$$t = s + \phi, \qquad z = cs.$$

Conversely, we have:

$$\phi = t - z/c, \qquad s = z/c.$$

(c) *Eigenvectors associated with a characteristic plane.* We know that the equation

$$(A^z - cI)^T \mathbf{l} = 0,$$

where the superscript T denotes the transpose, possesses n vector solutions $\mathbf{l}$ associated with the n possible values of c. These eigenvectors of $(A^z)^T$ can be made to form an orthonormal set which constitutes a basis of the n-space; this entitles us to assume that A^z is symmetric without restriction.

(d) *Change of variables associated with a characteristic plane.* Now set

$$\phi(z, t) = t - z/c, \qquad s(z) = z/c;$$

then $\phi(z, t) = 0$ is the equation of a characteristic plane. Next set

$$\tilde{U}(x, y, z, t) = U(x, y, \phi(z, t), s(z)).$$

With the independent variables (x, y, ϕ, s) and since the initial state is rest, the operator $\tilde{L}$ takes on the new form L such that

$$cLU = (cI - A^z)\frac{\partial}{\partial \phi} U + A^z \frac{\partial}{\partial s} U + cB_\phi U$$

where B_ϕ is the integro-differential operator defined by

$$B_\phi U = \left| A^x \frac{\partial}{\partial x} + A^y \frac{\partial}{\partial y} + D(\phi) \circledast^\phi \right| U$$

in which the superscript ϕ denotes that the Stieltjes product is with respect to the variable ϕ.

3. One-dimensional summed progressing wave. Consider temporarily the restricted case where $A^x = A^y = 0$ and the data are such that the solution U does not depend on (x, y).

First assume the characteristic $\phi(z, t) = 0$ to be simple and such that $c \neq 0$. Let $\mathbf{l}$ be the eigenvector of A^z associated with c. Our key idea is to seek solutions of the system $LU = 0$ which admit the decomposition:

$$U(\phi, s) = \Gamma(\phi) \circledast b(\phi, s) \circledast w(\phi) \tag{1}$$

where w is a scalar function of the variable ϕ, in class K for instance (that is piecewise continuous together with its first derivative), b is a scalar function of the two variables ϕ and s which is infinitely differentiable, Γ is a n-dimensional vector function of the variable ϕ which is also infinitely differentiable, and which meet at $s = 0$ some boundary condition involving the value of the scalar product $\mathbf{s} \cdot U$, where $\mathbf{s}$ is a n-vector such that

$$\mathbf{s} \cdot \mathbf{l} = 1. \tag{2}$$

The operation $\circledast$ bears upon the variable ϕ; it requires a generalized derivative, if necessary. In the sequel α will denote an auxiliary scalar function of the variable ϕ related to Γ.

Seeking solutions of the form (1) leads to the following result.

FUNDAMENTAL THEOREM. Let Γ, α and b be three functions that meet the following relations and conditions:

$$(cI - A^z)\frac{d}{d\phi}\Gamma + A^z\Gamma \circledast \alpha + cB_\phi \Gamma = 0, \tag{3}$$

$$\Gamma(0) = \mathbf{l}, \tag{4}$$

$$\mathbf{s} \cdot \Gamma(\phi) = 1, \tag{5}$$

$$\frac{\partial}{\partial s} b(\phi, s) = \alpha(\phi) \circledast b(\phi, s), \tag{6}$$

$$b(\phi, 0) = 1. \tag{7}$$

Then the field U defined by (1) is a weak solution of the system $LU = 0$.

Proof. From the properties of the operation ⊛ we have

$$\frac{\partial}{\partial\phi} U = \Gamma(0)\frac{\partial}{\partial\phi}(b \circledast w) + \frac{d\Gamma}{d\phi} \circledast b \circledast w,$$

whence:

$$(cI - A^z)\frac{\partial}{\partial\phi} U = (cI - A^z)\frac{d\Gamma}{d\phi} \circledast b \circledast w$$

as

$$(cI - A^z)\Gamma(0) = (cI - A^z)\mathbf{l} = 0$$

from Sec. 2(c). Moreover:

$$\frac{\partial}{\partial s} U = \Gamma \circledast \frac{\partial b}{\partial s} \circledast w = \Gamma \circledast \alpha \circledast b \circledast w$$

from (6). Thus:

$$cLU = \left\{(cI - A^z)\frac{d\Gamma}{d\phi} + A^z\Gamma \circledast \alpha + cB_\phi \Gamma\right\} \circledast b \circledast w$$

which is zero according to (3).

