ELASTIC REALIZABILITY OF FORCE AND DISPLACEMENT SYSTEMS IN STRUCTURES*

BY

W. R. SPILLERS AND E. LEFCOCHILOS

Rensselaer Polytechnic Institute

Abstract. This paper discusses the constraints imposed upon equilibrium and compatibility solutions in structures through the use of constitutive equations. It is shown, for example, that "realizable" force systems are bounded by statically determinate force systems for the case of trusses. The analysis used depends heavily upon the concept of "basic solutions" (statically determinate substructures) for linear systems which appears most commonly in the theory of linear programming.

Introduction. As used in this paper, the term "physically realizable" or simply "realizable" applies to force and displacement solutions which satisfy *all* the equations of structures. It is the intention here to study the properties of these solutions with respect to solutions of the equilibrium or compatibility equations which may or may not also satisfy the constitutive equations. The idea, of course, is that if equilibrium or compatibility systems are to be "realized" or built they must also satisfy constitutive equations.

While questions such as realizability have a certain academic interest, the motivation here is somewhat deeper and lies in structural optimization. Having tried unsuccessfully to take structural optimization problems head on, the next step is to attempt to simplify matters. In some cases [1] a convenient simplification is just to neglect the constitutive equations and work with, for example, objective functions and constraints written in terms of forces. When the constitutive equations are dropped from the set of constraints it can happen—notably in cases of multiple loading conditions—that the resulting force system cannot be realized (built). Rather than a practical solution, these results must be regarded as bounds.

The question then to be addressed here is the effect of constitutive equations on force and compatibility solutions. It will be shown that the answer lies within the segment of the theory of linear systems which deals with the question of non-negative solutions of linear equations and some concepts of convex analysis. In spite of the fact that this type of analysis is basic to the study of linear programming, much of the supporting material is not easily available. There is, however, an excellent summary of this material in the first chapter of Gale's book on economic models [2] which is highly recommended to the reader.

In an earlier effort [3] the authors examined some aspects of the realizability problem for a simple example while here it is hoped to develop a general theory of realizability. In this earlier case it was in fact true that realizable forces were bounded by statically determinate force systems, although this fact was not noted. It is proposed to show here that

* Received November 16, 1978. This work has been supported by the National Science Foundation.

in general realizable solutions are bounded by statically determinate force systems. Other properties of realizable solutions will also be developed.

The work presented here proceeds in the following manner. First of all a notation is introduced. Then the case of a single redundant is considered. Finally the general case of an arbitrary number of redundants is developed from the case of the single redundant. The entire paper relies heavily on the concept of a "basic" solution of a linear system which of course corresponds to a statically determinate substructure of a given structure. When specific examples are considered they will be trusses, but it may be added that the notation used applies to any type of structure including continuous systems. (The case of the truss, which has a diagonal primitive stiffness matrix, must, however, be extended to the general case in which the primitive stiffness matrix is partitioned-diagonal.)

Notation. It is proposed to present the node and mesh methods of structural analysis in the following form (see [5] or any good book on matrix structural analysis):

The Node Method:

$$\begin{aligned} \tilde{N}F &= P\text{—node equilibrium,} \\ F &= K\Delta\text{—constitutive equation,} \\ \Delta &= N\delta\text{—member/joint displacement equation,} \end{aligned} \tag{1}$$

The Mesh Method:

$$\begin{aligned} \tilde{C}\Delta &= 0\text{—compatibility equations,} \\ \Delta &= K^{-1}F\text{—constitutive equation,} \\ F &= F^{\circ}+CF_m\text{—member/mesh force equation.} \end{aligned} \tag{2}$$

In these equations

F, Δ—member force and displacement,

P, δ—node force and displacement,

K—primitive stiffness matrix,

F°—any equilibrium force system ($\tilde{N}\, F^{\circ} = P$),

F_m—mesh force matrix,

N—generalized incidence matrix,

C—generalized branch-mesh matrix.

Ordinarily the node and mesh methods are solved as

$$(\tilde{N}KN)\delta = P \quad \text{or} \quad (\tilde{C}K^{-1}C)F_m = -\tilde{C}K^{-1}F^{\circ} \tag{3}$$

in which δ and F_m are to be computed given the other matrices. In this paper the interest lies then in determining the *ranges* of δ and F_m as K varies in some arbitrary manner while remaining positive definite (as required by the particular class of structure under study).

