PLANE STRAIN PROBLEM OF TWO COPLANAR CRACKS IN AN INITIALLY STRESSED NEO-HOOKEAN ANISOTROPIC INFINITE MEDIUM*

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Abstract We consider the plane strain problem of determining the crack energy in an initially stressed neo-Hookean anisotropic infinite medium containing two coplanar cracks. We assume that the cracks are opened by a constant internal pressure. By using a Fourier transform solution of the equilibrium equations the problem is reduced to solving a set of triple integral equations with a cosine kernel. These integral equations are solved and a closed-form expression for the crack energy W is obtained. The numerical values of W are graphed in Figs. 1, 2.

1. Introduction. Incremental deformation theory concerns the infinitesimal deformation of a solid with a known initial finite deformation. The basic equations of such an incremental deformation theory have been derived by Trefftz [1], Biot [2,3], Neuber [4], Green, Rivlin and Shield [5] and Green and Zerna [6]. More references on this type of work may be found in an excellent monograph by Biot [7].

In this paper we use the basic equations derived by Biot [7] for initially stressed neo-Hookean anisotropic solids for solving the problem of two coplanar cracks in an infinite medium. In Sec. 2, we obtain a Fourier transform solution of the equilibrium equations and obtain expressions for the components of displacement and stress. In Sec. 3, we give the boundary conditions and reduce the problem to a set of simple triple integral equations. These integral equations are solved exactly and the closed form expression for the crack energy W is obtained. The numerical values of QW/p_0^2 are graphed in Figs. 1, 2 where p_0 is the constant internal pressure on the crack surfaces.

2. Basic equations and their solution. We consider an infinite elastic medium which is incompressible, homogeneous and of orthotropic symmetry. The coordinate axes are oriented along the directions of elastic symmetry. The principal initial stresses S_{11}, S_{22}, S_{33} are also oriented along the same directions. Incremental stresses s_{11}, s_{12}, s_{22} corresponding to plane strain in the x, y plane satisfy the equilibrium equations

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{12}}{\partial y} - P\frac{\partial \omega}{\partial y} = 0, \qquad \frac{\partial s_{12}}{\partial x} + \frac{\partial s_{22}}{\partial y} - P\frac{\partial \omega}{\partial x} = 0, \tag{2.1}$$

where P is the constant initial stress in the x, y plane represented by

$$S_{11} = -P, \qquad S_{22} = 0 \tag{2.2}$$

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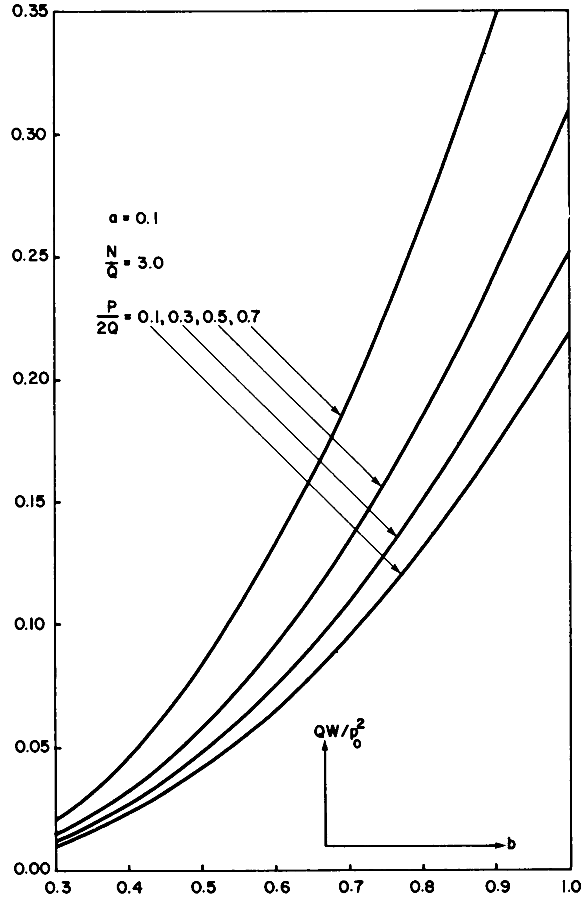

FIG. 1. Numerical values of QW/p_0^2 against b for $a = 0.1$, $(N/Q) = 3.0$, $(P/2Q) = 0.1, 0.3, 0.5, 0.7$.

and ω is the rotation about z-axis given by

$$\omega = \tfrac{1}{2}(\partial v/\partial x - \partial u/\partial y); \tag{2.3}$$

u and v are the displacements in the x and y directions respectively.