This enables us to propose the following definition.

Definition. We call "summed progressing wave" every weak solution of $LU = 0$ which is of the form (1) where Γ, α and b meet ((3) to (7)).

Consequences. (a) Let $\boldsymbol{\sigma}$ denote a n-vector such that

$$A^z\boldsymbol{\sigma} = c\mathbf{s}. \tag{8}$$

On premultiplication of (3) by $\boldsymbol{\sigma}$, we see that α can be expressed in terms of Γ through:

$$\alpha(\phi) = -\boldsymbol{\sigma}\left(\frac{d}{d\phi}\Gamma + B_\phi \Gamma\right). \tag{9}$$

(b) Let $(s\alpha\circledast)$ denote the operator which associates the function $s\alpha \circledast u$ with every differentiable $u = u(\phi)$ and let $\exp(s\alpha\circledast)$ denote its exponential which is formally defined by [9]:

$$\exp(s\alpha \circledast)u = \left(1 + s\alpha + \frac{s^2}{2!}\alpha \circledast \alpha + \cdots\right) \circledast u.$$

Then it is clear from (6) and (7) that

$$b(\phi, s) = \exp(s \circledast)b(\phi, 0) \quad \text{with} \quad b(\phi, 0) = 1, \tag{10}$$

or, more generally,

$$b(\phi, s) = \exp((s - s_0)\alpha \circledast)b(\phi, s_0). \tag{11}$$

Comparing (10) and (11) gives

$$b(\phi, s) = b(\phi, s - s_0) \circledast b(\phi, s_0), \tag{12}$$

which constitutes the *semigroup property* of b with respect to s.

4. Properties of the summed progressing wave. Consider the specific case when $w(\phi)$ takes on the form $w(\phi) = H(\phi)\omega(\phi)$, where H denotes the Heaviside step function and ω is infinitely differentiable. Then U defined by (1) meets:

the system $LU = 0$ in the weak sense in the domain $(z > 0, t > 0)$;

rest initial conditions;

the boundary condition at $z = 0$ (i.e. $s = 0$, $\phi = t$);

$$\mathbf{s} \cdot U(t, 0) = \mathbf{s} \cdot \Gamma(t) \circledast b(t, 0) \circledast w(t) = w(t), \tag{13}$$

from (5) and (7), where $\circledast$ is with respect to t.

Set $z = 0$; then $s = 0$, $\phi = t$ and, from (1), $U(t, 0) = \Gamma(t) \circledast \omega(t)$ where $\circledast$ is with respect to t.

This shows that Γ fully defines the *coupling* between the various components of U on the boundary $z = 0$, without regard to ω, which, as will be seen later, represents a prescribed excitation.

Moreover, for an observer traveling along the axis $z > 0$ with the velocity c of the wavefront, the value of ϕ remains constant and (1) states that the variation of U with $z = cs$ is entirely governed by the function $b(\cdot, s)$.

It follows that the summed progressing wave enables one to identify three important effects in the solution:

(1) the scalar function w represents the datum on the boundary on one hand and the initial state (here rest) on the other;

(2) the vector function Γ fully accounts for the "vector character" of the solution and thus defines the coupling between the unknown functions, without regard to the form of the "loading" prescribed on the boundary $z = 0$ or to the initial state;

(3) the scalar function b describes the attenuation of the wave during its propagation, without regard to initial or boundary data.

5. A more convenient formulation. Due to the fact that matrix $cI - A^z$ is singular, the system (3)–(7) does not easily lend itself to computation in its current form.