For the case of trusses the primitive stiffness matrix is particularly simple: it is just a diagonal matrix with non-negative diagonal terms. For this case the terms in the system matrices in Eq. (3) are linear in either the elements of K or K^{-1} and these equations can be rewritten in the form

$$Dk = p \quad \text{and} \quad \mathscr{F}k^{-1} = 0 \tag{4}$$

where k and k^{-1} are simply column matrices whose elements are the diagonal terms of K and K^{-1} are simply column matrices whose elements are the diagonal terms of K and K^{-1} respectively. The elements of the matrices D and $\mathscr{F}$ are the linear in the displacements δ and forces F_m. From Eq. (4) the realizability problem is reduced to finding all values of δ and F_m for which these equations have semi-positive solutions k and k^{-1}. (In Gale's terminology, x is semi-positive if $x \geq 0$ but $x \neq 0$.)

There are basic differences in the forces and displacement formulations as they appear in Eqs. (3–4). From one point of view the displacement formulation deals with the solutions of a non-homogeneous system while the force formulation deals with a homogeneous system; from another point of view the displacements δ vary inversely along a ray in K-space while the forces are constant along such a ray.

Finally it should be noted that the form of Eq. (4) is reminiscent of work on linear inequalities. But the fact that the coefficients of D and $\mathscr{F}$ are linear in the displacements δ and the forces F_m adds a degree of difficulty not common in this area.

Structures with a single redundant. In this section it will be shown that realizable force systems are bounded by statically determinant force systems for structures which are statically indeterminant to the first degree. Displacement realizability will also be discussed.

In this case it is convenient to start with the node equilibrium equation,

$$\tilde{N}F = P. \tag{5}$$

If n is the number of nodal degrees of freedom and b is the number of branch forces (b is the number of bars in the case of the truss), a single redundant implies that $b = n + 1$. It is furthermore assumed that the structure geometrically stable, which implies that the rank of the matrix N is n.

Fig. 1 shows a plane truss which will be useful in discussing this case of a single redundant. The loading itself is of a certain interest since it corresponds to a situation in which the sign of a bar force can be changed by changing the values of the member stiffnesses. For example, as $k_3 \to 0$ (bar 3 is removed from the structure) bar 2 goes into tension, while as $k_1 \to 0$ it goes into compression. Some of the appropriate matrices are also indicated in this figure.

At this point it is convenient to invoke the following theorem. ([5, theorem 2.9]):

THEOREM 1. Exactly one of the following alternatives holds. Either the equation

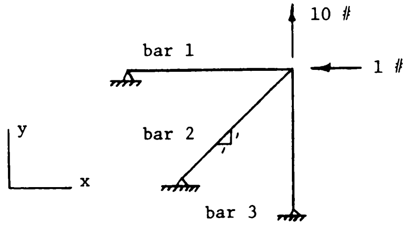
$$Ax = 0$$

has a semi-positive solution or the inequality

$$\tilde{A}y > 0$$

has a solution.

When Theorem 1 is applied to the system $\mathscr{F}k^{-1} = 0$ for the case of a single redundant, it simply states that either (a) the system is realizable or (b) all the terms in the row matrix $\mathscr{F}$ must have the same sign. The regions of realizability are therefore defined by points at which the terms in $\mathscr{F}$ change sign (pass through zero). Since a term in $\mathscr{F}$ becoming zero corresponds to the formation of a statically determinate substructure, the region of real-



$(\tilde{N}\ K\ N)\ \delta = P$

$$\begin{bmatrix} k_1 + k_2/2 & k_2/2 \\ k_2/2 & k_2/2 + k_3 \end{bmatrix} \begin{bmatrix} \delta_{1x} \\ \delta_{1y} \end{bmatrix} = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$$

$(\tilde{C}\ K^{-1}\ C)\ F_m = -\ \tilde{C}\ K^{-1}\ F°$

$$\frac{F_2}{\sqrt{2}} \left(\frac{2}{k_2} + \frac{1}{k_1} + \frac{1}{k_3}\right) = -\frac{1}{k_1} + \frac{10}{k_3}$$

