SUTURES IN STRETCHED MEMBRANES*

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Abstract. A suture function defining a one-to-one mapping between the opposite sides of equal or unequal lengths of a hole is introduced. This function specifies the precise manner in which the opposite sides of the hole are sutured. The problem is first formulated in the general context of finite plane-stress theory, and then specialized for large stretching. The suturing of an elliptic hole is worked out as an example.

1. Introduction. When an elastic solid containing a crack is deformed, the crack is usually opened up into a hole. The shape of the hole is either infinitesimally "flat" (linear theory) or completely arbitrary (nonlinear theory). This problem has received much attention in the theory of elasticity because of the prominent role it plays in the theory of fracture mechanics.

The opposite problem is the closing up of a hole by suturing the opposite sides of the hole together. This problem has received very little attention in the literature although the "operation" has been widely used in plastic surgery as well as in manufacturing inflatable structures. A striking feature of this operation is that it is by nature a nonlinear problem; and, as such, a completely nonlinear formulation must be used. While the theory of nonlinear elasticity has been well established, the boundary conditions, and hence the boundary-value problem, associated with a suture operation are not of the commonly encountered types. This is particularly so when the opposite sides of a hole are taken to be of unequal lengths. We introduce the concept of a suture function which assigns a one-to-one mapping between the sides to be sutured together. Once this is done, the usual continuity requirements may be applied without any ambiguity.

There are two difficulties involved in solving such a boundary value problem. The first has to do with the nonlinearity of the governing equations which, of course, is not a consequence of this particular problem. The second originates from the fact that a thin membrane wrinkles under compression, a situation that is likely to appear around a sutured hole unless the membrane is sufficiently stretched. To kill both birds with one stone, we restrict ourselves to the cases where the applied stretches are indeed very large. That this assumption actually has the effect of simplifying the governing equations was discussed in [1, 2].

Finite plane-stress theory is reviewed in Sec. 2. The concept of a suture function as well as the associated boundary conditions are presented in Sec. 3. These general results are then specialized for large stretching problems in Sec. 4, where general solutions are

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expressed in terms of two holomorphic functions. The suturing of an elliptic hole is worked out in great detail as an example in Sec. 5.

2. Finite plane-stress theory. In this section we outline the theory to be used, largely without derivation. Our exposition follows closely that of [2, 3, 4].

Let (Z_1, Z_2, Z_3) be rectangular cartesian coordinates, and let D be the domain of the (Z_1, Z_2)-plane defining the shape of the middle plane of a thin elastic membrane in its undeformed configuration. We assume a plane-stress deformation such that the position of a point $(Z_1, Z_2, 0)$ after deformation is $(z_1, z_2, 0)$. The deformation may be characterized by a transformation

$$z_\alpha = z_\alpha(Z_1, Z_2) \quad \text{for all} \quad (Z_1, Z_2) \in D \tag{2.1}**$$

which maps D onto a domain d of the $Z_3 = 0$ plane.

Let $F_{\alpha\beta}$ be the components of the deformation-gradient tensor $\mathbf{F}$ associated with (2.1) and J its Jacobian determinant, whence

$$F_{\alpha\beta} = z_{\alpha,\beta}, \qquad J = \det \mathbf{F} > 0 \quad \text{on} \quad D. \tag{2.2}†$$

The right-Cauchy–Green deformation tensor $\mathbf{G}$ is just

$$\mathbf{G} = \mathbf{F}^T\mathbf{F}; \tag{2.3}$$

where the superscript T indicates transposition, and its fundamental scalar invariants may be taken as

$$J = (\det \mathbf{G})^{1/2} = \det \mathbf{F} = \Lambda_1 \Lambda_2, \tag{2.4}$$

$$I = (F_{\alpha\beta}F_{\alpha\beta} + 2J)^{1/2} = \Lambda_1 + \Lambda_2, \tag{2.5}$$

where Λ_1 and Λ_2 are the in-plane principal stretch ratios. The transverse stretch ratio Λ_3 may be deduced from the usual plane-stress assumptions.

