

## FLOWS WITH CONDENSATION\*

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**Abstract.** Some simple flows with condensation are considered and their solutions given. For the vapor phase, the nonlinearity of the equations of motion and of heat diffusion, and of the equation of state, and the dissipation due to shear and volume viscosities are taken into account. For the liquid phase the density is assumed constant. At the interface, where condensation takes place, the velocity, the stress, and the temperature gradient are all discontinuous. The same approach can be used for flows with evaporation.

**1. Introduction.** In the literature there are very few solutions of the Navier-Stokes equations for fluid flows with condensation. Studies of gas flows with evaporation at a wet boundary are much more common. But in these studies, with a few exceptions (notably the studies by Milton Plesset), the evaporating surface is treated at most as a source of vapor and a heat sink, and the concentration of vapor carried away by the flowing gas is considered to be a passive quantity with no effect on the flow except possibly through the action of gravity, since the Boussinesq approximation is invariably used. So far as I am aware, few attempts have been made to study the flow of the liquid phase and to match it with the flow of the vapor (or vapors) in contact with it, with the effects of the viscosities and the thermal diffusivity, the equation of state, and the condition of the vapor at the condensation surface all taken into account.

In this paper, a simple flow of the vapor of a pure substance downward toward a cooling porous plate will first be considered (Fig. 1). The cooling plate causes condensation and the liquid formed flows away through the pores of the plate. The solution of the Navier-Stokes and diffusion equations for this flow will be given. Then an inclined flow with condensation will be treated and flows with evaporation briefly discussed.

As will be seen presently, the solutions to be given have several novel features arising from the necessity of abandoning the many usual conditions of continuity at the vapor-liquid interface, such as the continuity of velocity, of stresses, and of heat flux.

**2. Equations governing normal flow of a vapor against a cooling plate.** Let  $y$  be measured in the direction opposite to that of the gravitational acceleration, and let the velocity component in the direction of increasing  $y$  be denoted by  $v$ . We consider steady downward flows of a vapor toward a cold plate. Far upstream, where  $y$  is taken to be (positive) infinite for convenience, the vapor has density  $\rho_0$ , velocity  $-v_0$ , temperature  $T_0$ , and pressure  $p_0$ . The position of the vapor-liquid interface will be taken to be the origin of  $y$ . At the plate, where the temperature is maintained at  $T_0 - \Delta T$ ,

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$$y = -d,$$

$d$  being the thickness of the liquid above the plate, to be determined by calculation.

As usual,  $\mu$  will denote the viscosity and  $\lambda$  the volume viscosity minus  $2\mu/3$ . The thermal conductivity, the specific heat at constant pressure, and the specific heat at constant volume will be denoted by  $k$ ,  $c_p$ , and  $c_v$ , respectively, and will be assumed constant. The ratio  $c_p/c_v$  will be denoted by  $\gamma$ . The density will be denoted by  $\rho$ , the pressure by  $p$ , and the absolute temperature by  $T$ .

The surface representing the equation of state of a pure substance, say water or water vapor, is well known. There is a surface of zero curvature in the space of pressure, temperature and specific volume, on which, if the temperature is maintained constant, the pressure will remain constant as the specific volume  $\rho^{-1}$  changes. This is the region of condensation or evaporation, in which the liquid phase and the vapor of the substance coexist. This developable surface is bounded on the right by the vapor-saturation curve and on the left by the liquid-saturation curve, meeting at the critical point above which the liquid state and the vapor state are separated by a curve on which the two states merge and are indistinguishable. The pressure and the temperature at the critical point are called the critical pressure and the critical temperature. Condensation or evaporation can take place only when the pressure is lower than the critical pressure  $p_{cr}$  and the temperature is lower than the critical temperature  $T_{cr}$ . The change of state at a pressure higher than the critical pressure does not involve evaporation or condensation but takes place through a state of indistinguishable phases without the release or absorption of latent heat.

The temperature at the plate, of course, must be below  $T_{cr}$  for condensation to take place. The problem is to determine the temperature and velocity fields and the thickness  $d$  of the liquid phase.