In (3) the term $A^z\Gamma \circledast \alpha$ can be transformed using the fact that $\Gamma \circledast \alpha = \Gamma(0)\alpha + (d\Gamma/d\phi) * \alpha$ where $*$ is the usual Riemann convolution product such that

$$f(\phi) * g(\phi) = \int_0^{\phi} f(\phi - \psi) g(\psi)\, d\psi.$$

Thus $A^z\Gamma \circledast \alpha = c\mathbf{l}\alpha + A^z(d\Gamma/d\phi) * \alpha$ as $\Gamma(0) = \mathbf{l}$ and $A^z\mathbf{l} = c\mathbf{l}$.

From (9) it can be shown that

$$A^z\Gamma \circledast \alpha = -c\mathbf{l} \otimes \boldsymbol{\sigma}\left(\frac{d}{d\phi}\Gamma + B_\phi \Gamma\right) + A^z \frac{d\Gamma}{d\phi} * \alpha,$$

where $\otimes$ denotes the dyadic product of n-vectors. (3) becomes:

$$(cI - A^z - c\mathbf{l} \otimes \boldsymbol{\sigma})\frac{d}{d\phi}\Gamma + c(I - \mathbf{l} \otimes \boldsymbol{\sigma})B_\phi \Gamma + A^z \frac{d\Gamma}{d\phi} * \alpha = 0.$$

Now it can be demonstrated that the matrix $A^z - cI + c\mathbf{l} \otimes \boldsymbol{\sigma}$ possesses the remarkable property that it is *invertible*; let M denote its inverse. Then the system (3)–(5) is profitably replaced by:

$$\frac{d}{d\phi}\Gamma = cM(I - \mathbf{l} \otimes \boldsymbol{\sigma})B_\phi \Gamma + MA^z \frac{d\Gamma}{d\phi} * \alpha; \qquad \Gamma(0) = \mathbf{l}, \tag{14}$$

$$\alpha = -\boldsymbol{\sigma} \cdot \left(\frac{d}{d\phi}\Gamma + B_\phi \Gamma\right). \tag{15}$$

By eliminating α between (14) and (15), and after some manipulation, it is clear that the vector function $G = (d/d\phi)\Gamma$ satisfies a surprisingly *nonlinear* Volterra integro-differential system of the second kind.

6. Multiple characteristics. Most of the physical theories referred to in the introduction lead to systems that possess multiple characteristics. Now assume the characteristic $\phi(z, t) = 0$ to have multiplicity two: $\phi(z, t) = t - z/c$ where $c = c_2 = c_3 \neq 0$ is a double root of the equation $\det(A^z - cI) = 0$.

Let w^2 and w^3 be two arbitrary functions of ϕ in class K; let $\mathbf{l}_2$ and $\mathbf{l}_3$ denote two mutually orthogonal eigenvectors of A^z associated with the characteristic $\phi(z, t) = 0$ and $\mathbf{s}_2$, $\mathbf{s}_3$ denote two n-vectors such that

$$\mathbf{s}_i \cdot \mathbf{l}_j = \delta_{ij} \qquad (i, j = 2, 3)$$

where δ_{ij} is the Kronecker delta; $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$ are two n-vectors such that

$$A^z \boldsymbol{\sigma}_i = c\mathbf{s}_i \qquad (i = 2, 3).$$

It can then be verified that the matrix $A^z - cI + c(\mathbf{l}_2 \otimes \boldsymbol{\sigma}_2 + \mathbf{l}_3 \otimes \boldsymbol{\sigma}_3)$ is invertible; its inverse is denoted by N.