$Dk = P$

$$\begin{bmatrix} \delta_{1x} & \frac{\delta_{1x} + \delta_{1y}}{2} & 0 \\ 0 & \frac{\delta_{1x} + \delta_{1y}}{2} & \delta_{1y} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$$

$F\ k^{-1} = 0$

$$[F_2/\sqrt{2} + 1 \quad \sqrt{2}\ F_2 \quad F_2/\sqrt{2} - 10] \begin{bmatrix} 1/k_1 \\ 1/k_2 \\ 1/k_3 \end{bmatrix} = 0$$

FIG. 1. A plane truss.

izability must be bounded by statically determinant substructures. It remains to show that the region is finite and contains no holes.

The fact that the region of feasibility is finite can be argued from the fact that the system matrix $(\tilde{C}K^{-1}C)$ is positive definite which implies that the coefficients of F_m in Eq. (3) must all have the same sign. Since the terms in $\mathscr{F}$ are all linear in F_m, F_m can be made sufficiently large—either positive or negative—so that the linear part of each term dominates the constant part. Then for this sufficiently large F_m the terms must all have the same sign and F_m must be unrealizable.

The absence of holes within the domain of realizable solutions can be argued in the following manner. Each of the terms of the matrix F varies linearly with the scalar parameter F_m. As F_m, for example, proceeds from $-\infty$ to $+\infty$ the terms of $\mathscr{F}$ go from the condition of all having the same sign at $-\infty$ to all having the same, but different sign at $+\infty$. Since each term only changes sign once in this range, there can be no holes within it in which all the terms have the same sign. Fig. 2 shows some of these properties for the example indicated in Fig. 1.

For displacement realizability it is convenient to invoke Theorem 2 ([5, theorem 2.6]):
THEOREM 2. Exactly one of the following alternatives holds. Either the equation

$$Ax = b$$

has a non-negative solution or the inequalities

$$\tilde{A}y \geq 0, \qquad \tilde{b}y < 0$$

have a solution.
For displacement realizability Theorem 2 can have the following interpretation. Since $Ax = b$ corresponds to $Dk = P$, it is convenient to interpret y in Theorem 2 as a virtual displacement. Then the inequality $\tilde{b}y < 0$ corresponds to negative virtual work.

The inequality $\tilde{A}y \geq 0$ is somewhat more difficult to deal with. First of all, it is convenient to partition the matrix N by rows and write

$$\tilde{N}KN\,\delta = P \Rightarrow (\sum \tilde{N}_i k_i N_i)\,\delta = P \tag{6}$$

or

$$\sum_i N_i k_i \Delta_i = P = \sum N_i \Delta_i k_i \,. \tag{7}$$

Then

$$\tilde{A}y \geq 0 \Rightarrow \Delta_i (Ny)_i \geq 0. \tag{8}$$

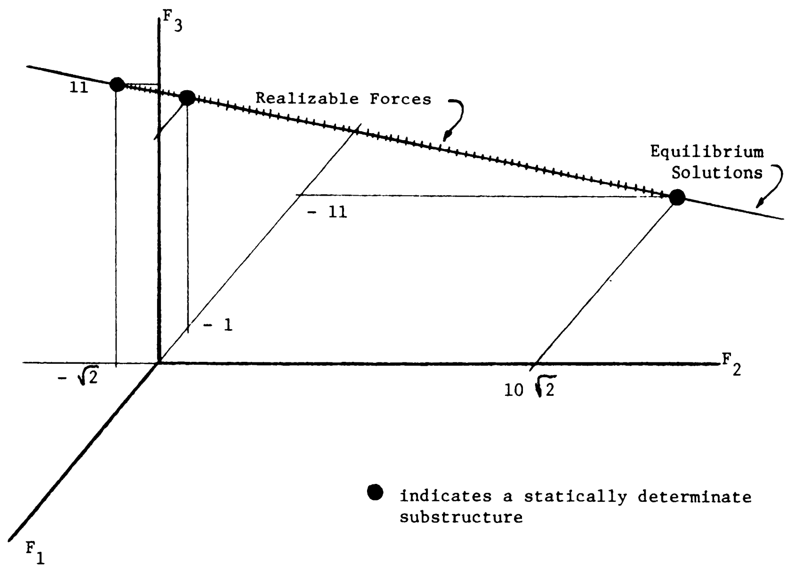


FIG. 2.