For a homogeneous isotropic elastic material, the strain energy density per unit volume of the undeformed solid is a function of the three (three-dimensional) invariants involved, and the corresponding plane-stress strain energy density W per unit area of the undeformed middle surface may be taken as a function of I and J.

In terms of W, the components $t_{\alpha\beta}$ of the Cauchy stress-resultant tensor $\mathbf{t}$ are

$$t_{\alpha\beta} = \frac{1}{IJ}\frac{\partial W}{\partial I} F_{\alpha\gamma}F_{\beta\gamma} + \left(\frac{1}{I}\frac{\partial W}{\partial I} + \frac{\partial W}{\partial J}\right)\delta_{\alpha\beta} \tag{2.6}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. The Piola stress-resultant tensor $\mathbf{T}$ associated with $\mathbf{t}$ is defined by

$$\mathbf{T} = J\mathbf{t}(\mathbf{F}^{-1})^T, \tag{2.7}$$

where $\mathbf{F}^{-1}$ is the inverse of $\mathbf{F}$. The components $T_{\alpha\beta}$ of $\mathbf{T}$ are just

$$T_{\alpha\beta} = \frac{1}{I}\frac{\partial W}{\partial I} F_{\alpha\beta} + \left(\frac{1}{I}\frac{\partial W}{\partial I} + \frac{\partial W}{\partial J}\right)e_{\alpha\mu}e_{\beta\nu}F_{\mu\nu}, \tag{2.8}$$

** Greek subscripts range over the integers (1, 2) and summation over repeated subscripts is taken for granted.

† Subscripts preceded by a comma indicate differentiation with respect to the corresponding cartesian coordinates.

where $e_{\alpha\beta}$ are the components of the two-dimensional alternator. In the absence of body forces, the stress equations of equilibrium are either

$$\frac{\partial T_{\alpha\beta}}{\partial Z_\beta} = 0 \quad \text{on} \quad D, \tag{2.9}$$

or

$$\frac{\partial t_{\alpha\beta}}{\partial z_\beta} = 0, \qquad t_{\alpha\beta} = t_{\beta\alpha} \quad \text{on} \quad d. \tag{2.10}$$

Using (2.2) and (2.8), we obtain from (2.9) the displacement equations of equilibrium

$$\left(\frac{1}{I}\frac{\partial W}{\partial I} z_{\alpha,\beta}\right)_{,\beta} + e_{\alpha\mu} e_{\beta\nu}\left(\frac{1}{I}\frac{\partial W}{\partial I} + \frac{\partial W}{\partial J}\right)_{,\beta} z_{\mu,\nu} = 0 \quad \text{on} \quad D. \tag{2.11}$$

We find it convenient to use either (2.9) or (2.11) as the basis of our analyses. We note in passing that (2.9) are identically satisfied if

$$T_{\alpha\beta} = e_{\alpha\mu} e_{\beta\nu} \Phi_{\mu,\nu}, \tag{2.12}$$

where $\Phi_\alpha = \Phi_\alpha(Z_1, Z_2)$ are two stress functions. This representation, however, is not that useful unless W is of such a form that a simple compatibility condition can be derived. This is the case for the class of problems discussed in Sec. 4.

Let C be a curve in D defined by

$$Z_\alpha = C_\alpha(L), \tag{2.13}$$

where L measures the arc length along C. The components N_α of the unit normal vector $\mathbf{N}$ of C are defined by

$$N_\alpha = e_{\beta\alpha} C'_\beta(L), \tag{2.14}$$

where primes indicate differentiation of functions of a single variable. The sense of L is chosen in such a way that $\mathbf{N}$ is pointing away from D.