Let Γ^i and α^{ij} on one hand, b^{ij} on the other $(i, j = 2, 3)$ denote the functions of ϕ and (ϕ, s) respectively, with $s = z/c$, such that:

$$\frac{d}{d\phi}\Gamma^i = cN(I - \mathbf{l}_2 \otimes \boldsymbol{\sigma}_2 - \mathbf{l}_3 \otimes \boldsymbol{\sigma}_3)B_\phi \Gamma^i + NA^z \sum_{j=2}^{3} \left(\frac{d\Gamma^j}{d\phi}\right) * \alpha^{ij}, \tag{16}$$

$$\Gamma^i(0) = \mathbf{l}_i, \tag{17}$$

$$\gamma^{ij} = -\sigma_j \cdot \left(\frac{d}{d\phi}\Gamma^i + B_\phi \Gamma^i\right), \tag{18}$$

$$\frac{\partial}{\partial s} b^{ij} = \sum_{k=2}^{3} \alpha^{kj} \circledast b^{ik}, \tag{19}$$

$$b^{ij}(\phi, 0) = \delta_{ij} \qquad (i, j = 2, 3). \tag{20}$$

It can be demonstrated that the summed progressing wave associated with the double characteristic $\phi(z, t) = 0$, weak solution of $LU = 0$, is:

$$U(\phi, s) = \sum_{i,j=2}^{3} \{\Gamma^i(\phi) \circledast b^{ij}(\phi, s) \circledast\} w^j(\phi). \tag{21}$$

After some lengthy manipulation, it can be verified that

$$\mathbf{s}_i \cdot \Gamma^j = \delta_{ij} \qquad i, j = 2, 3. \tag{22}$$

By putting $z = 0$ (whence $s = 0$, $\phi = t$), and taking into account (20), (22), the above solution is seen to satisfy the two boundary conditions:

$$\mathbf{s}_2 \cdot U(t, 0) = w^2(t), \qquad \mathbf{s}_3 \cdot U(t, 0) = w^3(t).$$

7. Mixed initial-boundary-value problem. To illustrate this approach, assume that the characteristics emanating from the origin are the two straight lines: $\phi_1(z, t) = 0$ associated with the simple root $c = c_1$ and the eigenvector $\mathbf{l}_1$ on one hand, $\phi_2(z, t) = 0$ associated with the double root $c = c_2$ and the two mutually orthogonal eigenvectors $\mathbf{l}_2$ and $\mathbf{l}_3$ on the other. Next, set $s_1 = z/c_1$ and $s_2 = z/c_2$.

Consider the following mixed problem: the initial conditions pertain to the rest state (which is not very restrictive for practical applications)

$$\tilde{U}(z, 0) = 0;$$

the Boundary conditions consist in prescribing the classical data for hyperbolic systems (e.g. see (10)):

$$\mathbf{s}_k \cdot \tilde{U}(0, t) = g_k(t), \qquad k = 1, 2, 3,$$

with

$$\mathbf{s}_1 \cdot \mathbf{l}_1 = 1, \qquad \mathbf{s}_i \cdot \mathbf{l}_j = \delta_{ij} \qquad (i, j = 2, 3).$$

The solution is known to be of the form

$$\tilde{U}(z, t) = U^1(\phi_1(z, t), s_1(z)) + U^2(\phi_2(z, t), s_2(z)), \tag{23}$$

where U^1 is of the form (1) and U^2 is of the form (21). $g_k (k = 1, 2, 3)$ are given functions in class K for instance that characterize the prescribed "excitation."

Let w^1 on one hand, w^2 and w^3 on the other, denote the functions w associated with U^1 and U^2 respectively. The three functions w^1, w^2, w^3 are determined by writing the boundary conditions at $z = 0$, where $s_1 = s_2 = 0$ and $\phi_1 = \phi_2 = t$.

Taking into account conditions (5), (7), (20), (22) it can be seen that, once the functions Γ^k, α_1, α^{ij}, b^1, b^{ij} $(k = 1, 2, 3;\ i, j = 2, 3)$ are known, the functions w^k are of the form

$$w^k(\phi_k) = H(\phi_k)\omega^k(\phi_k) \qquad (k = 1, 2, 3 \text{ with } \phi_3 = \phi_2)$$

where the new functions ω^k solve the partial integro-differential system

$$\mathbf{s}_k \cdot \sum_{j=1}^{3} \Gamma^j(t) \circledast \omega^j(t) = g_k(t), \qquad k = 1, 2, 3 \tag{24}$$

in which the operation $\circledast$ bears upon the variable t (it is understood that the functions Γ^2 and Γ^3 are associated with the summed wave U^2, whereas Γ^1 is associated with U^1). As will be demonstrated later, the system (24) enables one to determine the functions w^k completely.