Since the flow is vertically downward,  $v$ , the only non-zero velocity component, is

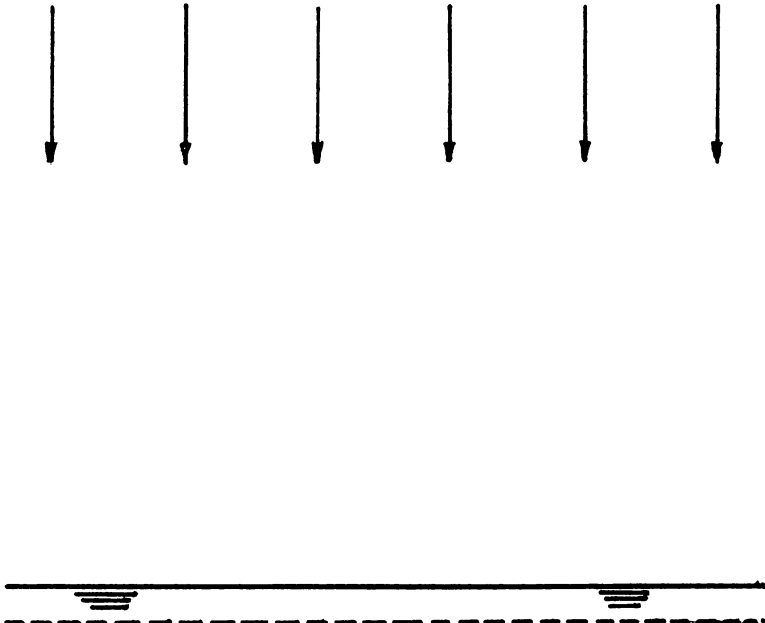


FIG. 1. Sketch showing vapor flow downward toward a porous cooling plate, with an overlying liquid layer.

everywhere negative, including the liquid region. The flow considered being steady,  $y$  is the only independent variable, and the equation of motion is, with primes indicating  $d/dy$ ,

$$\rho v v' = -g\rho - (p - \lambda v)' + (2\mu v')', \quad (1)$$

where  $g$  is the gravitational acceleration.

For the vapor phase we shall neglect the effect of gravity on the pressure and on the flow. This amounts to neglecting the variation in the hydrostatic pressure in comparison with the prevailing pressure, or neglecting the effect of the height of the apparatus, in any experiment done for the problem under study, on the pressure in the vapor. We continue to use  $y = \infty$  for the position far upstream, with the understanding that  $y$  is very large compared with a length scale to be specified later. Hence for the vapor we have

$$\rho v v' = -p' + [(\lambda + 2\mu)v']'. \quad (2)$$

The heat equation is

$$\rho c_v T' + p v' = (kT)' + (\lambda + 2\mu)v'^2, \quad (3)$$

where the last term represents the effect of viscous dissipation.

The equation of continuity is

$$\rho v = -\rho_0 v_0 = m, \quad (4)$$

and the equation of state for the vapor will be taken to be that of the ideal gas:

$$p = R\rho T, \quad (5)$$

where  $R$  is the gas constant. This equation is not strictly satisfied near the vapor-saturation curve, but the latent heat for the evaporation of water calculated by the use of this equation, for instance, agrees very well with the measured values. Hence the use of (5) is expected to introduce very little error.

**3. Preliminary integration of the equation of motion and of heat diffusion.** For the vapor phase, then, (4) allows the left-hand side of (2) to be written as  $-(mv)'$ , and a first integration of (2) gives

$$-mv = -p + (\lambda + 2\mu)v' + C_1, \quad (6)$$

where

$$C_1 = p_0 + mv_0 = R\rho_0 T_0 + mv_0. \quad (7)$$

Eqs. (4) and (6) then reduce (3) to the simple form

$$-mc_v T' + mvv' = (kT)' - C_1 v'. \quad (8)$$

Assuming  $c_v$  to be constant, we can integrate (8) to

$$mc_v(T - T_0) + \frac{m}{2} v^2 = kT' - C_1 v + C_2, \quad (9)$$

where

$$C_2 = \frac{m}{2} v_0^2 - C_1 v_0 = -mRT_0 - \frac{m}{2} v_0^2, \quad (10)$$

since  $T' = 0$  far upstream.

