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TRANSLATIONAL ADDITION THEOREMS FOR SPHEROIDAL SCALAR AND VECTOR WAVE FUNCTIONS*

BY

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Abstract. The translational addition theorems for spheroidal scalar wave functions $R_{mn}^{(i)}(h, \xi)S_{mn}(h, \eta)\exp(jm\phi)$; $i = 1, 3, 4$ and spheroidal vector wave functions $\mathbf{M}_{mn}^{x,y,z(i)}(h; \xi, \eta, \phi)$, $\mathbf{N}_{mn}^{x,y,z(i)}(h; \xi, \eta, \phi)$; $i = 1, 3, 4$, with reference to the spheroidal coordinate system at the origin O, have been obtained in terms of spheroidal scalar and vector wave functions with reference to the translated spheroidal coordinate system at the origin O', where O' has the spherical coordinates (r_0, θ_0, ϕ_0) with respect to O. These addition theorems are useful in acoustics and electromagnetics in those cases involving spheroidal radiators and scatterers.

1. Introduction. The translational addition theorems for spherical scalar wave functions were developed by Friedman and Russek [1] and those for spherical vector wave functions were given by Stein [2] and Cruzan [3]. These addition theorems were applied by Bruning and Lo [4] to the problem of scattering of a plane electromagnetic wave from a system of two spheres. In general the boundary-value scattering problem involving many bodies will require transformation of an outgoing wave from one body and its associated origin O (the center of the body) and coordinate system X into the incoming wave to another body with its own associated origin O' (center of the body) and coordinate system X'. These transformations are accomplished by addition theorems. In particular, when the geometries of interacting bodies are the same and the axes of symmetry are oriented parallel to each other, the transformations are by translational addition theorems for wave functions corresponding to the coordinate geometry of the interacting bodies. The motivation here is to obtain an exact solution in terms of spheroidal wave functions for the problems (acoustic and electromagnetic) of plane wave scattering from two or more spheroids with parallel axial configurations as an extension of multipole solutions given by the authors [5] for electromagnetic plane wave scattering from a single prolate spheroid.

The first expansion treated in this paper is that for the standing spheroidal wave $R_{mn}^{(1)}(h, \xi)S_{mn}(h, \eta)\exp(jm\phi)$ with reference to the coordinate system X at origin O in terms of spheroidal waves $R_{mn}(h', \xi')S_{mn}(h', \eta')\exp(jm\phi')$ with reference to the translated coordinate system X' at origin O'; (ξ, η, ϕ) are the spheroidal coordinates, $h = kF$† where k

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† h corresponds to c and F corresponds to $d/2$ of Flammer's text [6].

is the wave number and F is the semi-interfocal distance. A detailed account of spheroidal functions R_{mn} (radial function) and S_{mn} (angle function) can be obtained in Flammer [6].

The second expansion is that for the outgoing spheroidal wave $R_{mn}^{(3)}(h, \xi)S_{mn}(h, \eta)\exp(jm\phi)$.

From the first two expansions we then obtain an expansion for $R_{mn}^{(4)}(h, \xi)S_{mn}(h, \eta)\exp(jm\phi)$ with reference to the coordinate system (O, X) in terms of the spheroidal waves referred to the translated system (O', X') by using the relationship $R_{mn}^{(4)}(h, \xi) = 2R_{mn}^{(1)}(h, \xi) - R_{mn}^{(3)}(h, \xi)$.

The transformations of the vector functions $\mathbf{M}_{mn}^{x, y, z(1, 3, 4)}(h; \xi, \eta, \phi)$ and $\mathbf{N}_{mn}^{x, y, z(1, 3, 4)}(h; \xi, \eta, \phi)$ from (h, X) to (O', X') are then obtained from the expansions of the scalar functions $R_{mn}^{(1, 3, 4)}(h, \xi)S_{mn}(h, \eta)\exp(jm\phi)$.

To evaluate the integrals which are instrumental in the above transformations $(O, X \to O', X')$, a formula is needed which expresses the product of two associated Legendre functions in terms of a sum of associated Legendre functions such as given by Stein [2, pp. 22–23]. A fast and convenient computer algorithm has been developed in Appendix I to evaluate the coefficients of this expansion.

In this paper all derivations are for prolate spheroidal functions, although the terminology "spheroidal" instead of "prolate spheroidal" has been used since the derivations for the oblate case are very similar. In fact, the results for the oblate system are obtained from those for the prolate system by the transformation $\xi \to j\xi$, $h \to -jh$ (or $F \to -jF$).

2. The expansion for $\psi_{mn}^{(1)}(h; \xi, \eta, \phi) = R_{mn}^{(1)}(h, \xi)S_{mn}(h, \eta)\exp(jm\phi)$. The Cartesian coordinate system associated with a spheroidal coordinate system has its origin at the center and z-axis along the positive direction of the spheroidal axis of symmetry. The translation moves the Cartesian system at the origin O corresponding to the spheroidal system $(h; \xi, \eta, \phi)$ to the origin O' corresponding to the spheroidal system $(h'; \xi', \eta', \phi')$. The cartesian coordinates under translation are shown in Fig. 1. The point P has the spheroidal coordinates $(h; \xi, \eta, \phi)$ and $(h'; \xi', \eta', \phi')$ in the two coordinate systems. The polar coordinates of the origin O' with reference to the origin O are (r_0, θ_0, ϕ_0).

A plane wave to the direction $[\theta = \theta_i, \phi = \phi_i]$ can be expanded in terms of the spheroidal scalar waves in the $(h; \xi, \eta, \phi)$ system at O [6, p. 48] as

$$\exp(jkr\cos\gamma) = 2\sum_{m=0}^{\infty}\sum_{n=m}^{\infty} j^n \frac{\varepsilon_m}{N_{mn}(h)} \cdot S_{mn}(h, \cos\theta_i)S_{mn}(h, \eta)R_{mn}^{(1)}(h, \xi)\cos[m(\phi - \phi_i)] \tag{1}$$

where $N_{mn}(h)$ = normalization constant [6, p. 22] for the angle function and

$$\varepsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m \neq 0 \end{cases}.$$

If we wish to use the complete range of m, i.e. $-\infty \le m \le +\infty$, then from expressions given by Flammer [6, p. 22] we note

$$\frac{S_{mn}(h, \eta)S_{mn}(h, \cos\theta_i)}{N_{mn}(h)} = \frac{S_{-m,n}(h, \eta)S_{-m,n}(h, \cos\theta_i)}{N_{-m,n}(h)}$$

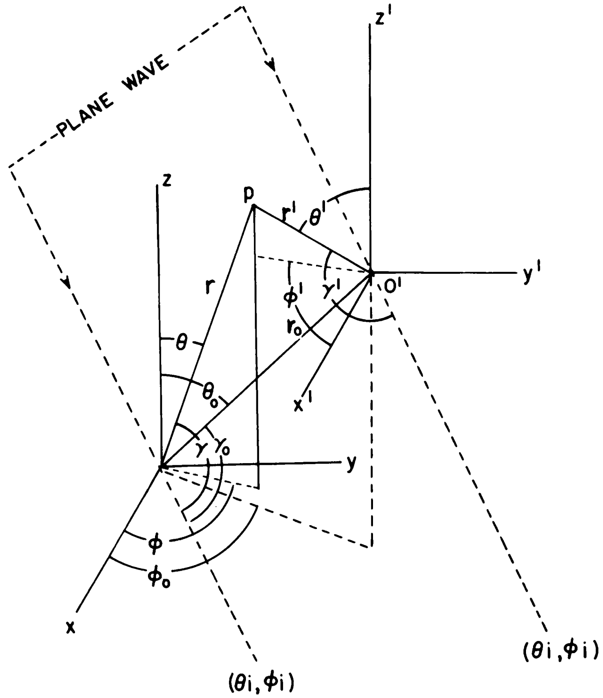

FIG. 1. Coordinate translation.