The interpretation of the theorem follows as either $Dk = P$ has a non-negative solution or there exists a virtual node displacement y which produces member displacements $(Ny)_i$ of the same sign as $\Delta_i = (N\delta)_i$ but which corresponds to negative virtual work.

The general case. It is now possible to develop the general case of n redundants using the results for the case of a single redundant. Fig. 3 indicates an example to which it will be convenient to refer in this section. This example is of course obtained from the example of Fig. 1 by simply adding a bar, and is therefore statically indeterminant to the second degree.

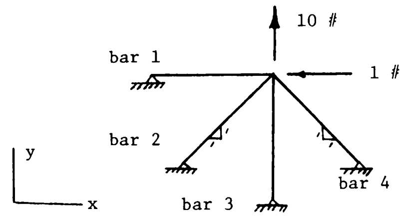


Figure 3

A Plane Truss

$(\tilde{N}\ K\ N) = P$

$$\begin{bmatrix} k_1 + \frac{k_2 + k_4}{2} & \frac{k_2 - k_4}{2} \\ \frac{k_2 - k_4}{2} & k_3 + \frac{k_2 + k_4}{2} \end{bmatrix} \begin{bmatrix} \delta_{1x} \\ \delta_{1y} \end{bmatrix} = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$$

$(\tilde{C}\ K^{-1}\ C)\ F_m = -\ \tilde{C}\ K^{-1}\ F^\circ$

$$\begin{bmatrix} 2/k_2 + 1/k_1 + 1/k_3 & -(1/k_1 - 1/k_3) \\ -(1/k_1 - 1/k_3) & 2/k_4 + 1/k_1 + 1/k_3 \end{bmatrix} \begin{bmatrix} \frac{F_2}{\sqrt{2}} \\ \frac{F_4}{\sqrt{2}} \end{bmatrix} \qquad \begin{bmatrix} -1/k_1 + 10/k_3 \\ 1/k_1 + 10/k_3 \end{bmatrix}$$

$Dk = P$

$$\begin{bmatrix} \delta_{1x} & \frac{\delta_{1x} + \delta_{1y}}{2} & 0 & \frac{\delta_{1x} - \delta_{1y}}{2} \\ 0 & \frac{\delta_{1x} + \delta_{1y}}{2} & \delta_{1y} & -\frac{\delta_{1x} - \delta_{1y}}{2} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$$

$F\ k^{-1} = 0$

$$\begin{bmatrix} F_2/\sqrt{2} + 1 - F_4/\sqrt{2} & \sqrt{2}\,F_2 & F_2/\sqrt{2} - 10 + F_4/\sqrt{2} & 0 \\ -(F_2/\sqrt{2} + 1 - F_4/\sqrt{2}) & 0 & F_2/\sqrt{2} - 10 + F_4/\sqrt{2} & \sqrt{2}\,F_4 \end{bmatrix} \begin{bmatrix} 1/k_1 \\ 1/k_2 \\ 1/k_3 \\ 1/k_4 \end{bmatrix} = 0$$

FIG. 3. A plane truss.

It is convenient at this point to introduce the concept of *basic solutions* [2]. A system of linear equations

$$Ax = b \tag{9}$$

can be written as

$$\sum_i A_i x_i = b \tag{10}$$

when the matrix A is partitioned by columns. Any solution x is termed basic if the columns associated with non-zero x_i form a complete independent set. Applied to the equilibrium equations, this concept, of course, leads to identifying the statically determinant substructures of a given structure as basic solutions of the equilibrium equations. Simonnard [5] points out that the maximum number of basic solutions is the combinatorial problem of, for example, b bars taken n degrees of freedom at a time, or

$$\binom{b}{n} = \frac{b!}{n!\,(b-n)!}\,.$$

The actual number of basic solutions must be determined by examining the matrix N. Since there are frequently statically determinate pieces of a given structure, it would be expected that the maximum number of basic solutions might not be achieved frequently.