Eq. (6) can be written as

$$R\rho T = mv + (\lambda + 2\mu)v' + C_1,$$

or, by virtue of (4), as

$$-RmT = mv^2 + (\lambda + 2\mu)\nu v' + C_1\nu. \quad (11)$$

Eqs. (9) and (11) govern the flow of vapor. After  $\nu$  and  $T$  are found, (4) gives  $\rho$  and (5) gives  $p$ .

It is desirable to put (9) and (11) in dimensionless forms. For this purpose we use the new variables

$$V = \frac{\nu}{\nu_0} + 1, \quad \theta = \frac{T_0 - T}{T_0}, \quad \eta = \frac{mc_\nu}{k} y, \quad (12)$$

so that  $V$  and  $\theta$  vanish far upstream. Dividing (9) by  $m\nu_0^2$ , then multiplying the result by  $\gamma(\gamma - 1)M^2$ , we obtain

$$\theta + \frac{\gamma(\gamma - 1)}{2} M^2 V^2 = -\theta' - (\gamma - 1)V, \quad (13)$$

where

$$M = \nu_0/c_0, \quad c_0^2 = \gamma RT_0, \quad (14)$$

$c_0$  being the speed of sound at temperature  $T_0$ , so that  $M$  is the Mach number far upstream. The prime in (13) now indicates differentiation with respect to  $\eta$ .

Eq. (11), upon division by  $m\nu_0^2$  and then multiplication by  $\gamma M^2$ , becomes

$$\theta = \gamma M^2 (V - 1)(V + \sigma V') + V, \quad (15)$$

where

$$\sigma = \frac{\lambda + 2\mu}{k} c_\nu \quad (16)$$

is a kind of Prandtl number. Eqs. (13) and (15) will be solved together with the equations governing liquid flow and the boundary and interfacial conditions. The vanishing of  $\theta$  and  $V$  far upstream gives

$$\theta(\infty) = V(\infty) = 0. \quad (17)$$

The interfacial conditions will be given later.

**4. The flow of the liquid.** We shall use the subscript  $l$  to indicate the properties of the liquid, and shall take  $\rho_l$ , the density of the liquid, to be constant. The  $\nu_l$  is also constant, for

$$\rho_l \nu_l = -m. \quad (18)$$

Therefore only the heat equation needs to be solved for the liquid. This equation is still (3), except that  $\nu$  there is now the constant  $\nu_l$ . The definition of  $\theta$  is still given by (12), except that  $T$  is  $T_l$ , so that

$$\theta_l = (T_0 - T_l)/T_0.$$

Retaining the meaning of  $\eta$  defined in (13), we have

$$-\beta\theta_i' = \theta_i'', \tag{19}$$

where

$$\beta = \frac{k}{k_i} \frac{c_{vt}}{c_v}.$$

Since (19) is simple, we shall give its solution immediately. At the interface, where  $\eta = 0$ ,

$$\theta_i = (T_0 - T_i)/T_0 = \theta,$$

and

$$\theta_i = \Delta T/T_0 \equiv \alpha$$

on the plate, where

$$\eta = -\frac{mc_v}{k} d = -\delta. \tag{20}$$

Two integrations of (19) give

$$\theta_i = C_4 + C_3 \exp(-\beta\eta), \tag{21}$$

where the constants of integration are determined by the boundary conditions to be

$$C_3 = \frac{\alpha - \theta_i}{\exp(\beta\delta) - 1}, \quad C_4 = \theta_i - C_3. \tag{22}$$

**5. Interfacial conditions.** On the vapor-saturation curve, with  $F$  standing for “function,”

$$T_i = F(\rho_i), \tag{23}$$

where and henceforth the subscript  $i$  denotes the interface, and this condition has to be satisfied by the density and temperature of the *vapor* at the interface. In a numerical calculation, (23) can be used without approximation. For an analytical solution, and perhaps even for a numerical one, it is better to approximate (23) with

$$T_i = \frac{A}{\rho_i} + B = -\frac{A}{m} v_i + B,$$

in which, for a typical substance (such as water),  $A$  is negative and  $B$  positive. In terms of  $\theta$  and  $V$ , this can be written as

$$\theta_i = aV_i + b, \tag{24}$$

in which, for a typical substance (such as water), both  $a$  and  $b$  are negative. (Remember the definition of  $\theta$ .) We can use (24) with  $a$  and  $b$  chosen to approximate (23) as closely as possible. Then, when  $\theta$  is determined on the interface, we can go back to (23) to seek a pair of values of  $A$  and  $B$  for the tangent of the curve (23) for the temperature  $T_i$  determined. Then we use the new  $a$  and  $b$  corresponding to the new  $A$  and  $B$ , and so on. With the understanding that this iteration is always available, we shall assume  $a$  and  $b$  to be given and proceed with the calculation. We shall use  $h$  to denote the latent heat.