The image of C under the mapping (2.1) is a curve c in d defined by

$$z_\alpha = z_\alpha(\mathbf{C}(L)) = z_\alpha(C_1(L), C_2(L)), \qquad L \in C. \tag{2.15}$$

If we denote the arc length along c by l, then

$$(dl)^2 = z_{\alpha,\beta} z_{\alpha,\gamma} C'_\beta C'_\gamma (dL)^2, \tag{2.16}$$

which may be used to define

$$l = l(L) \quad \text{and} \quad L = L(l) \quad \text{for} \quad L \in C \quad \text{and} \quad l \in c. \tag{2.17}$$

It follows that c has the parametric representation

$$z_\alpha = c_\alpha(l) = z_\alpha(\mathbf{C}(L(l))). \tag{2.18}$$

The components n_α of the unit normal vector $\mathbf{n}$ of c are just

$$n_\alpha = e_{\beta\alpha} c'_\beta(l) \tag{2.19}$$

where, as in (2.14), the sense of l is chosen in such a way that $\mathbf{n}$ is pointing away from d.

Let p_α and P_α be respectively the components of the Cauchy and Piola traction vectors $\mathbf{p}$ and $\mathbf{P}$, then

$$p_\alpha(l) = t_{\alpha\beta}(\mathbf{c}(l))n_\beta(l) \quad \text{for all} \quad l \in c, \tag{2.20}$$

$$P_\alpha(L) = T_{\alpha\beta}(\mathbf{C}(L))N_\beta(L) \quad \text{for all} \quad L \in C. \tag{2.21}$$

It follows from

$$p_\alpha(l)\, dl = P_\alpha(L)\, dL \tag{2.22}$$

that

$$p_\alpha(l(L))l'(L) = P_\alpha(L) \quad \text{for all} \quad L \in C. \tag{2.23}$$

3. Boundary conditions pertinent to a suture. Let C be a simple contour describing the shape of a hole in a thin elastic membrane in its undeformed configuration. Then C is a part of the boundary ∂D of D. The purpose of this section is to determine the boundary conditions along C when the hole defined by C is closed up by a suture operation. The boundary conditions along the remaining portion of ∂D are assumed to be of the types commonly encountered in the theory of elasticity.

To define the contour C, we use the parametric representation:

$$C\colon Z_\alpha = C_\alpha(L), \qquad -L_1 \le L \le L_0, \tag{3.1}$$

where L measures the arc length along C so that $L_1 + L_0$ is the total length of the contour and

$$C_\alpha(-L_1) = C_\alpha(L_0). \tag{3.2}$$

The functions C_α are assumed to be continuous but are otherwise arbitrary. Thus, holes with sharp corners are permissible configurations.

Before proceeding to the determination of the boundary conditions, we must give the term "suture operation" a precise meaning. For this purpose we define intervals

$$I^+ = \{L \mid 0 \le L \le L_0\}, \qquad I^- = \{L \mid -L_1 \le L \le 0\}, \tag{3.3}$$

and introduce a function $S(L)$ for all $L \in I^+$. The function $S(L)$, together with its first derivative, is assumed to be continuous and satisfies the conditions:

$$\begin{aligned} &S(L) \in I^-, \qquad S'(L) \ne 0, \infty \quad \text{for all} \quad L \in I^+, \\ &S(0) = 0, \qquad S(L_0) = -L_1. \end{aligned} \tag{3.4}$$

We shall call $S(L)$ the *suture function* and define:

Suture operation. A suture operation performed on a hole defined by (3.1) is uniquely determined by a suture function $S(L)$. It requires that the pair of points $C_\alpha(L)$ and $C_\alpha(S(L))$ be sutured together. We shall also use the term *uniform suture* to indicate the special operation defined by $S(L) = -L$. It is clear that for such an operation to be possible the condition $L_0 = L_1$ must be satisfied.

To give the word *suture* a precise meaning, we shall henceforth use it to mean the curve assumed by the sutured hole in the deformed configuration. The displacements associated with the pair of points $C_\alpha(L)$ and $C_\alpha(S(L))$ must be the same. Recalling that the position of a point $(Z_1, Z_2, 0)$ after deformation is $(z_1, z_2, 0)$, the above condition becomes

$$z_\alpha(\mathbf{C}(L)) = z_\alpha(\mathbf{C}(S(L))) \quad \text{for all} \quad L \in I^+. \tag{3.5}$$