and from the integral equations [6, pp. 48–50] involving $R^{(i)}_{mn}(h, \xi)$, $i = 1, 3, 4$ we see that $R^{(i)}_{mn}(h, \xi) = R^{(i)}_{-m,n}(h, \xi)$. These relations change Eq. (1) to

$$\exp(jkr\cos\gamma) = 2\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{j^n}{N_{mn}(h)} \cdot S_{mn}(h, \cos\theta_i)S_{mn}(h, \eta)R^{(1)}_{mn}(h, \xi)\exp[jm(\phi - \phi_i)]. \tag{2}$$

Multiplying both sides of Eq. (2) by $S_{mn}(h, \cos\theta_i)\exp(jm\phi_i)\sin\theta_i$, integrating on θ_i and ϕ_i, and using the orthogonality property of the angle functions [6, p. 22], we obtain

$$\begin{aligned}\psi^{(1)}_{mn}(h; \xi, \eta, \phi) &= S_{mn}(h, \eta)R^{(1)}_{mn}(h, \xi)\exp(jm\phi)\\ &= (4\pi j^n)^{-1}\int_0^{2\pi}\int_0^{\pi}\exp(jkr\cos\gamma)S_{mn}(h, \cos\theta_i)\\ &\quad\cdot\exp(jm\phi_i)\sin\theta_i\, d\theta_i\, d\phi_i.\end{aligned} \tag{3}$$

Similarly, the standing spheroidal waves referred to O' are given by

$$\begin{aligned}\psi'^{(1)}_{mn}(h'; \xi', \eta', \phi') &= S_{mn}(h', \eta')R^{(1)}_{mn}(h', \xi')\exp(jm\phi')\\ &= (4\pi j^n)^{-1}\int_0^{2\pi}\int_0^{\pi}\exp(jkr'\cos\gamma')S_{mn}(h', \cos\theta_i)\\ &\quad\cdot\exp(jm\phi_i)\sin\theta_i\, d\theta_i\, d\phi_i.\end{aligned} \tag{4}$$

Now since the radius vectors are such that (see Fig. 1) $\mathbf{r} = \mathbf{r}' + \mathbf{r}_0$, it follows that if $\mathbf{k}_i$ is the propagation vector of the incident plane wave, then

$$\mathbf{k}_i \cdot \mathbf{r} = \mathbf{k}_i \cdot (\mathbf{r}' + \mathbf{r}_0)$$

or equivalently

$$r \cos \gamma = r' \cos \gamma' + r_0 \cos \gamma_0 . \tag{5}$$

From Eqs. (3) and (5) we get

$$\psi_{mn}^{(1)}(h; \xi, \eta, \phi) = (4\pi j^n)^{-1} \int_0^{2\pi} \int_0^{\pi} \exp(jkr' \cos \gamma')\exp(jkr_0 \cos \gamma_0) \cdot S_{mn}(h, \cos \theta_i)\exp(jm\phi_i) \cdot \sin \theta_i \, d\theta_i \, d\phi_i . \tag{6}$$

Now consider the plane wave expansion

$$\exp(jkr' \cos \gamma') = 2 \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \frac{j^\nu}{N_{\mu\nu}(h')} S_{\mu\nu}(h', \cos \theta_i) S_{\mu\nu}(h', \eta') \cdot R_{\mu\nu}^{(1)}(h', \xi')\exp[j\mu(\phi' - \phi_i)]. \tag{7}$$

It can be shown [1, p. 18] that (7) is a uniformly convergent series; hence, on substituting (7) into (6), the order of summation and integration may be interchanged. Consequently we have

$$\psi_{mn}^{(1)}(h; \xi, \eta, \phi) = (4\pi j^n)^{-1} 2 \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \frac{j^\nu}{N_{\mu\nu}(h')} \psi_{\mu\nu}'^{(1)}(h'; \xi', \eta', \phi') \cdot \int_0^{2\pi} \int_0^{\pi} \{\exp(jkr_0 \cos \gamma_0) S_{mn}(h, \cos \theta_i) S_{\mu\nu}(h', \cos \theta_i) \cdot \exp[j(m - \mu)\phi_i]\sin \theta_i\} \, d\theta_i \, d\phi_i . \tag{8}$$

Now from the definition of S_{mn} [6] we have

$$\begin{aligned} S_{mn}(h, \cos \theta_i) & S_{\mu\nu}(h', \cos \theta_i) \\ &= \sum_{q=0,1}^{\infty\prime} d_q^{mn}(h) P_{|m|+q}^m(\cos \theta_i) \sum_{s=0,1}^{\infty\prime} d_s^{\mu\nu}(h') P_{|\mu|+s}^{\mu}(\cos \theta_i) \\ &= \sum_{q=0,1}^{\infty\prime} \sum_{s=0,1}^{\infty\prime} d_q^{mn}(h) \, d_s^{\mu\nu}(h')(-1)^\mu \frac{(|\mu| + \mu + s)!}{(|\mu| - \mu + s)!} \\ &\quad \cdot P_{|m|+q}^m(\cos \theta_i) P_{|\mu|+s}^{-\mu}(\cos \theta_i), \end{aligned} \tag{9}$$

where the d's are the expansion coefficients for the spheroidal angle function.

Using the linearization formula [2, pp. 22–23]

$$P_n^m(\cos \theta_i) P_\nu^\mu(\cos \theta_i) = \sum_p a(m, n | \mu, \nu | p) P_p^{m+\mu}(\cos \theta_i) \tag{10}$$

where $p = n + \nu, n + \nu - 2, n + \nu - 4, \ldots, |n - \nu|$, in (9) we obtain

$$S_{mn}(h, \cos\theta_i)S_{\mu\nu}(h', \cos\theta_i) = \sum_{q=0,1}^{\infty\,\prime} \sum_{s=0,1}^{\infty\,\prime} \sum_{p} d_q^{mn}(h)\, d_s^{\mu\nu}(h')(-1)^{\mu} \frac{(|\mu| + \mu + s)!}{(|\mu| - \mu + s)!} \cdot a(m, |m| + q \,|\, -\mu, |\mu| + s \,|\, p) P_p^{m-\mu}(\cos\theta_i) \tag{11}$$

with

$$p = |m| + |\mu| + q + s,\ |m| + |\mu| + q + s - 2, \ldots, \big||m| - |\mu| + q - s\big|.$$

The determination of the expansion coefficients $a(\cdots)$ is discussed in Appendix I of this paper.