On the interface, where  $\eta = 0$ ,

$$\theta = \theta_i = \theta_{i'} \quad (25)$$

Furthermore, at the interface the heat-flux condition is

$$\theta'(0) - \frac{h}{c_p T_0} = \frac{k_i}{k} \theta_{i'}(0). \quad (26)$$

We note that the equation of state has been used in obtaining (8) and in going from (6) to (11). If  $p$  is found from (6) after  $v$  is determined and  $p_i$  is calculated, it will automatically satisfy the equation of state together with  $T_i$  and  $\rho_i$ .

**6. Solution of the differential system governing the flow.** Since the Mach number is expected to be small for most cases of interest, the forms of (15) and (16) suggest that the solution can be obtained by expansions of  $\theta$  and  $V$  in power series of  $M^2$ . Thus we take

$$\theta = \theta_0 + M^2 \theta_1 + M^4 \theta_2 + \dots, \quad (26)$$

$$V = V_0 + M^2 V_1 + M^4 V_2 + \dots. \quad (27)$$

Furthermore, since  $d$  will change when terms of higher orders in  $M$  are taken into account, we shall expand  $\delta$  in the power series

$$\delta = \delta_0 + M^2 \delta_1 + M^4 \delta_2 + \dots. \quad (28)$$

Then from (13) and (15) we have

$$\theta_0 = -\theta_0' - (\gamma - 1)V_0, \quad \theta_0 = V_0,$$

the solution of which is

$$\theta_0 = V_0 = a_0 \exp(-\gamma\eta), \quad (29)$$

which satisfies (17).

The solution for  $\theta_i$  is always given by (21). But  $C_3$  and  $C_4$  contain  $\theta_i$  and  $\delta$ . In particular,

$$C_3 = C_{30} + M^2 C_{31} + M^4 C_{32} + \dots, \quad (30)$$

in which

$$C_{30} = \frac{\alpha - \theta_{0i}}{\exp(\beta\delta_0) - 1},$$

$$C_{31} = \frac{1}{\exp(\beta\delta_0) - 1} \left[ \frac{\beta\delta_1}{\exp(\beta\delta_0) - 1} (\theta_{0i} - \alpha) - \theta_{1i} \right],$$

etc.

Substituting (29) into (24), we have

$$a_0 = \frac{b}{1 - a},$$

which is negative since both  $a$  and  $b$  are negative. The interfacial heat-flux condition (26) then gives

$$-\gamma a_0 - \frac{h}{\gamma T_0} = -\frac{k_i}{k} \beta C_{30},$$

which determines the  $\delta_0$  in  $C_3$ .

Extracting terms of order  $M^2$  in (13) and (15), and combining the two equations so obtained, we have

$$\theta_1' + \gamma\theta_1 = -2\gamma D \exp(-2\gamma\eta) + E \exp(-\gamma\eta), \tag{31}$$

where

$$D = \frac{1}{4}(\gamma - 1)(2\sigma\gamma - 1)a_0^2, \quad E = \gamma(\gamma - 1)(\sigma\gamma - 1)a_0.$$

The solution of (31) is

$$\theta_1 = a_1 \exp(-\gamma\eta) + E\eta \exp(-\gamma\eta) + D \exp(-2\gamma\eta), \tag{32}$$

which satisfies (17). Then from the equation extracted from (15) by taking terms of  $O(M^2)$ , we have

$$V_1 = \left( a_1 - \frac{E}{\gamma - 1} + E\eta \right) \exp(-\gamma\eta) + G \exp(-2\gamma\eta), \tag{33}$$

in which

$$G = \frac{1}{4}[6\sigma\gamma^2 - (2\sigma + 5)\gamma + 1]a_0^2.$$

It is evident that  $V_1$  satisfies (17).