Let l measure the arc length along the contour c formed by the "two sides" of a suture. Then we may use the representation (2.17) to write

$$l = l(L) \quad \text{for} \quad -L_1 \le L \le L_0, \tag{3.6}$$

where $l(0) = 0$. In view of (3.5), the function $l(L)$ must satisfy the condition

$$l(L) = -l(S(L)) \quad \text{for all} \quad L \in I^+ \tag{3.7}$$

and hence

$$l'(L) = -l'(S(L))S'(L) \quad \text{for all} \quad L \in I^+. \tag{3.8}$$

The Cauchy traction vector $\mathbf{p}(l)$ on the two sides of a suture must be equal and opposite. It follows from this requirement and the relations (2.23) and (3.8) that

$$P_\alpha(L) = S'(L)P_\alpha(S(L)) \quad \text{for all} \quad L \in I^+. \tag{3.9}$$

Eqs. (3.5) and (3.9) are the required boundary conditions.

4. Sutures subjected to large stretching. The mathematical problem associated with a suture operation requires the solution of (2.11), (3.5), (3.9) together with certain additional boundary conditions of the ordinary types. While the conditions (3.5) and (3.9) do not seem to have posed additional complications to the problem, the basic difficulties involved in solving (2.11) are still there. Furthermore, it is intuitively clear that in a general situation wrinkles will be induced by a suture operation. The theory presented here cannot be used to predict the wrinkling phenomenon. It simply accepts a compressive stress resultant, large or small, as an admissible solution. To kill both birds, i.e., (2.11) and wrinkle, with one stone we assume that the membrane is subjected to a large stretching. It is again intuitively clear that when the stretching is sufficiently large the stresses will be tensile everywhere and hence the membrane will be taut everywhere. As to Eq. (2.11), it permits certain simplification when the strain-energy density function W possesses certain asymptotic property for large stretch ratios.

It was discussed in [2] that for a rather large class of materials the strain-energy density function W may be approximated by the form

$$W = C[I^2 - (2 + c)J], \tag{4.1}$$

where C and c are constants, provided that I^2 and J are large. The case $c = 0$ may be deduced from a neo-Hookean material and was first studied by Wong and Shield [1]. Some of their results indicate that the approximation is quite satisfactory even if the strains involved are only around 60%. The simplification resulted from (4.1) is very substantial.

Substituting (4.1) into the equations obtained in Sec. 2, one finds that the functions z_α as well as the stress functions Φ_α defined by (2.12) are all harmonic [2]. To this end we introduce complex variables and complex differentiations:

$$Z = Z_1 + iZ_2, \qquad z = z_1 + iz_2, \tag{4.2}$$

$$C(2 + c)\phi = \Phi = \Phi_1 + i\Phi_2, \tag{4.3}$$

$$2(\ )_{,Z} = \partial(\ )/\partial Z_1 - i\,\partial(\ )/\partial Z_2, \tag{4.4}$$

$$2(\ )_{,\bar{Z}} = \partial(\ )/\partial Z_1 + i\,\partial(\ )/\partial Z_2, \tag{4.5}$$

where, and throughout this paper, $(\bar{\ })$ indicates complex conjugate of ( ). The functions z and ϕ may be obtained in terms of two holomorphic functions $\Omega(Z)$ and $\Psi(Z)$. The derivations may be found in [2] and the results are:

$$z = \Omega + \bar{\Psi} + \sum_k \frac{1}{4\pi C}(X_k + iY_k)\ln|Z - Z_k^0|, \tag{4.6}$$

$$\phi = \frac{2-c}{2+c}\Omega - \bar{\Psi} + \sum_k \frac{1}{8\pi C}(X_k + iY_k)\left[\frac{2-c}{2+c}\ln(Z - Z_k^0) - \ln(\bar{Z} - \bar{Z}_k^0)\right], \tag{4.7}$$

where Z_k^0 is a point inside the kth hole in a multiply connected region, and $X_k + iY_k$ the resultant force on the kth hole. The resultant force on a sutured hole is, of course, zero.