From (8) and (11) it follows that

$$\psi_{mn}^{(1)}(h; \xi, \eta, \phi) = 2 \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{q=0,1}^{\infty\,\prime} \sum_{s=0,1}^{\infty\,\prime} \sum_{p} \frac{(-1)^{\mu}}{N_{\mu\nu}(h')} j^{p+\nu-n} \cdot \frac{(|\mu| + \mu + s)!}{(|\mu| - \mu + s)!} d_q^{mn}(h)\, d_s^{\mu\nu}(h') \cdot a(m, |m| + q \,|\, -\mu, |\mu| + s \,|\, p) \cdot \{(4\pi j^{p})^{-1} \int_0^{2\pi} \int_0^{\pi} \exp(jkr_0 \cos\gamma_0) P_p^{m-\mu}(\cos\theta_i) \cdot \exp[j(m-\mu)\phi_i] \sin\theta_i \, d\theta_i \, d\phi_i\} \psi_{\mu\nu}^{\prime(1)}(h'; \xi', \eta', \phi'). \tag{12}$$

Now from Stratton [7]

$$j_n(kr_0)P_n^m(\cos\theta_0)\exp(jm\phi_0) = (4\pi j)^{-n} \int_0^{2\pi} \int_0^{\pi} \exp(jkr_0 \cos\gamma_0) P_n^m(\cos\theta_0)\exp(jm\phi_i)\sin\theta_i \, d\theta_i \, d\phi_i. \tag{13}$$

Hence, (12) reduces to

$$\psi_{mn}^{(1)}(h; \xi, \eta, \phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} A_{\mu\nu}^{(1)} \psi_{\mu\nu}^{\prime(1)}(h'; \xi', \eta', \phi') \tag{14}$$

where

$$A_{\mu\nu}^{(1)} = 2 \frac{(-1)^{\mu}}{N_{\mu\nu}(h')} \sum_{q=0,1}^{\infty\,\prime} \sum_{s=0,1}^{\infty\,\prime} \sum_{p} j^{p+\nu-n} \cdot \frac{(|\mu| + \mu + s)!}{(|\mu| - \mu + s)!} d_q^{mn}(h)\, d_s^{\mu\nu}(h') a(m, |m| + q \,|\, -\mu, |\mu| + s \,|\, p) \cdot j_p(kr_0) P_p^{m-\mu}(\cos\theta_0) \exp[j(m-\mu)\phi_0]. \tag{15}$$

The equation (14) is the required transformation from the O-system to O'-system. Since any of the equations given above does not include any restriction on relative size of r' and r_0, Eq. (14) is valid for any r'.

3. The expansion for $\psi_{mn}^{(3,4)}(h;\xi,\eta,\phi)=R_{mn}^{(3,4)}(h,\xi)S_{mn}(h,\eta)\exp(jm\phi)$. To obtain the addition theorem for the outgoing spheroidal wave $R_{mn}^{(3)}(h,\xi)S_m(h,\eta)\exp(jm\phi)$ we start with an integral equation given by Flammer [6, p. 50, Eq. (5.3.23)]:

$$\begin{aligned}\psi_{mn}^{(3)}(h;\xi,\eta,\phi)&=R_{mn}^{(3)}(h,\xi)S_{mn}(h,\eta)\exp(jm\phi)\\&=(2\pi j^n)^{-1}\int_0^{2\pi}\int_0^{\pi/2-j\infty}\exp(jkr\cos\gamma)S_{mn}(h,\cos\theta_i)\\&\quad\cdot\exp(jm\phi_i)\sin\theta_i\,d\theta_i\,d\phi_i.\end{aligned}\tag{16}$$

Proceeding in a similar fashion as before and using Eqs. (16), (5), (7) and (11), we get

$$\begin{aligned}\psi_{mn}^{(3)}(h;\xi,\eta,\phi)&=2\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sideset{}{'}\sum_{q=0,1}^{\infty}\sideset{}{'}\sum_{s=0,1}^{\infty}\sum_{p}\frac{(-1)^\mu}{N_{\mu\nu}(h')}j^{p+\nu-n}d_q^{mn}(h)\,d_s^{\mu\nu}(h')\\&\quad\cdot\frac{(|\mu|+\mu+s)!}{(|\mu|-\mu+s)!}a(m,|m|+q|-\mu,|\mu|+s|p)\\&\quad\cdot\left\{(2\pi j^p)^{-1}\int_0^{2\pi}\int_0^{\pi/2-j\infty}\exp(jkr_0\cos\gamma_0)P_p^{m-\mu}(\cos\theta_i)\right.\\&\quad\left.\cdot\exp[j(m-\mu)\phi_i]\sin\theta_i\,d\theta_i\,d\phi_i\right\}\psi_{\mu\nu}^{\prime(1)}(h';\xi',\eta',\phi').\end{aligned}\tag{17}$$

Eq. (17) has been obtained by exchanging the order of summation of Eq. (7) and integration of Eq. (16), which is possible, provided that $r'<r_0$ [1, pp. 19–22]. Hence Eq. (17) is valid in the region $r'<r_0$.

Using a relation given by Friedman and Russek [1, Eq. (15)] we can write

$$\begin{aligned}(2\pi j^p)^{-1}\int_0^{2\pi}\int_0^{\pi/2-j\infty}&\exp(jkr_0\cos\gamma_0)P_p^{m-\mu}(\cos\theta_i)\exp[j(m-\mu)\phi_i]\sin\theta_i\,d\theta_i\,d\phi_i\\&=h_p^{(1)}(kr_0)P_p^{m-\mu}(\cos\theta_0)\exp[j(m-\mu)\phi_0],\end{aligned}\tag{18}$$

where $h_p^{(1)}$ is the spherical Hankel function of the first kind. Substitution of (18) into (17) leads to

$$\psi_{mn}^{(3)}(h;\xi,\eta,\phi)=\sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}A_{\mu\nu}^{(3)}\psi_{\mu\nu}^{\prime(1)}(h';\xi',\eta',\phi'),\qquad r'<r_0\tag{19}$$

where

$$\begin{aligned}A_{\mu\nu}^{(3)}&=2\frac{(-1)^\mu}{N_{\mu\nu}(h')}\sideset{}{'}\sum_{q=0,1}^{\infty}\sideset{}{'}\sum_{s=0,1}^{\infty}\sum_{p}j^{p+\nu-n}\frac{(|\mu|+\mu+s)!}{(|\mu|-\mu+s)!}\\&\quad\cdot d_q^{mn}(h)\,d_s^{\mu\nu}(h')a(m,|m|+q|-\mu,|\mu|+s|p)\\&\quad\cdot h_p^{(1)}(kr_0)P_p^{m-\mu}(\cos\theta_0)\exp[j(m-\mu)\phi_0].\end{aligned}\tag{20}$$