8. Numerical implementation. Computations take place according to the following stages:

(1) the first step consists in calculating Γ^k and α^k or α^{ij}, the only data being A^z, $D(t)$, and $\mathbf{s}_k$;

(2) once Γ^k, α^k and α^{ij} have been determined, the next two steps can be achieved simultaneously; these are:

(a) calculation of w^k from Γ^k, $\mathbf{s}_k$ and g_k

(b) calculation of b^k or b^{ij} from the values of α^k or α^{ij}.

As regards the functions Γ^k, α^k and α^{ij}, it has already been verified that the systems (14)–(15) or (16)–(18) lead to systems of nonlinear Volterra integral equations of the second kind; for their solution, the iteration method turned out to be very efficient.

As regards the functions w^k, taking into account the conditions (5) and (22), the relations (24) lead to a system of linear Volterra integral equations of the second kind; the iteration method can again be used successfully.

As regards the functions b^k or b^{ij}, their computation amounts to the calculation of the generalized exponential (10). In order to perform it, we resorted to the following procedure:

(1) calculate b for a small value of s by using the first terms of the expansion of b in the powers of s;

(2) then calculate b for the values $2s$, $4s$, ..., through a repeated use of the semigroup property (12).

This procedure allows one considerably to reduce the number of convolution products to be computed. Thanks to it, the asymptotic solution for large values of s could be reached easily while solving a specific problem: we investigated the propagation of dispersive stress waves along a beam of rectangular cross-section whose end is subjected to a shock [11]. The phenomenon is described by either Volterra's or Medick's theory which take into account the lateral contraction of cross-sections. The pertinent system $\tilde{L}\tilde{U} = 0$ has dimension eight. With the help of the summed progressing wave formalism, and after repeated use of the semigroup property (12), we could reach both short-term and long-term solutions through the same computation, which could not be performed as easily by other means; this example illustrates the potential of the proposed method.

9. Extension to three-dimensional problems. Consider now the following problem. The solution is sought in the domain $\{z > 0, t > 0\}$ of the four-space (x, y, z, t). The initial data are homogeneous:

$$\tilde{U}(x, y, z, 0) = 0.$$

The boundary data are of the form $\mathbf{s}_k \cdot \tilde{U}(x, y, 0, t) = g_k(x, y, t)$ where the vectors $\mathbf{s}_k$ are defined as above and the functions g_k characterize the loading prescribed on the plane $z = 0$ (such problems are often encountered in seismology).

The solution can still be constructed by superimposing several "summed progressing waves," each summed wave being associated with a characteristic plane $\phi(z, t) = t - z/c = 0$ that contains the straight line $(t = 0, z = 0)$ in four-space.

In the case where the characteristic plane $\phi(z, t) = 0$ is simple, the relevant summed progressing wave is now defined as follows:

$$U(x, y, \phi, s) = \Gamma\left(\phi, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \circledast b\left(\phi, s, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \circledast w(\phi, x, y)$$

in which:

w is some function related to the boundary data,

b is *no longer a function*, but a "generalized" differential operator with respect to the variables x and y, depending upon parameters ϕ and s. The term "generalized" will soon be specified;

Γ is also a generalized differential operator with respect to x and y, like b, but it depends on the parameter ϕ only and it is vector-valued.

The operation $\circledast$ can now be defined as:

$$b\left(\phi, s, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \circledast w(\phi, x, y) = \frac{\partial}{\partial \phi} \int_0^{\phi} b\left(\phi - \psi, s, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) w(\psi, x, y)\, d\psi.$$

In the cases of interest here, b and Γ have the form:

$$b\left(\phi, s, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \sum_{\nu=0}^{\infty} \frac{\phi^{\nu}}{\nu!} b_{\nu}\left(s, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

$$\Gamma\left(\phi, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \sum_{\nu=0}^{\infty} \frac{\phi^{\nu}}{\nu!} \gamma_{\nu}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

where b_{ν} and γ_{ν} are operators, differential with respect to x and y, of order ν at most, which are number-valued and vector-valued respectively. A more rigorous definition of such operators could be attempted by using the concept of "pseudo-differential" operator [13].