In terms of the functions z and ϕ, the Piola stresses are

$$(T_{11} + T_{22}) + i(T_{21} - T_{12}) = 2C(2 + c)\phi_{,Z} = 2C(2 - c)z_{,Z}, \tag{4.8}$$

$$(T_{22} - T_{11}) - i(T_{21} + T_{12}) = 2C(2 + c)\phi_{,\bar{Z}} = -2C(2 + c)z_{,\bar{Z}}. \tag{4.9}$$

Finally, the Piola traction vector (2.21) may be expressed in terms of the derivative of ϕ,

$$\frac{d\phi}{dL} = -\frac{i}{C(2 + c)}[P_1(L) + iP_2(L)]. \tag{4.10}$$

Consider now the hole defined by (3.1). It is to be sutured according to a suture function $S(L)$. The condition (3.5) is simply

$$z(C_1(L) + iC_2(L)) = z(C_1(S(L)) + iC_2(S(L))), \qquad L \in I^+. \tag{4.11}$$

Using (4.10), we obtain from (3.9)

$$\frac{d\phi}{dL} = S'(L)\frac{d\phi}{dL}\bigg|_{S(L)} = \frac{d}{dL}\phi(S(L)), \tag{4.12}$$

It follows that

$$\phi(C_1(L) + iC_2(L)) = \phi(C_1(S(L)) + iC_2(S(L))), \qquad L \in I^+. \tag{4.13}$$

This completes the general complex formulation for a suture subjected to large stretching. The results are applicable as long as (4.1) is a valid approximation.

5. Suturing of an elliptic hole in an infinite membrane. Let D be the infinite plane exterior to an elliptic hole C having the parametric representation:

$$C\colon Z = a\left(\cos\theta + i\frac{b}{a}\sin\theta\right), \qquad -\pi \le \theta \le \pi, \tag{5.1}$$

where a and b are, respectively, the major and minor axes. The arc length $L(\theta)$ along C is defined by

$$\frac{dL}{d\theta} = a\left[\sin^2\theta + \left(\frac{b}{a}\right)^2\cos^2\theta\right]^{1/2}, \qquad L(0) = 0. \tag{5.2}$$

The hole is to be sutured uniformly, i.e., $S(L) = -L$. The elastic membrane is then stretched at infinity according to:

$$z = AZ + \bar{B}\bar{Z} \quad \text{as} \quad Z \to \infty, \tag{5.3}$$

where $A = \bar{A}$, implying that the rotation at infinity is zero, and

$$B = B_1 + iB_2 \tag{5.4}$$

is a complex constant. The three constants A, B_1 and B_2 are related to the behavior of the solution at infinity:

$$A = \tfrac{1}{2}(\Lambda_1^\infty + \Lambda_2^\infty) = \tfrac{1}{2}(T_1^\infty + T_2^\infty)/C(2-c) \tag{5.5}$$

$$B = \tfrac{1}{2}(\Lambda_1^\infty - \Lambda_2^\infty)e^{-i2\beta} = \tfrac{1}{2}(T_1^\infty - T_2^\infty)e^{-i2\beta}/C(2+c) \tag{5.6}$$

where T_α^∞ and Λ_α^∞ are, respectively, the principal Piola stresses and stretch ratios at infinity, and β the angle between the Λ_1^∞-direction and the Z_1-axis.

Since the resultant force on C is zero, the solution is (cf. (4.6) and (4.7))

$$z = \Omega + \bar{\Psi}, \qquad \phi = \frac{2-c}{2+c}\Omega - \bar{\Psi}, \qquad Z \in D, \tag{5.7}$$

where the holomorphic functions Ω and Ψ must satisfy (5.3), (4.11) and (4.13) which, for the present problem, become

$$\Omega(Z) + \overline{\Psi(Z)} = AZ + \bar{B}\bar{Z} \quad \text{as} \quad Z \to \infty, \tag{5.8}$$

$$\Omega(Z) - \Omega(\bar{Z}) + \overline{\Psi(Z)} - \overline{\Psi(\bar{Z})} = 0, \qquad Z \in C, \tag{5.9}$$

$$\frac{2-c}{2+c}[\Omega(Z) - \Omega(\bar{Z})] - \overline{\Psi(Z)} + \overline{\Psi(\bar{Z})} = 0, \qquad Z \in C. \tag{5.10}$$