From the relation $R_{mn}^{(4)}(h,\xi)=2R_{mn}^{(1)}(h,\xi)-R_{mn}^{(3)}(h,\xi)$ we obtain

$$\psi_{mn}^{(4)}(h;\xi,\eta,\phi)=2\psi_{mn}^{(1)}(h;\xi,\eta,\phi)-\psi_{mn}^{(3)}(h;\xi,\eta,\phi)$$

which on substitution of the equation $h_p^{(2)}(kr_0) = 2j_p(kr_0) - h_p^{(1)}(kr_0)$, where $h_p^{(2)}$ is the spherical Hankel function of the second kind, gives rise to

$$\psi_{mn}^{(4)}(h; \xi, \eta, \phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} A_{\mu\nu}^{(4)} \psi_{\mu\nu}^{\prime(1)}(h'; \xi', \eta', \phi'), \qquad r' < r_0 \tag{21}$$

where

$$\begin{aligned} A_{\mu\nu}^{(4)} = 2\,\frac{(-1)^\mu}{N_{\mu\nu}(h')} \sum_{q=0,1}^{\infty\,\prime} \sum_{s=0,1}^{\infty\,\prime} \sum_{p} j^{p+\nu-n} \frac{(|\mu| + \mu + s)!}{(|\mu| - \mu + s)!} \\ \cdot d_q^{mn}(h)\, d_s^{\mu\nu}(h') a(m, |m| + q \,| -\mu, |\mu| + s \,| p) \\ \cdot h_p^{(2)}(kr_0) P_p^{m-\mu}(\cos\theta_0) \exp[j(m-\mu)\phi_0]. \end{aligned} \tag{22}$$

In order to obtain the transformation for $r' > r_0$ we substitute Eq. (5) into Eq. (16), then express $\exp(jkr_0 \cos\gamma_0)$ as the following summation [1, Eq. (15)]:

$$\begin{aligned} \exp(jkr_0 \cos\gamma_0) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} j^\nu (2\nu + 1) j_\nu(kr_0) \frac{(\nu - \mu)!}{(\nu + \mu)!} \\ \cdot P_\nu^\mu(\cos\theta_0) P_\nu^\mu(\cos\theta_i) \exp[j\mu(\phi_0 - \phi_i)] \end{aligned} \tag{23}$$

after which we exchange the order of summation and integration (valid if $r' > r_0$) and obtain

$$\begin{aligned} \psi_{mn}^{(3)}(h; \xi, \eta, \phi) = (2\pi j^n)^{-1} \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} j^\nu (2\nu + 1) \frac{(\nu - \mu)!}{(\nu + \mu)!} j_\nu(kr_0) \\ \cdot P_\nu^\mu(\cos\theta_0) \exp(j\mu\phi_0) \int_0^{2\pi} \int_0^{\pi/2 - j\infty} \exp(jkr' \cos\gamma') \\ \cdot S_{mn}(h, \cos\theta_i) P_\nu^\mu(\cos\theta_i) \exp[j(m-\mu)\phi_i] \\ \cdot \sin\theta_i \, d\theta_i \, d\phi_i, \qquad r' > r_0. \end{aligned} \tag{24}$$

Now, by expressing $S_{mn}(h, \cos\theta_i)$ in terms of associated Legendre functions, viz. $S_{mn}(h, \cos\theta_i) = \sum_{q=0,1}' d_q^{mn}(h) P_{|m|+q}^m(\cos\theta_i)$, converting $P_\nu^\mu(\cos\theta_i)$ to $P_\nu^{-\mu}(\cos\theta_i)$ by the well-known relation $P_\nu^\mu(\cos\theta_i) = (-1)^\mu[(\nu+\mu)!/(\nu-\mu)!] P_\nu^{-\mu}(\cos\theta_i)$ and finally applying the linearization expansion given by Eq. (10), we get

$$\begin{aligned} \psi_{mn}^{(3)}(h; \xi, \eta, \phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{q=0,1}^{\infty\,\prime} \sum_{p} (-1)^\mu j^{p+\nu-n} (2\nu + 1) \\ \cdot \frac{(\nu + \mu)!}{(\nu - \mu)!} d_q^{mn}(h) a(m, |m| + q \,| -\mu, \nu \,| p) j_\nu(kr_0) \\ \cdot P_\nu^\mu(\cos\theta_0) \exp(j\mu\phi_0) (2\pi j^p)^{-1} \\ \cdot \int_0^{2\pi} \int_0^{\pi/2 - j\infty} \exp(jkr' \cos\gamma') P_p^{m-\mu}(\cos\theta_i) \exp[j(m-\mu)\phi_i] \\ \cdot \sin\theta_i \, d\theta_i \, d\phi_i, \qquad r' > r_0 \end{aligned} \tag{25}$$

where $p = |m| + q + \nu,\ |m| + q + \nu - 2, \ldots, \big||m| + q - \nu\big|$. Using a relation similar to (18), Eq. (25) can be reduced to

$$\psi_{mn}^{(3)}(h;\xi,\eta,\phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{q=0,1}^{\infty\,\prime} \sum_{p} (-1)^{\mu} j^{p+\nu-n} \cdot (2\nu+1)\frac{(\nu+\mu)!}{(\nu-\mu)!} d_q^{mn}(h) a(m, |m|+q|-\mu, \nu|p) \cdot j_\nu(kr_0) P_\nu^{\mu}(\cos\theta_0)\exp(j\mu\phi_0) h_p^{(1)}(kr') \cdot P_p^{m-\mu}(\cos\theta')\exp[j(m-\mu)\phi'], \qquad r' > r_0. \tag{26}$$

In order to obtain explicit transformation in terms of spheroidal waves we expand the spherical wave function $h_p^{(1)}(kr')P_p^{m-\mu}(\cos\theta')\exp[j(m-\mu)\phi']$ in terms of a complete set of spheroidal wave functions $R_{m-\mu,t}^{(3)}(h',\xi')$, $S_{m-\mu,t}(h',\eta')\exp[j(m-\mu)\phi']$, i.e.

$$h_p^{(1)}(kr')P_p^{m-\mu}(\cos\theta') = \sum_{t=|m-\mu|}^{\infty} \alpha_t(h') R_{m-\mu,t}^{(3)}(h',\xi') S_{m-\mu,t}(h',\eta'). \tag{27}$$

When $h'\xi'$ (or kr') $\to \infty$, then $\eta' \to \cos\theta'$, $h'\xi' \to k'r'$, $h_p^{(1)}(kr') \to (e^{jkr'}/kr')j^{p+1}$ and $R_{m-\mu,t}^{(3)}(h',\xi') \to (e^{jh'\xi'}/h'\xi')j^{t+1} \to (e^{jkr'}/kr')j^{t+1}$, so that Eq. (27) asymptotically reduces to

$$P_p^{m-\mu}(\cos\theta')j^{p+1} = \sum_t j^{t+1}\alpha_t(h') S_{m-\mu,t}(h',\cos\theta'). \tag{28}$$

Multiplying both sides of Eq. (28) by $S_{m-\mu,t}(h',\cos\theta')$ and using the orthogonality properties of associated Legendre functions and spheroidal angle functions [6, p. 22], we get

$$\alpha_t(h') = \frac{2}{2p+1}\frac{(p+m-\mu)!}{(p-m+\mu)!}\frac{j^{p-t}}{N_{m-\mu,t}(h')} d_{p-|m-\mu|}^{m-\mu,t}(h'). \tag{29}$$