The interfacial condition (24) demands

$$\theta_1(0) = aV_1(0),$$

so that

$$a_1 = \frac{1}{a - 1} \left( D + \frac{aE}{\gamma - 1} - aG \right). \tag{34}$$

The value of  $\theta_1$  at the interface is  $a_1 + D$ . The interfacial heat-flux condition is now

$$-a_1\gamma + E - 2\gamma D = (k_i/k) C_{31},$$

which determines the  $\delta_1$  in  $C_{31}$ .

Solutions for higher-order terms in the expansions (26), (27), and (28) proceed in the same way. But these are not necessary if the Mach number is less than  $\frac{1}{4}$ , say.

**7. A special solution.** If  $T_0$  and  $\rho_0$  already satisfy the relationship (23) for the evaporation or condensation curve, the solution is especially simple, for the solution of (15) and (13) is simply

$$\theta = 0 = V$$

for all values of  $y$  from the interface to positive infinity. The solution for  $\theta_1$  being always (21), the condition (25) gives

$$\frac{k_i}{k} C_3 = - \frac{h}{c_v T_0},$$

and the  $\delta$  in  $C_3$  (negative) can be determined once for all, whatever the value of the Mach number  $M$ .

For  $T_0$  and  $\rho_0$  that are not far from the  $T_i$  and  $\rho_i$  satisfying (23),  $V$  is nearly zero, and ignoring the nonlinear terms in (13) and (15), we can get the solutions once and for all, without using the Mach number expansions (26) to (28).

**8. Discussion.** Since (15) contains  $V'$ , the procedure of solution by expansion in power series of  $M^2$  reduces the order of the simultaneous differential equations (15) and (13), and this needs some clarification, which follows.

If one considers a flow in the neighborhood of the special solution just given, one can linearize (15) and (13) and obtain from them a linear second-order differential equation. This equation has two independent solutions, one of which becomes infinite at infinite  $y$ , and hence must be discarded. The useful one is the one produced by the expansion in power series in  $M$ .

We note also that use of the momentum equation between a section far upstream and the surface of the plate gives

$$p_0 + mv_0 + g\rho_0 d = p_p + mv_p,$$

where  $p_p$  is the pressure at the plate. For steady flows to exist the pressure at the surface of the porous plate must be maintained at the  $p_p$  so determined. Otherwise the interface cannot remain stationary, or there will be a moving shock wave in the vapor phase.

It is interesting, but ultimately not very useful, that the two nonlinear differential equations (13) and (15) can be combined to give two first-order nonlinear differential equations that can be solved in sequence. The first equation is of the form

$$[\theta + f_1(V)] \frac{d\theta}{dV} = f_2(V)\theta + f_3(V),$$

and the second equation is just (15), which can be solved once  $\theta$  is determined as a function of  $V$  from the above equation. Unfortunately a simple analytical solution is not possible, and in any numerical calculation one will encounter the difficulty of having to start at infinity. Hence the Mach-number expansion is still the best way to solve the problem.

**9. Inclined flow of vapor toward a cold plate.** If the velocity at infinite  $y$  is  $(u_0, -v_0)$ ,  $u$  being the velocity in the (horizontal) direction of increasing  $x$ , the two velocity components are uncoupled:  $v$  is given by the solution given in the foregoing sections, and  $u = u_0$  for the vapor. For the liquid

$$u_l = u_{il} \left(1 + \frac{y}{d}\right),$$

where  $u_{il}$  is the interfacial value of  $u_l$ . This interfacial value is determined by the equation for tangential momentum at the interface, i.e.,

$$\frac{u_l u_{il}}{d} = m(u_0 - u_{il}).$$

These results follow immediately from the equations of motion, which will not be presented in detail.

The streamlines start as inclined straight lines far upstream, become curvilinear as



they approach the cold plate, suffer a discontinuity in their slope at the interface, and intersect the plate at right angles.

**10. Flows with evaporation.** We have discussed flows with condensation. If a porous plate is heated, and liquid approaches it normally or obliquely from below, the treatment is similar. We shall not give the details since all the points have been made in this paper.

We note that at the surface of the porous plate permitting the vapor to flow through, the normal stress must be specified to permit a solution. This normal stress is to be matched to that in the porous medium. The specification of the normal stress amounts to the specification of the velocity gradient, since the pressure is known once the temperature and the density (via the velocity) are specified.