To this end, we introduce a complex ζ $(=\zeta_1 + i\zeta_2)$-plane and define a mapping function

$$Z = m(\zeta) = \frac{R}{2}\left(\zeta + \frac{k}{\zeta}\right) \quad \text{for all} \quad \zeta \in D^+\{|\zeta| > 1\}, \tag{5.11}$$

where

$$R = a\left(1 + \frac{b}{a}\right), \qquad k = \left(1 - \frac{b}{a}\right)\bigg/\left(1 + \frac{b}{a}\right). \tag{5.12}$$

The function $m(\zeta)$ maps D^+ onto D. Further, we define

$$\omega(\zeta) = \Omega(m(\zeta)), \qquad \psi(\zeta) = \Psi(m(\zeta)), \tag{5.13}$$

which are holomorphic in D^+. The condition (5.8) now becomes

$$\omega(\zeta) + \overline{\psi(\zeta)} = \frac{R}{2}(A\zeta + \bar{B}\bar{\zeta}) \quad \text{as} \quad \zeta \to \infty. \tag{5.14}$$

Since $\overline{m(\zeta)} = m(\bar{\zeta})$, the conditions (5.9) and (5.10) are just

$$\omega(\zeta) - \omega(\bar{\zeta}) + \overline{\psi(\zeta)} - \overline{\psi(\bar{\zeta})} = 0 \quad \text{for} \quad |\zeta| = 1, \tag{5.15}$$

$$\frac{2-c}{2+c}[\omega(\zeta) - \omega(\bar{\zeta})] - \overline{\psi(\zeta)} + \overline{\psi(\bar{\zeta})} = 0 \quad \text{for} \quad |\zeta| = 1. \tag{5.16}$$

A straightforward calculation reveals that

$$\omega = \frac{R}{2}A\left(\zeta + \frac{1}{\zeta}\right), \qquad \psi = \frac{R}{2}B\left(\zeta + \frac{1}{\zeta}\right), \qquad \zeta \in D^+ \tag{5.17}$$

This completes the solution in the ζ-plane.

To examine the deformation, we set $\zeta = \rho \exp(i\theta)$ and use (ρ, θ) as curvilinear coordinates in both the undeformed and deformed configurations. The coordinate curves in the undeformed configuration may be determined from (5.11). They are

$$(Z_1/X_\rho)^2 + (Z_2/Y_\rho)^2 = 1, \tag{5.18}$$

$$(Z_1/X_\theta)^2 - (Z_2/Y_\theta)^2 = 1, \tag{5.19}$$

where

$$X_\rho = \frac{a}{2}\left[\left(\rho + \frac{1}{\rho}\right) + \frac{b}{a}\left(\rho - \frac{1}{\rho}\right)\right], \qquad Y_\rho = \frac{a}{2}\left[\left(\rho - \frac{1}{\rho}\right) + \frac{b}{a}\left(\rho + \frac{1}{\rho}\right)\right], \tag{5.20}$$

$$X_\theta = a\left[1 - \left(\frac{b}{a}\right)^2\right]^{1/2} \cos\theta, \qquad Y_\theta = a\left[1 - \left(\frac{b}{a}\right)^2\right]^{1/2} \sin\theta. \tag{5.21}$$

For the deformed configuration, we first use (5.7) and (5.17) to obtain

$$z = \frac{R}{2}\left\{A\left(\zeta + \frac{1}{\zeta}\right) + \bar{B}\left(\bar{\zeta} + \frac{1}{\bar{\zeta}}\right)\right\}, \tag{5.22}$$

which indicates that the coordinate curves $\theta = 0$, $\theta = \pi$ and $\rho = 1$ fall on the straight line:

$$z_2/z_1 = \tan\alpha = (1 - \Lambda)\sin 2\beta/[(1 + \Lambda) + (1 - \Lambda)\cos 2\beta], \tag{5.23}$$ ††