The stress-strain relations are

$$s_{11} - s = 2Ne_{xx}, \qquad s_{22} - s = 2Ne_{yy}, \tag{2.4}$$

$$s_{12} = 2Qe_{xy}, \qquad (2s = s_{11} + s_{22}),$$

where N and Q are the elastic moduli of the laminated medium along the x and y directions respectively and Q is small in comparison with N. To the above relations we must add the condition of incompressibility:

$$e_{xx} + e_{yy} = 0. \tag{2.5}$$

Since

$$e_{xx} = \partial u/\partial x, \qquad e_{yy} = \partial v/\partial y, \qquad e_{xy} = \tfrac{1}{2}(\partial u/\partial y + \partial v/\partial x), \tag{2.6}$$

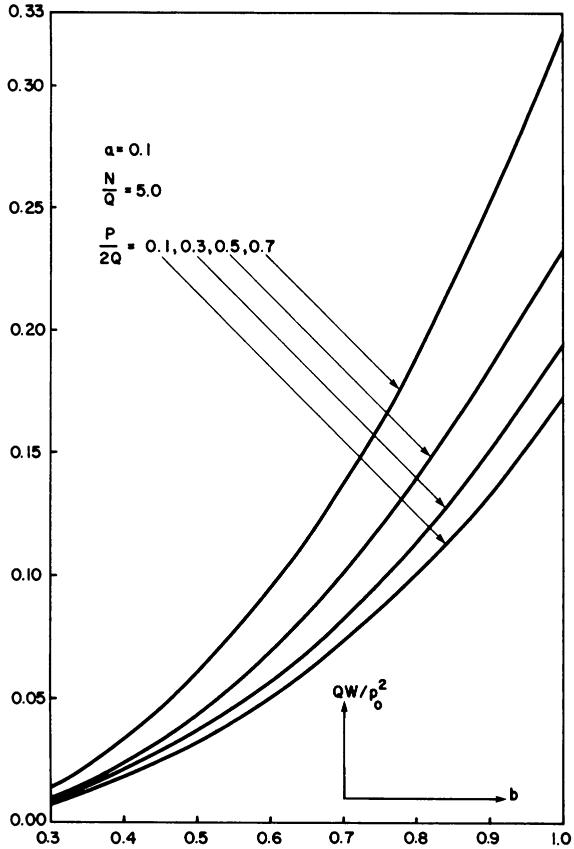

FIG. 2. Numerical values of $QW/p_0{}^2$ against b for $a = 0.1$, $(N/Q) = 5.0$, $(P/2Q) = 0.1, 0.3, 0.5, 0.7$.

Eq. (2.5) is satisfied by writing the displacement as

$$u = -\partial\phi/\partial y, \qquad v = \partial\phi/\partial x, \tag{2.7}$$

where ϕ is a function of x and y.

Expressing the strain components in terms of ϕ through Eq. (2.6) and (2.7) and substituting the values (2.3) and (2.4) of the rotation ω and the stress components into the equilibrium equations (2.1), we obtain

$$\begin{aligned} &\frac{\partial s}{\partial x} - \frac{\partial}{\partial y}\left[\left(2N - Q + \frac{P}{2}\right)\frac{\partial^2\phi}{\partial x^2} + \left(Q + \frac{P}{2}\right)\frac{\partial^2\phi}{\partial y^2}\right] = 0, \\ &\frac{\partial s}{\partial y} + \frac{\partial}{\partial x}\left[\left(2N - Q - \frac{P}{2}\right)\frac{\partial^2\phi}{\partial y^2} + \left(Q - \frac{P}{2}\right)\frac{\partial^2\phi}{\partial x^2}\right] = 0. \end{aligned} \tag{2.8}$$

Elimination of s in the above two equations leads to a single equation for ϕ:

$$k^2\frac{\partial^4\phi}{\partial x^4} + 2m\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0, \tag{2.9}$$

where

$$m = \frac{2N - Q}{Q + \frac{1}{2}P}, \qquad k^2 = \frac{Q - \frac{1}{2}P}{Q + \frac{1}{2}P}. \tag{2.10}$$