Now if $p - |m-\mu|$ is even, then in order that $d_{p-|m-\mu|}^{m-\mu,t}(h')$ and hence α_t do not vanish, $t - |m-\mu|$ must be even, i.e. $t = |m-\mu|, |m-\mu|+2, |m-\mu|+4, \ldots$. Similarly if $p - |m-\mu| =$ odd, then $t - |m-\mu|$ must be odd for nonvanishing $\alpha_t(h')$, i.e. $t = |m-\mu|+1, |m-\mu|+3, |m-\mu|+5, \ldots$. Hence, from Eq. (28) and Eq. (29) we write

$$h_p^{(1)}(kr')P_p^{m-\mu}(\cos\theta') = \sum_{t=|m-\mu|,\,|m-\mu|+1}^{\infty\,\prime} \frac{2}{2p+1}\frac{(p+m-\mu)!}{(p-m+\mu)!}\frac{j^{p-t}}{N_{m-\mu,t}(h')} \cdot d_{p-|m-\mu|}^{m-\mu,t}(h') R_{m-\mu,t}^{(3)}(h',\xi') S_{m-\mu,t}(h',\eta'). \tag{30}$$

From Eqs. (26) and (30) we have finally

$$\psi_{mn}^{(3)}(h;\xi,\eta,\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_{t=|m-\mu|,\,|m-\mu|+1}^{\infty\,\prime} B_{\mu\nu t}\,\psi'^{(3)}_{m-\mu,t}(h';\xi',\eta',\phi'), \qquad r' > r_0 \tag{31}$$

where

$$B_{\mu\nu t} = 2\frac{(-1)^{\mu}}{N_{m-\mu,t}(h')}(2\nu+1)\frac{(\nu+\mu)!}{(\nu-\mu)!} j_\nu(kr_0) P_\nu^{\mu}(\cos\theta_0)\exp(j\mu\phi_0) \cdot \sum_{q=0,1}^{\infty\,\prime}\sum_p \frac{j^{2p+\nu-n-t}}{(2p+1)}\frac{(p+m-\mu)!}{(p-m+\mu)!} d_q^{mn}(h) \cdot d_{p-|m-\mu|}^{m-\mu,t}(h') a(m, |m|+q|-\mu, \nu|p). \tag{32}$$

In a similar manner (as for $\psi^{(3)}$, $r' > r_0$) it can be shown that without any restriction on r',

$$\psi_{mn}^{(1)}(h;\, \xi, \eta, \phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{t=|m-\mu|,\, |m-\mu|+1}^{\infty} B_{\mu\nu t} \psi'^{(1)}_{m-\mu,\, t}(h';\, \xi', \eta', \phi') \tag{33}$$

since the interchange of integrals and summations does not require any restriction in this case. In fact, Eqs. (14) and (33) are equivalent. Finally from Eqs. (31) and (33) we obtain

$$\psi_{mn}^{(4)}(h;\, \xi, \eta, \phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{t=|m-\mu|,\, |m-\mu|+1}^{\infty} B_{\mu\nu t} \psi'^{(4)}_{m-\mu,\, t}(h';\, \xi', \eta', \phi'), \qquad r' > r_0. \tag{34}$$

In summary, we have shown that

$$\psi_{mn}^{(i)}(h;\, \xi, \eta, \phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} A_{\mu\nu}^{(i)} \psi'^{(1)}_{\mu\nu}(h';\, \xi', \eta', \phi'), \qquad r' \leq r_0 \tag{35}$$

$$= \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{t=|m-\mu|,\, |m-\mu|+1}^{\infty\,\prime} B_{\mu\nu t} \psi'^{(i)}_{m-\mu,\, t}(h';\, \xi', \eta', \phi'), \qquad r' \geq r_0 \tag{36}$$

and have provided explicit expressions for $A_{\mu\nu}^{(i)}$ and $B_{\mu\nu t}$.

In Appendix II it is shown that in the limit when the spheroidal coordinates become spherical coordinates, Eqs. (35) and (36) reduce to the addition theorems for spherical coordinates given by Cruzan [3, p. 40].

4. Spheroidal vector wave functions. The vector wave functions are the solutions of the vector wave equation

$$\nabla\nabla \cdot \mathbf{A} - \nabla \times \nabla \times \mathbf{A} + k^2\mathbf{A} = 0 \tag{37}$$

The divergenceless solutions of (37) are designated as **M** and **N** which are related to each other by

$$k\mathbf{N} = \nabla \times \mathbf{M}, \qquad k\mathbf{M} = \nabla \times \mathbf{N}. \tag{38}$$

Together with the constraint $\nabla \cdot \mathbf{M} = 0$, **M** satisfies Eq. (37) only when

$$\mathbf{M} = \nabla\psi \times \mathbf{a}$$

where ψ is the solution of the scalar Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0$$

and **a** is a constant or radius vector. Now each of the coordinate unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$ is a constant unit vector. For the sake of simplicity of coordinate transformations between Cartesian and spheroidal coordinates, the Cartesian coordinate system is chosen to have its origin at the center and z-axis along the positive direction of the axis of symmetry of the spheroidal system.

The three Cartesian unit vectors generate three distinct classes of **M** vectors:

$$\mathbf{M}_{mn}^{a(i)}(h;\, \xi, \eta, \phi) = \nabla\psi_{mn}^{(i)}(h;\, \xi, \eta, \phi) \times \hat{a} \tag{39}$$

where $i = 1, 3, 4$ and $\hat{a} = \hat{x}, \hat{y}, \hat{z}$. Similarly the three distinct classes of **N** vectors generated are:

$$\mathbf{N}_{mn}^{a(i)}(h;\, \xi, \eta, \phi) = \frac{1}{k} \nabla \times \mathbf{M}_{mn}^{a(i)}(h;\, \xi, \eta, \phi). \tag{40}$$

In the following section we shall establish the translational addition theorems for $\mathbf{M}_{mn}^{a(i)}$ and $\mathbf{N}_{mn}^{a(i)}$ vectors. The explicit expressions for **M** and **N** are given in detail in Flammer's text [6, pp. 69–78] as even (e) and odd (o) functions—

$$\mathbf{M}_{\substack{e\\o}mn}^{a(i)} \quad \text{and} \quad \mathbf{N}_{\substack{e\\o}mn}^{a(i)}.$$

$\mathbf{M}_{mn}^{a(i)}$ and $\mathbf{N}_{mn}^{a(i)}$ are related to these by the equations:

$$(\mathbf{M}_{mn}^{a(i)}, \mathbf{N}_{mn}^{a(i)}) = (1, j)\begin{pmatrix} \mathbf{M}_{emn}^{a(i)} & \mathbf{N}_{emn}^{a(i)} \\ \mathbf{M}_{omn}^{a(i)} & \mathbf{N}_{omn}^{a(i)} \end{pmatrix}. \tag{41}$$

5. Transformations of $\mathbf{M}_{mn}^{x,y,z(1,3,4)}(h;\ \xi,\ \eta,\ \phi)$ and $\mathbf{N}_{mn}^{x,y,z(1,3,4)}(h;\ \xi,\ \eta,\ \phi)$. From Eqs. (35), (36) and (39) we immediately write the addition theorems for the **M** vectors, viz.