where

$$\Lambda = \Lambda_2^\infty/\Lambda_1^\infty. \tag{5.24}$$

It is now convenient to introduce a new set of coordinates (y_1, y_2) defined by (see Fig. 1)

$$y_1 + iy_2 = \exp(-i\alpha)z. \tag{5.25}$$

To facilitate further calculations, we also introduce a set of oblique coordinates (x_1, x_2) defined by (see Fig. 1)

$$x_1 = y_1 - y_2 \cot\gamma, \qquad x_2 = y_2 \csc\gamma, \tag{5.26}$$

$$\tan\gamma = 2\Lambda/(1 - \Lambda^2)\sin 2\beta. \tag{5.27}$$

The coordinate curves (ρ, θ) may now be expressed in terms of (x_1, x_2) by using the above relations. They are

$$(x_1/x_\rho)^2 + (x_2/y_\rho)^2 = 1, \tag{5.28}$$

$$(x_1/x_\theta)^2 - (x_2/y_\theta)^2 = 1, \tag{5.29}$$

†† (5.5) and (5.6) have been used in deriving (5.23).

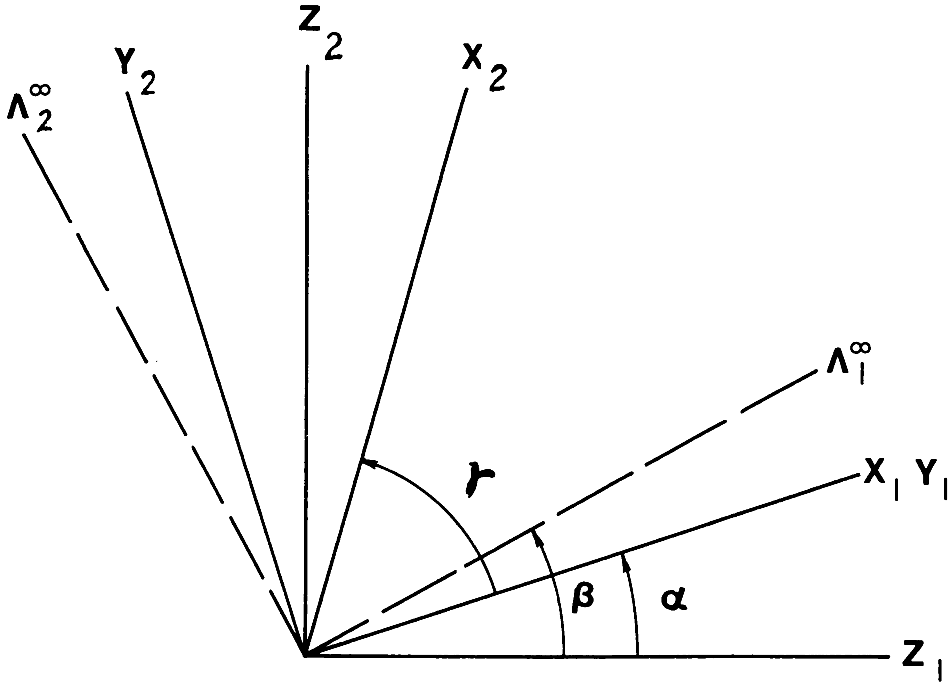

FIG. 1.

where

$$\left.\begin{matrix} x_\rho \\ y_\rho \end{matrix}\right\} = a\left(1 + \frac{b}{a}\right)\left(\rho + \frac{1}{\rho}\right)\Lambda_1^\infty[\tfrac{1}{2}(1 + \Lambda^2) \pm \tfrac{1}{2}(1 - \Lambda^2)\cos 2\beta]^{1/2}, \tag{5.30}$$

$$\left.\begin{matrix} x_\theta \\ y_\theta \end{matrix}\right\} = a\left(1 + \frac{b}{a}\right)\Lambda_1^\infty[\tfrac{1}{2}(1 + \Lambda^2) \pm \tfrac{1}{2}(1 - \Lambda^2)\cos 2\beta]^{1/2}\left\{\begin{matrix} \cos\theta \\ \sin\theta \end{matrix}\right\}. \tag{5.31}$$

A typical example is shown in Fig. 2 which gives a clear picture of the deformation of the curvilinear net (ρ, θ).