Fig. 4 shows the feasible region for the case of example 2 which is regarded as typical of problems of higher dimension. In this figure the heavy dots indicate statically determinant substructures. It is proposed to argue that in general the feasible region is defined by (and limited by) statically determinate substructures. In particular, it will be argued that

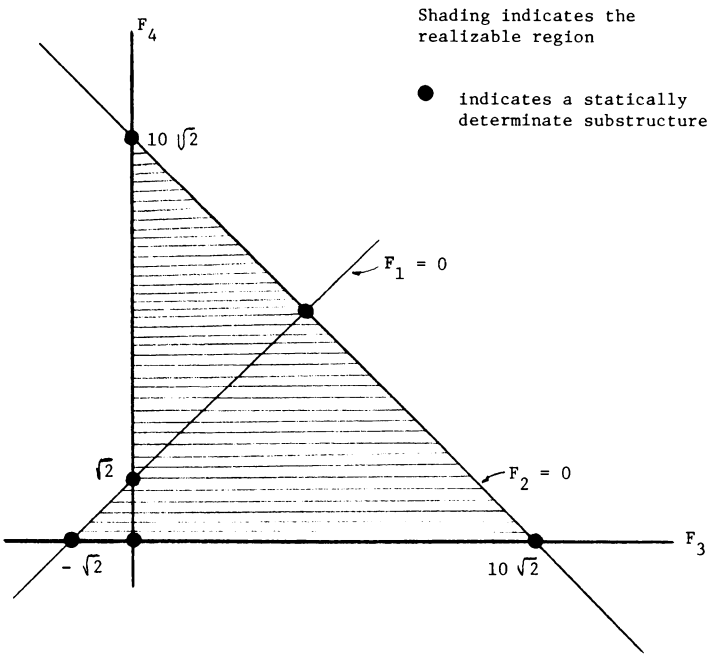


FIG. 4. Realizable region for example 2.

any feasible force solution can be written as a convex combination of statically determinate substructures. This of course does not imply that the feasible region itself is convex. In fact in Fig. 2 the feasible region is not convex.

Several properties of feasible solutions will now be listed.

Property 1. Given a stable structure and loads, a feasible interior solution always exists. (This can be shown by simply taking $K = I$ in Eq. (1).)

Property 2. Given any feasible interior point, there exists a feasible neighborhood bounded by lines or surfaces along which member forces are zero. (This neighborhood can be developed by keeping the displacement fixed, varying the member forces, and simply computing the member stiffness as $K_i = F_i/\Delta_i$.)

Property 3. Given a feasible region, it is possible to develop the other feasible regions through systematic variation of the primitive stiffness matrix K. (Starting with the point $K = I$ it is possible to cross surfaces on which member forces go through zero by moving along lines in K space from this point to the neighborhood of any statically determinate substructure.)

Property 4. If member forces change sign within the feasible region, this change of sign also occurs in the statically determinate substructures of this structure. (This follows from arguments of continuity using the statically determinate substructures as limiting points of the feasible region. See the Appendix.)

Property 5. Any feasible force system can be written as a convex combination of statically determinate solutions. (This follows from the fact that feasible regions can be subdivided into convex regions in which the member forces do not change sign. These regions are bounded by statically determinate substructures.)

Topological analysis. There are some interesting combinatorial aspects of the realizability problem. For example, Fig. 4 can be constructed using the analyses of the four possible substructures which are statically indeterminate to the first degree. In each of these cases the feasible region can be represented by two lines joining three nodes. Since it is known from graph theory that

$$B = M + N - 1 \tag{11}$$

where B is the number of branches, M is the number of meshes, and N is the number of nodes, it follows that in the case shown in Fig. 4,

$$4 \times 2 = M + 6 - 1, \qquad M = 3,$$

or that there are three feasible subregions as indicated.

In general, the equilibrium equations are satisfied within some subspace of the space of the bar forces, F. (This subspace is actually the space of the mesh forces F_m.) The statically determinate substructures, $F^{(i)}$ (basic solutions of the equilibrium equations) are points in both F and F_m. Obviously any convex combination $\bar{F}$ of these substructures,

$$\bar{F} = \sum_i \alpha_i F^{(i)} \quad \text{with} \quad \sum \alpha_i = 1, \tag{12}$$

satisfies the equilibrium equations since $\tilde{N}F^{(i)} = P$.

