ADDITIONAL PSEUDO-SIMILARITY SOLUTIONS OF THE HEAT EQUATION IN THE PRESENCE OF MOVING BOUNDARIES*

By

R. J. TAIT

University of Alberta, Edmonton

1. Introduction. Moving and free boundary problems for the heat equation have a number of practical applications. Since exact solutions of the free boundary problem, except in a few cases, are not available, a number of authors have considered the problem of finding exact solutions for boundary conditions given on a prescribed moving boundary. When such solutions are available we have an approximation to the free boundary problem in the sense that the solution provides a possible state of the medium, provided that heat can be supplied externally in a prescribed way. Again, it may be of interest to ask whether we can force the boundary to move in a given way by satisfying these additional external requirements. Such a situation might arise for example in the thawing of pipes, the dyeing of fibres, the immersion of plant stems in solution, or in the freezing of foodstuffs. Progress has been made in this direction by introducing similarity variables and transformations.

Langford [1] considered the equation

$$u_{rr} + \frac{2\nu + 1}{r} u_r - u_t = 0, \quad 0 < r < R(t), \quad 0 < t < T, \tag{1.1}$$

with suitable initial conditions on the interval $(0, A)$ and given boundary conditions on $r = 0$ and on the moving boundary $r = R(t)$, where

$$R(t) = (A + Bt)^{1/2}. \tag{1.2}$$

The important physical cases are $\nu = \pm 1/2, 0$.

Langford's solutions have been extended by Bluman [2]. In [2] general invariance properties of a class of equations, including (1.1) in the case $\nu = -1/2$ have been investigated and similarity solutions found where

$$R^2(t) = \alpha - 2\beta t - \gamma t^2. \tag{1.3}$$

There are a number of particular results on problems of the above type in the literature, and in particular Gibson [3] has obtained a class of solutions of (1.1) when forcing terms are present by apparently unrelated methods.

In the present paper we consider Eq. (1.1) for general $\nu \geq -1/2$, subject to moving boundaries of the form (1.3), and indicate how a number of problems can be solved. In particular, Gibson's results are included.

* Received July 6, 1978.

2. Transformations of the equation. Consider the equation

$$u_{rr} + \frac{2\nu + 1}{r} u_r - u_t = 0, \tag{2.1}$$

in a suitable region of (r, t) space. The substitutions

$$u = A(r, t)\, w(\xi, \tau),\ \xi = r/R(t), \tag{2.2}$$

transform Eq. (2.1) to the form

$$w_{\xi\xi} + \frac{2\nu + 1}{\xi} w_\xi - \frac{\delta^2\xi^2}{4} w - w_\tau = 0, \tag{2.3}$$

where

$$R^2(t) = \alpha - 2\beta t - \gamma t^2,\ A(r, t) = R^{-(\nu+1)} \exp\left(-\frac{r^2\dot{R}}{4R}\right),\ \frac{d\tau}{dt} = \frac{1}{R^2(t)},\ \delta = (\alpha\gamma + \beta^2)^{1/2} \geq 0. \tag{2.4}$$

In terms of variable separable solutions,

$$w = \xi^{-(\nu+1/2)} \exp(-\chi\tau) Q(\xi), \tag{2.5}$$

$$Q'' + \left(\chi - \frac{\nu^2 - \frac{1}{4}}{\xi^2} - \frac{\delta^2\xi^2}{4}\right) Q = 0. \tag{2.6}$$

For $\delta = 0$, Eq. (2.6) is Bessel's equation, and if $\delta \neq 0$ we obtain solutions in terms of Whittaker functions

$$\xi^{-1/2} M_{\chi/2\delta,\nu/2}\left(\frac{\delta\xi^2}{2}\right),\quad \xi^{-1/2} W_{\chi/2\delta,\nu/2}\left(\frac{\delta\xi^2}{2}\right). \tag{2.7}$$

The effect of the similarity variable ξ is to transform conditions on the moving boundary $r = R(t)$ to conditions on a fixed boundary. The form of $R(t)$ given by (2.4) allows us to examine interior and exterior problems in the following cases: if $\delta > 0$, $\gamma = 0$, we have parabolas, opening down for $\beta > 0$, and up for $\beta < 0$. For $\gamma > 0$, we have ellipses, and for $\gamma < 0$, hyperbolas. If $\delta = 0$, we have the limiting case of straight lines. In all cases $\alpha \geq 0$.