$$\mathbf{M}_{mn}^{a(i)}(h;\ \xi,\ \eta,\ \phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} A_{\mu\nu}^{(i)} \mathbf{M}_{\mu\nu}^{a'(i)}(h';\ \xi',\ \eta',\ \phi'), \qquad r' \leq r_0 \tag{42}$$

$$= \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{t=|m-\mu|,\ |m-\mu|+1}^{\infty} B_{\mu\nu t} \mathbf{M}_{m-\mu,t}^{a'(i)}(h';\ \xi',\ \eta',\ \phi'), \qquad r' \geq r_0 \tag{43}$$

where $a = x,\ y,\ z,\ a' = x',\ y',\ z'$ and $i = 1,\ 3,\ 4$. The above relations follow because $(\hat{x},\ \hat{y},\ \hat{z}) = (\hat{x}',\ \hat{y}',\ \hat{z}')$ and the gradient (∇) is invariant under coordinate transformations. From Eqs. (41), (42) and (43) **N** vectors are given as

$$\mathbf{N}_{mn}^{a(i)}(h;\ \xi,\ \eta,\ \phi) = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} A_{\mu\nu}^{(i)} \mathbf{N}_{\mu\nu}^{a'(1)}(h';\ \xi',\ \eta',\ \phi'), \qquad r' \leq r_0 \tag{44}$$

$$= \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \sum_{t=|m-\mu|,\ |m-\mu|+1}^{\infty\,\prime} B_{\mu\nu t} \mathbf{N}_{m-\mu,t}^{a'(i)}(h';\ \xi',\ \eta',\ \phi'), \qquad r' \geq r_0. \tag{45}$$

Eqs. (44) and (45) are the required translational expansions of spheroidal vector wave functions from O-system to O'-system.

Appendix I. Determination of linearization coefficients. The linearization expansion of (10) can be restated as follows:

$$P_n^m(x) p_\nu^\mu(x) = \sum_p a(m,\ n\,|\,\mu,\ \nu\,|\,p) P_p^{m+\mu}(x), \qquad -1 \leq x \leq +1 \tag{I-1}$$

where $p = n + \nu,\ n + \nu - 2,\ n + \nu - 4, \ldots,\ |n - \nu|$ if $|n - \nu| \geq |m + \mu|$. If $|n - \nu| < |m + \mu|$, the lower limit $|n - \nu|$ for p is replaced by $|m + \mu|$ or $|m + \mu| + 1$ according as $n + \nu + |m + \mu|$ is even or odd. This is due to the fact that $P_{m+\mu}^p(x) \equiv 0$ for $p < |m + \mu|$. The lower limit of p will be designated as $p_{\min}$.

The coefficients a may be identified with a product of two Wigner 3-j symbols which are associated with the coupling of two angular momentum eigenvectors:

$$a(m,\ n\,|\,\mu,\ \nu\,|\,p) = (-1)^{m+\mu}(2p+1)\left[\frac{(n+m)!\,(\nu+\mu)!\,(p-m-\mu)!}{(n-m)!\,(\nu-\mu)!\,(p+m+\mu)!}\right]^{1/2} \cdot \begin{pmatrix} n & \nu & \mathrm{p} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} n & \nu & p \\ m & \mu & -m-\mu \end{pmatrix} \tag{I-2}$$

where $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the Wigner 3-j symbol of which there are several definitions, all involving summations of a multitude of factorials. Consequently straightforward calculations using (I-2) are very inefficient. Obviously a recursion relation for the $a(\cdot)$ in which only the index p cycles would be highly desirable, especially for machine computation; such a recursion relation has been given by Bruning and Lo [4, pp. 389–390] for the special cases of $\mu = -m$. In this appendix, using Bruming and Lo's recursion and the recursion formulas derived in the Appendix A of [3], recursion formulas are developed for machine computation of $a(\cdot)$ for any values of μ and m.

From Cruzan [3, Eq. A-16], for m, n and ν held fixed, we get

$$b_1(\mu, p)a(\mu \mid p) = b_2(\mu)a(\mu - 1 \mid p) + b_3(\mu, p)a(\mu + 1 \mid p) \tag{I-3}$$

where

$$b_1(\mu, p) = (p + m + \mu)(p - m - \mu + 1) + (\nu - \mu)(\nu + \mu + 1) - (n + m)(n - m + 1),$$
$$b_2(\mu) = (\nu + \mu)(\nu - \mu + 1), \qquad b_3(\mu, p) = (p - m - \mu)(p + m + \mu + 1),$$
$$a(\mu \mid p) = a(m, n \mid \mu, \nu \mid p).$$

Now, replacing μ by $\mu + 1$ in Eq. (I-3), we obtain

$$a(\mu + 2 \mid p) = \frac{1}{b_3(\mu + 1, p)}[b_1(\mu + 1, p)a(\mu + 1 \mid p) - b_2(\mu + 1)a(\mu \mid p)]. \tag{I-4}$$

Further, from Eq. (I-3) we can write the following relations:

$$a(\mu - 1 \mid p + 1) = \frac{1}{b_2(\mu)}[b_1(\mu, p + 1)a(\mu \mid p + 1) - b_3(\mu, p + 1)a(\mu + 1 \mid p + 1)], \tag{I-5}$$

$$a(\mu - 1 \mid p - 1) = \frac{1}{b_2(\mu)}[b_1(\mu, p - 1)a(\mu \mid p - 1) - b_3(\mu, p - 1)a(\mu + 1 \mid p - 1)]. \tag{I-6}$$

From Eq. A-10 of [3] we get

$$c_1(\mu, p)a(\mu + 1 \mid p + 1) - c_2(\mu, p)a(\mu - 1 \mid p + 1) - c_3(\mu, p)a(\mu \mid p + 1)$$
$$= d_1(\mu, p)a(\mu + 1 \mid p - 1) - d_2(\mu, p)a(\mu - 1 \mid p - 1) + d_3(\mu, p)a(\mu \mid p - 1) \tag{I-7}$$

where

$$c_1(\mu, p) = (2p - 1)(p + m + \mu + 1)(p + m + \mu + 2),$$
$$c_2(\mu, p) = (2p - 1)(\nu + \mu)(\nu - \mu + 1), \qquad c_3(\mu, p) = 2\mu(2p - 1)(p + m + \mu + 1),$$
$$d_1(\mu, p) = (2p + 3)(p - m - \mu)(p - m - \mu - 1),$$
$$d_2(\mu, p) = (2p + 3)(\nu + \mu)(\nu - \mu + 1), \qquad d_3(\mu, p) = 2\mu(2p + 3)(p - m - \mu).$$

By eliminating the terms containing $\mu - 1$ in Eq. (I-7) with the help of Eqs. (I-5) and (I-6) we get

$$e_1(\mu, p)a(\mu + 1 \mid p + 1) - e_2(\mu, p)a(\mu \mid p + 1)$$
$$= e_3(\mu, p)a(\mu + 1 \mid p - 1) + e_4(\mu, p)a(\mu \mid p - 1) \tag{I-8}$$

where

$$e_1(\mu, p) = c_1(\mu, p) + \frac{c_2(\mu, p)}{b_2(\mu)} b_3(\mu, p + 1) = c_1(\mu, p) + (2p - 1)b_3(\mu, p + 1)$$

$$e_2(\mu, p) = c_3(\mu, p) + \frac{c_2(\mu, p)}{b_2(\mu)} b_1(\mu, p + 1) = c_3(\mu, p) + (2p - 1)b_1(\mu, p + 1),$$

$$e_3(\mu, p) = d_1(\mu, p) + (2p + 3)b_3(\mu, p - 1),$$

$$e_4(\mu, p) = d_3(\mu, p) - (2p + 3)b_1(\mu, p - 1).$$

Replacing p by $p - 1$ in Eq. (I-8) we have

$$a(\mu + 1 \mid p - 2) = \frac{1}{e_3(\mu, p - 1)} [e_1(\mu, p - 1)a(\mu + 1 \mid p) - e_2(\mu, p - 1)a(\mu \mid p) - e_4(\mu, p - 1)a(\mu \mid p - 2)]. \quad \text{(I-9)}$$