Now let n, the rank of N, be equal to the nodal degrees of freedom and $k = b - n$ be the degree of statical indeterminacy. Here b, the number of bars in the truss, is the dimension of F, and k is of course the dimension of F_m. By definition, each of the points $F^{(i)}$ must have at least k zero elements.

The first step in the analysis is to compute, exhaustively, all the basic solutions $F^{(i)}$. Assume that there are s of these solutions. Each of these solutions must lie at the intersection of k hyperplanes which define the k zero elements in each $F^{(i)}$. (The load vector can be perturbed at this point to avoid the degenerate case in which there are more than k zeroes in any $F^{(i)}$.) From these basic solutions it is possible to construct the region of feasibility. This region is composed of subregions within which the member forces do not change sign which are themselves bounded by surfaces on which particular member forces are zero. These subregions can be constructed by analyzing the neighborhoods of the statically determinate substructures. For this analysis it is only necessary to identify the term in any vector as positive, negative or zero. The specific magnitude of any term is not important.

Let two statically determinate substructures be termed "adjacent" if their components "differ" (as implied above) in only two elements, one coming from zero and one going to zero (as in linear programming, one term entering the basis and one term leaving it). Adjacent substructures now define surfaces which in turn define regions in the neighborhood of a given substructure. For example, a statically determinate substructure together with its adjacent points form a set which must be subdivided into sets of "consistent" substructures which bound regions in which bar forces do not change sign. These open regions can be subsequently joined to form the closed global subregions above with which member forces do not change sign. These subregions themselves comprise the region of feasibility.

The point is that there is sufficient information within the list of statically determinate substructures to subdivide their neighborhood into feasible regions which can subsequently be combined to describe the entire feasible region. It is clear that the combinatorial aspects of this kind of problem can easily become overwhelming, at which point it may be worthwhile to have recourse to some of the techniques of logic circuit design (see, e.g., Roth [6]) in order to establish systematic procedures.

References

[1] William R. Spillers, *Iterative structural design,* North Holland Publishing Co., Amsterdam, 1975

[2] David Gale, *The theory of linear economic models,* McGraw-Hill, New York, 1960

[3] William R. Spillers and E. Lefcochilos, *Realizability of force and displacement systems,* Proc. ASCE **ST8,** 1193–1202 (1978)

[4] William R. Spillers, *Automated structural analysis: an introduction,* Pergamon Press, New York, 1972

[5] Michel Simonnard, *Linear programming,* Prentice Hall, Englewood Cliffs, N. J., 1966

[6] J. Paul Roth, *Algebraic topogical methods for the synthesis of switching systems,* Trans. American Mathematical Society **88**, 301–326 (1958)

Appendix 1. The derivative. It can be useful to examine the effect of small variations in the primitive stiffness matrix K upon the member force matrix F. In order to do so small variations are introduced into Eq. (1) as

$$\begin{aligned} \tilde{N}F = P &\rightarrow \tilde{N}(F + dF) = P, \\ F = K\Delta &\rightarrow F + dF = (K + dk)(\Delta + d\Delta) \\ \Delta = N\delta &\rightarrow \Delta + d\Delta = N(\delta + d\delta). \end{aligned} \tag{A.1}$$

After some algebraic manipulation it follows that

$$d\delta = (\tilde{N}KN)^{-1}(-\tilde{N}dK\Delta) \tag{A.2}$$

and

$$dF = [I - KN(\tilde{N}KN)^{-1}\,\tilde{N}]\,dK\Delta = A dk\Delta \tag{A.3}$$

where

$$A = I - KN\,(\tilde{N}KN)^{-1}\tilde{N}. \tag{A.4}$$

Of primary interest here is the fact that Eq. (A.3) implies that the variation dF is a well-behaved function of dK whenever the system matrix $\tilde{N}KN$ possesses an inverse. In other words, for any (geometrically) stable structure the member forces are well-behaved functions of the primitive stiffnesses.

Furthermore, it can be noted that the matrix A is idempotent, i.e. $A.A = A$ and that A goes to zero for any statically determinate structure. Finally, Eq. (A.3) implies the expected result that $dF = 0$ for the case in which $dk \sim k$.