We proceed to determine the Cauchy traction vector p_α along the "upper" side of the suture c defined by

$$c\colon z = R(A + \bar{B})\cos\theta, \qquad 0 \le \theta \le \pi \tag{5.32}$$

$(-\pi \le \theta \le 0$ defines the "lower" side of the suture). The arc length $l(\theta)$ along c is just

$$l(\theta) = R\,|A + \bar{B}|\,(1 - \cos\theta) \tag{5.33}$$

It follows from (2.23), (4.10) and the above that

$$\begin{aligned} p_1(\theta) + ip_2(\theta) &= iC(2 + c)\frac{d\phi}{d\theta}\frac{d\theta}{dl} \\ &= -iC[(2 - c)A - (2 + c)\bar{B}]/|A + \bar{B}|. \end{aligned} \tag{5.34}$$

The outward normal $\mathbf{n}$ to the upper side of the suture is

$$n_1 + in_2 = -i(A + \bar{B})/|A + \bar{B}|. \tag{5.35}$$

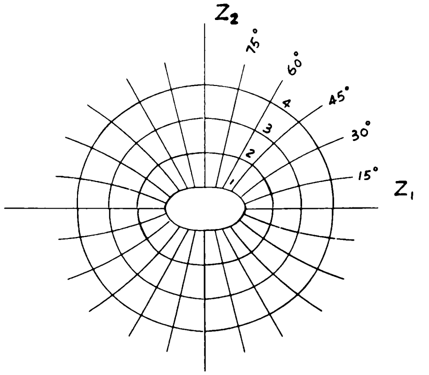

FIG. 2(a). An elliptic hole $(b/a) = 0.5$.

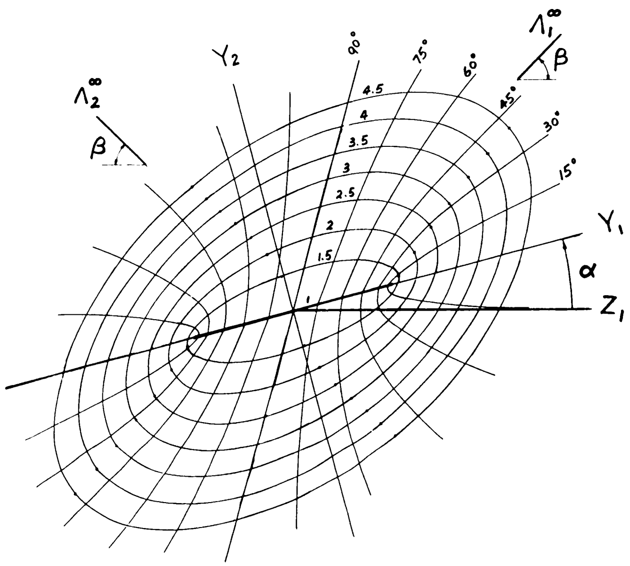

FIG. 2(b). Hole sutured and stretched. Given parameters: $\Lambda_1^\infty = 2$, $\Lambda_2^\infty = 2/\sqrt{3}$, $\beta = 45°$. Calculated parameters: $\alpha = 15°$, $\gamma = 60°$.

Combining the above relations, we get

$$\begin{aligned} p_n - ip_t &= (p_1 + ip_2)(n_1 - in_2) \\ &= C[(2 - c)A - (2 + c)B]/(A + \bar{B}). \end{aligned} \tag{5.36}$$

In fact, it can be shown that this expression holds for the $y_2 = 0$ (or $x_2 = 0$) line. It is interesting and perhaps surprising to note that when an infinite membrane without a hole is stretched according to (5.3), the orientation of the $Z_2 = 0$ line in the deformed state is the same as the $y_2 = 0$ (or $x_2 = 0$) line, and the Cauchy traction along the line is just (5.36). One must remember, though, that this conclusion has a lot to do with the assumption (4.1).

References

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