For an isotropic medium ($N = Q$) free of initial stress ($P = 0$) we obtain the well-known biharmonic equation

$$\nabla^4 \phi = 0. \tag{2.11}$$

If we assume a solution of Eq. (2.9) in the form

$$\phi(x, y) = \int_0^\infty F(\xi y) \sin(\xi x)\, d\xi, \tag{2.12}$$

we find that $F(\xi y)$ satisfies the equation

$$F^{(iv)}(\xi y) - 2mF''(\xi y) + k^2 F(\xi y) = 0. \tag{2.13}$$

If we assume F in the form

$$F(\xi y) = \exp(\xi \beta y), \tag{2.14}$$

then β satisfies the characteristic equation

$$\beta^4 - 2m\beta^2 + k^2 = 0. \tag{2.15}$$

The roots of the above equation are given by

$$\beta_1^2 = m + (m^2 - k^2)^{1/2}, \qquad \beta_2^2 = m - (m^2 - k^2)^{1/2}. \tag{2.16}$$

If $m > 0$, $m^2 > k^2$, both β_1^2 and β_2^2 are positive and hence β_1 and β_2 are real.

The solution of Eq. (2.13) for $-\infty < y \leq 0$ and vanishing at $y \to -\infty$ is given by

$$F(\xi y) = C(\xi) \exp(\beta_1 \xi y) + D(\xi) \exp(\beta_2 \xi y), \tag{2.17}$$

where $\beta_1 > 0$ and $\beta_2 > 0$.

From Eqs. (2.7), (2.12) and (2.17), we obtain

$$u(x, y) = -\int_0^\infty [\beta_1 C(\xi) e^{\beta_1 \xi y} + \beta_2 D(\xi) e^{\beta_2 \xi y}] \xi \sin(\xi x)\, d\xi, \tag{2.18}$$

$$v(x, y) = \int_0^\infty [C(\xi) e^{\beta_1 \xi y} + D(\xi) e^{\beta_2 \xi y}] \xi \cos(\xi x)\, d\xi. \tag{2.19}$$

From Eqs. (2.8), (2.12) and (2.17), we find that

$$s = \int_0^\infty \left[\left(2N - Q + \frac{P}{2}\right) (\beta_1 C e^{\beta_1 \xi y} + \beta_2 D e^{\beta_2 \xi y}) - \left(Q + \frac{P}{2}\right) (C \beta_1^3 e^{\beta_1 \xi y} + D \beta_2^3 e^{\beta_2 \xi y}) \right] \xi^2 \cos(\xi x)\, d\xi. \tag{2.20}$$

From Eq. $(2.4)_2$, $(2.6)_2$, (2.19) and (2.20), we obtain

$$s_{22} = \int_0^\infty \left[\left(4N - Q + \frac{P}{2}\right)(C\beta_1 e^{\beta_1 \xi y} + D\beta_2 e^{\beta_2 \xi y}) - \left(Q + \frac{P}{2}\right)(C\beta_1^3 e^{\beta_1 \xi y} + D\beta_2^3 e^{\beta_2 \xi y})\right]\xi^2 \cos(\xi x) d\xi. \quad (2.21)$$

From Eqs. $(2.4)_3$, $(2.6)_3$, (2.7), (2.12) and (2.17) we find that

$$s_{12} = Q(\partial^2\phi/\partial x^2 - \partial^2\phi/\partial y^2)$$

$$= -Q\int_0^\infty [(1 + \beta_1^2)Ce^{\beta_1 \xi y} + (1 + \beta_2^2)De^{\beta_2 \xi y}]\xi^2 \sin(\xi x) d\xi. \quad (2.22)$$

3. Statement and solution of the problem. We consider the plane strain problem in an infinite plane $-\infty < x < \infty$, $-\infty < y < \infty$ which is initially deformed in a manner given by Eqs. (2.2). If a pair of coplanar cracks develops by internal pressure in the xz-plane symmetrically located with respect to the yz-plane, we may consider the problem of a quarter plane $x \geq 0$, $y \leq 0$. The boundary conditions may be taken as:

$$\sigma_{yy}(x, 0) = -p_0, \qquad a < x < b, \quad (3.1)$$

$$v(x, 0) = 0, \qquad 0 \leq x < a, \qquad x > b, \quad (3.2)$$

$$\sigma_{xy}(x, 0) = 0, \qquad 0 \leq x < \infty. \quad (3.3)$$

The boundary condition (3.3) along with Eq (2.22) leads to the relation

$$D(\xi) = -\frac{1 + \beta_1^2}{1 + \beta_2^2} C(\xi). \quad (3.4)$$

Now from Eqs. (2.19), (2.21) and (3.4), we obtain

$$v(x, 0) = \left(\frac{\pi}{2}\right)^{1/2} \frac{\beta_2^2 - \beta_1^2}{1 + \beta_2^2} F_c[\xi C(\xi); \xi \to x], \quad (3.5)$$