The recursion relations given by Eqs. (I-4) and (I-9) are the required formulas which extend the Bruning and Lo's recursion to the general case. Bruning and Lo's recursion ($\mu = -m$) is given as follows:

$$a(-m \mid n + v) = \frac{(2n - 1)!!\,(2v - 1)!!}{(2n + 2v - 1)!!} \frac{(n + v)!}{(n - m)!\,(v + m)!}, \quad \text{(I-10)}$$

$$a(-m \mid n + v - 2) = \frac{(2n + 2v - 3)}{(2n - 1)(2v - 1)(n + v)} \cdot [nv - m^2(2n + 2v - 1)]a(-m \mid n + v), \quad \text{(I-11)}$$

where $(2q - 1)!! = (2q - 1)(2q - 3)$—$3.1$; $(-1)!! \equiv 1$ and a downward recursion scheme of

$$\alpha_{p-3}a(-m \mid p - 4) - (\alpha_{p-2} + \alpha_{p-1} - 4m^2)a(-m \mid p - 2) + \alpha_p a(-m \mid p) = 0, \quad \text{(I-12)}$$

where

$$\alpha_p = \frac{[(n + v + 1)^2 - p^2][p^2 - (n - v)^2]}{4p^2 - 1}.$$

Also, from Eq. (I-2) we get

$$a(-m + 1 \mid n + v) = -(2n + 2v + 1)\left[\frac{(n + m)!\,(v - m + 1)!\,(n + v - 1)!}{(n - m)!\,(v + m - 1)!\,(n + v + 1)!}\right]^{1/2} \cdot \begin{pmatrix} n & v & n + v \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} n & v & (n + v) \\ m & -m + 1 & -1 \end{pmatrix}, \quad \text{(I-13)}$$

where the last two factors are Wigner 3-j symbols. The first, according to Edmonds [8], is

$$\begin{pmatrix} n & v & n + v \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{n-v}\left[\frac{(2v)!\,(2n)!}{(2(n + v) + 1)!}\right]^{1/2} \frac{(n + v)!}{n!\,v!}$$

and the second [8] is

$$\begin{pmatrix} n & \nu & n+\nu \\ m & -m+1 & -1 \end{pmatrix} = (-1)^{n-\nu+1} \cdot \left[\frac{(2n)!\,(2\nu)!\,(n+\nu+1)!\,(n+\nu-1)!}{(2n+2\nu+1)!\,(n-m)!\,(n+m)!\,(\nu+m-1)!\,(\nu-m+1)!}\right]^{1/2}.$$

Now for given values of m and μ, $\mu = -m \pm \sigma$ where σ is a positive integer including zero.

(a) *Determination of* $a(\mu\,|\,p)$ *when* $\mu = -m + \sigma$. The machine computation is done according to the graphical schemes of Figs. 2 and 3. In each figure $a(\mu\,|\,p)$ are the grid points obtained by the intersection of columns (constant p) and rows (constant μ). The number at a grid point refers to the equation number of this appendix from which a at the grid point is obtained. The number on each arrow (the arrows represent the transition from one point to the other) refers to the equation number of this appendix representing a recursion. In Fig. 2 all the essential grid points, i.e. those situated on column

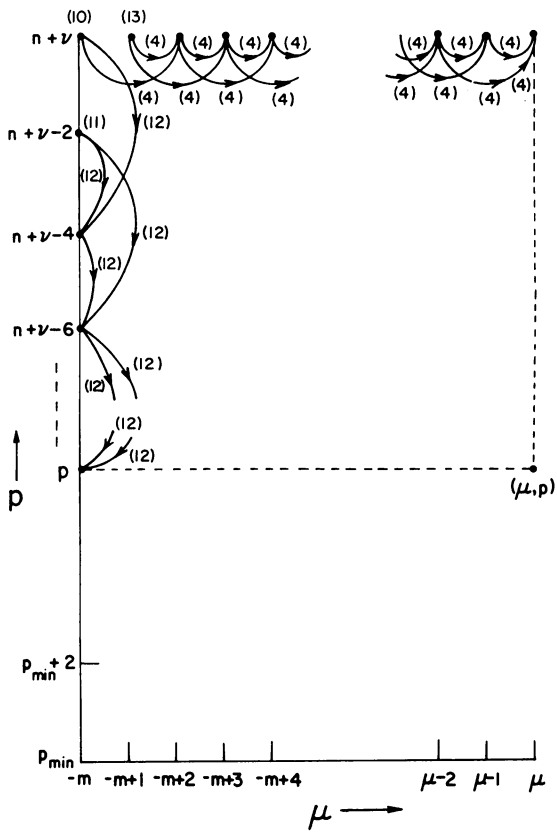

FIG. 2. Determination of $a(\mu\,|\,p)$ at key grid points.

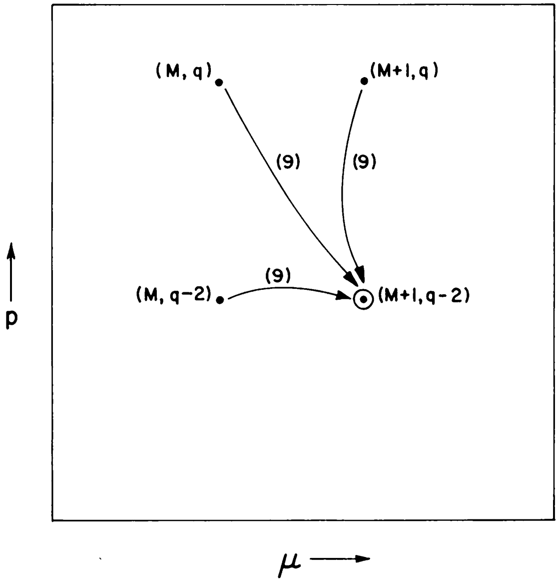

FIG. 3. General recursion scheme for $a(\mu\,|\,p)$.

TABLE 1. Expansion coefficients of the linearization expansion; $x = 0.3$.

I		II	III	IV
m, n, μ, ν	p	$a(m, n\,|\,\mu, \nu\,|\,p)$	$\sum_{p=p_{min}}^{n+\nu} a(m, n\,|\,\mu, \nu\,|\,p)P_p^{m+\mu}(x)$	$P_n^m(x)P_\nu^\mu(x)$
1, 1, 0, 2	1	−0.200000	−0.348188	−0.348188
	3	0.200000		
1, 2, 0, 3	1	−0.257142	−0.328393	−0.328393
	3	0.666667		
	5	0.190476		
1, 3, 0, 3	2	0.952375×10^{-1}	0.301027	0.301027
	4	0.116883		
	6	0.216450		
1, 2, 1, 3	3	0.200000	−0.675673	−0.675675
	5	0.142857		
2, 3, −1, 2	1	−0.857142	−0.585956	−0.585957
	3	−0.333333		
	5	0.190476		
2, 3, −1, 3	2	−0.238095	0.268563	0.268564
	4	−0.155844		
	6	0.108225		
2, 3, 0, 3	2	−0.476189	-0.156633×10^{1}	-0.156634×10^{1}
	4	−0.389609		
	6	0.865800×10^{-1}		
2, 3, 1, 3	4	0.779218×10^{-1}	-0.322276×10^{-1}	-0.322276×10^{-1}
	6	0.649350×10^{-1}		
2, 3, 2, 3	4	0.194804	0.167689×10^{2}	0.167690×10^{2}
	6	0.432899×10^{-1}		

$p = n + \mu$ and row $\mu = -m$, are obtained. In Fig. 3 a general scheme of determination of a at a grid point is shown.