The relations verified by these operators are formally the same as those verified by the corresponding functions in the one-dimensional case: after some obvious transpositions, the relations (3)–(7) met by functions become relations between operators (the order of the factors should be respected carefully). For instance, the relations (3), (6), (9) are left formally unaltered, whereas (4), (5), (7) go over to:

$$\Gamma\left(0, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \mathbf{I1}; \mathbf{s} \cdot \Gamma\left(\phi, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \mathbf{1}; b\left(\phi, 0, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = \mathbf{1}$$

where $\mathbf{1}$ is the identity operator in the space of number-valued functions. $\alpha(\phi)$ is replaced by $\alpha(\phi, \partial/\partial x, \partial/\partial y)$ and now:

$$\exp(s\alpha \circledast) = \mathbf{1} + (s\alpha \circledast) + \cdots.$$

After suitable transposition, the relations (14)–(15) enable one to readily determine the operators Γ and b through their expansions in powers of ϕ.

We applied this procedure to a problem involving three-dimensional elastodynamics. Here the system $\tilde{L}\tilde{U} = 0$ has dimension seven: the unknown functions are the three components of the velocity vector, the dilatation and the three components of the rotation vector. Boundary data consist in prescribing the values of the velocity vector on the plane $z = 0$. The summed progressing wave enabled us to obtain the solution in the form of a wave-front expansion more readily and more systematically than by using Recker's method [3] which requires the solution of the equations of transport of discontinuities. It is also interesting to note here that the corresponding viscoelastic problem could be treated in much the same way since the numerical procedure is not significantly altered by the introduction of viscoelasticity.

References

[1] R. M. Lewis, *Asymptotic theory of wave propagation*, Arch. Rat. Mech. Anal. **20**, 191–250 (1965)
[2] W. W. Recker, *A discontinuity expansion to plane problems in elastodynamics*, J. Sound Vibr. **23**, 41–53 (1972)
[3] W. W. Recker, *Series solution to problems in elastodynamics*, Proc. Am. Soc. Civ. Eng. **97 EM 4**, 1257–1275 (1971)
[4] W. H. Wittrick, *On elastic wave propagation in helical springs*, Int. J. Mech. Sci. **8**, 25–47 (1966)
[5] A. Blanchard, *Etude théorique des écoulements non permanents en conduite viscoélastique: application au phénomène du coup de bélier*, Thèse de Doctorat de Spécialité, Université Claude Bernard, Lyon (13 Octobre 1978)
[6] G. Turchetti and F. Mainardi, *Wave-front expansions and Padé approximants for transient wave in linear dispersive media*, in *Padé approximant method and its applications to mechanics*, edited by H. Cabannes, Springer-Verlag, Berlin, 1976, pp. 189–208
[7] G. M. C. Fisher, and M. E. Gurtin, *Wave propagation in the linear theory of viscoelasticity*, Quart. Appl. Math. **23**, 257–263 (1965)
[8] R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. 2, Interscience, New York, 1962
[9] K. Yosida, *Functional analysis*, 2d edition, Springer-Verlag, Berlin, 1968
[10] A. Majda and S. Osher, *Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary*, Comm. Pure Appl. Math. **28**, 607–675 (1975)
[11] D. Gamby and R. Lamy, *Propagation d'ondes dispersives dans une poutre longue de section rectangulaire dont une extrémité est soumise à un choc*, J. Mécanique Appliquée, forthcoming
[12] D. Gamby and R. Lamy, *Numerical implementation of the summed progressing wave formalism: computation of short-time and long-time transients*, Int. J. Numer. Methods in Eng., forthcoming
[13] M. Taylor, *Lectures on pseudo-differential operators*, Springer-Verlag, New York, 1974