$$\sigma_{yy}(x, 0) = -\left(\frac{\pi}{2}\right)^{1/2} L^{-1} F_c[\xi^2 C(\xi); \xi \to x], \quad (3.6)$$

where

$$L^{-1} = (1 + \beta_2^2)^{-1}(\beta_1 - \beta_2)\left[\left(4N - Q + \frac{P}{2}\right)(1 - \beta_1\beta_2) - \left(Q + \frac{P}{2}\right)(\beta_1^2 + \beta_2^2 + \beta_1\beta_2 + \beta_1^2\beta_2^2)\right], \quad (3.7)$$

and where F_c is a Fourier cosine transform (see [9]). From Eqs. (3.5) and (3.6) the boundary conditions (3.1) and (3.2) yield the following triple integral equations for the determination of $C(\xi)$:

$$F_c[\xi C(\xi); \xi \to x] = 0, \qquad 0 < x < a, \qquad x > b, \quad (3.8)$$

$$F_c[\xi^2 C(\xi); \xi \to x] = \left(\frac{2}{\pi}\right)^{1/2} p_0 L, \qquad a < x < b. \tag{3.9}$$

Solution of the triple integral equations (3.8) and (3.9) is given by (see Srivastava and Lowengrub [8]):

$$C(\xi) = \frac{1}{\xi^2}\int_a^b h(t^2)\sin \xi t\, dt, \int_a^b h(t^2)\, dt = 0, \tag{3.10}$$

$$h(t^2) = p_0 L(t^2 - b^2 E/F)[(t^2 - a^2)(b^2 - t^2)]^{-1/2}, \qquad a < t < b, \tag{3.11}$$

where

$$F = F\left(\frac{\pi}{2}, \frac{(b^2 - a^2)^{1/2}}{b}\right), \qquad E = E\left(\frac{\pi}{2}, \frac{(b^2 - a^2)^{1/2}}{b}\right), \tag{3.12}$$

are elliptic integrals of the first and second kind respectively.

From (3.10) and (3.5), we obtain

$$v(x, 0) = \frac{\beta_2^2 - \beta_1^2}{1 + \beta_2^2}\int_x^b h(t^2)dt, \qquad a < x < b. \tag{3.13}$$

Substituting for $h(t^2)$ from (3.11) into (3.13) and evaluating the resulting integrals (see Gradshteyn and Ryzhik [9], we get

$$v(x, 0) = p_0 b\, \frac{\beta_2^2 - \beta_1^2}{1 + \beta_2^2}\,[E(\lambda, q) - bF(\lambda, q)E/F]L, \tag{3.14}$$

where

$$q = [(b^2 - a^2)1/b^2]^{1/2}, \qquad \lambda = \sin\left[\left(\frac{b^2 - x^2}{b^2 - a^2}\right)^{1/2}\right]. \tag{3.15}$$

The total energy W required to open the cracks is given by

$$W = -2\int_a^b \sigma_{yy}(x, 0)v(x, 0)dx = 2p_0\int_a^b v(x, 0)dx\,. \tag{3.16}$$

Substituting for $v(x, 0)$ from (3.13) into (3.16) and making use of $(3.10)_2$, we obtain

$$W = 2p_0\,\frac{\beta_2^2 - \beta_1^2}{1 + \beta_2^2}\int_a^b th(t^2)dt \tag{3.17}$$

Now substituting for $h(t^2)$ from (3.7) into (3.17) and evaluating the resulting integrals, we obtain the following closed-form expression for the energy:

$$W = \frac{\pi}{2}p_0^2 L\,\frac{\beta_2^2 - \beta_1^2}{1 + \beta_2^2}\,(a^2 + b^2 - 2b^2 E/F)\,. \tag{3.18}$$

Numerical values of QW/p_0^2 have been graphed in Figs. 1, 2 against b for $a = 0.1$ and $(N/Q) = 3.0, 5.0$ and $(P/2Q) = 0.1, 0.3, 0.5, 0.7$. For incompressible isotropic elastic medium without initial stress, we have

$$N = Q = \mu, \qquad P = 0 \tag{3.19}$$

and hence from Eq. (2.10) and (2.16) we obtain

$$m = k = 1, \qquad \beta_1 = \beta_2 = 1 \tag{3.20}$$

From Eq. (3.18), (3.19) and (3.20), we obtain the following expression for crack energy:

$$W = \frac{\pi}{4} p_0^2(a^2 + b^2 - 2b^2E/F) \tag{3.21}$$

for an incompressible isotropic elastic medium. The expression (3.21) for W agrees with the corresponding expression obtained by Lowengrub and Srivastava [9].

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