(b) *Determination of* $a(\mu \mid p)$ *when* $\mu = -m - \sigma$. From the well-known formula

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)$$

we can write

$$P_n^m(x) P_\nu^\mu(x) = (-1)^{-(m+\mu)} \frac{(n+m)!\,(\nu+\mu)!}{(n-m)(\nu-\mu)!} P_n^{-m}(x) P_\nu^{-\mu}(x).$$

If we let $-m = m'$ and $-\mu = \mu'$, then $\mu' = -m' + \sigma$. Thus $a(m', n \mid \mu', \nu \mid p)$, $p = n + \nu$, $n + \nu - 2, \ldots, p_{\min}$ can be obtained from the computational procedure (a). When this is done, the required coefficients $a(m, n \mid \mu, \nu \mid p)$ are obtained by

$$a(m, n \mid \mu, \nu \mid p) = \frac{(n+m)!\,(\nu+\mu)!\,(p+\sigma)!}{(n-m)!\,(\nu-\mu)!\,(p-\sigma)!} a(m', n \mid \mu', \nu \mid p).$$

Some results of machine computation are presented in Table 1.

Appendix II. Reduction to spherical limit. When $h \to 0$, $h' \to 0$, the spheroidal coordinates $(h; \xi, \eta, \phi)$ and $(h'; \xi', \eta', \phi')$ go over to spherical coordinates (r, θ, ϕ) and (r', θ', ϕ'). In this limit

$$S_{mn}(h, \eta) = \sum_{q=0,1}^{\infty\,'} d_q^{mn}(h) P_{|m|+q}^m(\eta) \to P_n^m(\cos\theta),$$

$$S_{\mu\nu}(h', \eta') = \sum_{s=0,1}^{\infty\,'} d_s^{\mu\nu}(h) P_{|\mu|+s}^\mu(\eta') \to P_\nu^\mu(\cos\theta'),$$

$$N_{\mu\nu}(h) \to \frac{2}{(2\nu+1)} \frac{(\nu+\mu)!}{(\nu-\mu)!},$$

$$\{R_{mn}^{(1)}(h, \xi), R_{mn}^{(3)}(h, \xi), R_{mn}^{(4)}(h, \xi)\} \to \{j_n(kr), h_n^{(1)}(kr), h_n^{(2)}(kr)\},$$

$$\{R_{\mu\nu}^{(1)}(h', \xi'), R_{\mu\nu}^{(3)}(h', \xi'), R_{\mu\nu}^{(4)}(h', \xi')\} \to \{j_\nu(kr'), h_\nu^{(1)}(kr'), h_\nu^{(2)}(kr')\}.$$

From Eqs. (13), (15), (18), (20) and (22) we get

$$A_{\mu\nu}^{(i)} = (2\nu+1) \frac{(\nu-\mu)!}{(\nu+\mu)!} (-1)^\mu \sum_{q=0,1}^{\infty} \sum_{s=0,1}^{\infty} \sum_p j^{p+\nu-n} d_q^{mn}(h)$$

$$\cdot\, d_s^{\mu\nu}(h') \frac{(|\mu|+\mu+s)!}{(|\mu|-\mu+s)!} a(m, |m|+q \mid -\mu, |\mu|+s \mid p)$$

$$\cdot\, (4\pi j^p)^{-1} \int_0^{2\pi} \int_c \exp(jkr_0 \cos\gamma_0) P_p^{m-\mu}(\cos\theta_i) \exp[j(m-\mu)\phi_i] \sin\theta_i \, d\theta_i \, d\phi_i, \tag{II-1}$$

where $\int_c = \int_0^\pi$ for $i = 1$, $2\int_0^{\pi/2 - j\infty}$ for $i = 3$ and $2\int_{\pi/2 - j\infty}^{\pi}$ for $i = 4$.

From the linearization expansion of $P^m_{|m|+q}(\cos\theta_i)P^{-\mu}_{|\mu|+s}(\cos\theta_i)$ we get

$$A^{(i)}_{\mu\nu} = (-1)^\mu(2\nu+1)j^{\nu-n}(2\pi)^{-1}\int_0^{2\pi}\int_c \exp(jkr_0\cos\gamma_0) \cdot P^m_n(\cos\theta_i)P^{-\mu}_\nu(\cos\theta_i)\exp[j(m-\mu)\phi_i]\sin\theta_i\,d\theta_i\,d\phi_i. \tag{II-2}$$

Eq. (II-2), an application of linearization expansion for $P^m_n(\cos\theta_i)P^{-\mu}_\nu(\cos\theta_i)$ and Eqs. (13) and (18), yield

$$A^{(i)}_{\mu\nu} = (-1)^\mu(2\nu+1)\sum_p j^{\nu+p-n}a(m,n\,|\,-\mu,\nu\,|\,p) \cdot z_p(kr_0)P^{m-\mu}_p(\cos\theta_0)\exp[j(m-\mu)\theta_0] \tag{II-3}$$

where $z_p = j_p$, $h^{(1)}_p$, $h^{(2)}_p$ according as $i = 1, 3, 4$ and $p = n+\nu, n+\nu-2, \ldots, |n-\nu|$. Hence, Eq. (35) reduces to

$$z_n(kr)P^m_n(\cos\theta)\exp(jm\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_p(-1)^\mu j^{\nu+p-n}(2\nu+1) \cdot a(m,n\,|\,-\mu,\nu\,|\,p)j_\nu(kr')z_p(kr_0)P^\mu_\nu(\cos\theta') \cdot P^{m-\mu}_p(\cos\theta_0)\exp[j(m-\mu)\phi_0]\exp(j\mu\phi'), \qquad r' \le r_0. \tag{II-4}$$

Similarly, it can be shown that Eq. (36) yields in the spherical limit:

$$z_n(kr)P_n(\cos\theta)\exp(jm\phi) = \sum_{\nu=0}^{\infty}\sum_{\mu=-\nu}^{\nu}\sum_p(-1)^\mu j^{\nu+p-n} \cdot (2\nu+1)a(m,n\,|\,-\mu,\nu\,|\,p)j_\nu(kr_0)z_p(kr')P^\mu_\nu(\cos\theta_0) \cdot P^{m-\mu}_p(\cos\theta')\exp[j(m-\mu)\phi']\exp(j\mu\phi_0), \qquad r' \ge r_0. \tag{II-5}$$

Eqs. (II-4) and (II-5) are in fact the translational addition theorems for spherical wave functions given by Cruzan [3, p. 40].

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