3. Eigenfunction expansions. Solutions of (2.1) subject to suitable conditions on the moving and fixed boundary will follow from eigenfunction expansions of (2.3). We follow the notation of Titchmarsh [4].

If $\delta > 0$, set

$$\eta = \sqrt{\frac{\delta}{2}}\ \xi,\quad \lambda = 2\chi/\delta, \tag{3.1}$$

in Eq. (2.6), so that the equation reads

$$Q'' + \left(\lambda - \frac{\nu^2 - 1/4}{\eta^2} - \eta^2\right) Q = 0. \tag{3.2}$$

If $\delta = 0$, (2.6) is already of the form (3.2) with the last term inside the bracket omitted. On any interval (η_1, η_2), $0 < \eta_1 < \eta_2 < \infty$, the equation is regular and the procedure standard (see Langford [1]). We note the appropriate expansions on other intervals of interest.

If $\delta = 0$, then on $(0, \Lambda)$ we have a standard Fourier Bessel expansion, while on (Λ, ∞) we have

$$f(\eta) = \int_0^\infty \frac{sK(\eta, \Lambda; s)}{J_\nu^2(\Lambda s) + Y_\nu^2(\Lambda s)}\, ds \int_\Lambda^\infty K(y, \Lambda; s) f(y) dy, \tag{3.3}$$

with

$$K(y, \Lambda; s) = \eta^{1/2}\{J_\nu(ys)Y_\nu(\Lambda s) - Y_\nu(ys)J_\nu(\Lambda s)\}. \tag{3.4}$$

For $\delta > 0$, $\nu \geq -1/2$, the expansion on $(0, \Lambda)$ is

$$f(\eta) = \sum_{n=1}^{\infty} A_n \eta^{-1/2} M_{\lambda_n/4, \nu/2}(\eta^2), \tag{3.5}$$

with

$$A_n = -\left(\frac{\Lambda}{M_n' M_n^*}\right) \int_0^\Lambda y^{-1/2} M_{\lambda_n/4, \nu/2}(y^2) f(y) dy,$$

$$M_n' = \frac{\partial}{\partial \Lambda} M_{\lambda_n/4, \nu/2}(\Lambda^2),\ M_n^* = -\frac{\partial}{\partial \lambda_n} M_{\lambda_n/4, \nu/2}(\Lambda^2), \tag{3.6}$$

and λ_n are the zeros of $M_{\lambda/4, \nu/2}(\Lambda^2)$.

In most cases it is more convenient to work with the confluent hypergeometric function $F(a, b, z)$. Write

$$M_{\lambda_n/4, \nu/2}(\eta^2) = \eta^{\nu+1} \exp\left(-\frac{\eta^2}{2}\right) \phi_n^\nu(\eta^2), \tag{3.7}$$

where

$$\phi_n^\nu(\eta^2) = F\left(\frac{\nu+1}{2} - \frac{\lambda_n}{4}, \nu + 1, \eta^2\right). \tag{3.8}$$

Then we have

$$g(\eta) = \sum_{n=1}^{\infty} C_n \phi_n^\nu(\eta^2), \qquad 0 < \eta < \Lambda, \tag{3.9}$$

with

$$C_n = -\frac{\Lambda^{1-2\nu} \exp(\Lambda^2)}{\theta_n' \cdot \theta_n^*} \int_0^\Lambda y^{1+2\nu} \exp(-y^2) \phi_n^\nu(y^2) g(y)\, dy,$$

$$\theta_n' = \frac{\partial}{\partial \Lambda} \phi_n^\nu(\Lambda^2), \quad \theta_n^* = -\frac{\partial}{\partial \lambda_n} \phi_n^\nu(\Lambda^2). \tag{3.10}$$

In particular it is useful to note that

$$\frac{F(\nu + 1 + \alpha, \nu + 1, \eta^2)}{F(\nu + 1 + \alpha, \nu + 1, \Lambda^2)} = \sum_{n=1}^{\infty} \left\{\lambda_n + 4\left(\alpha + \frac{\nu+1}{2}\right)\right\}^{-1} \theta_n^{*-1}(\Lambda^2) \phi_n^\nu(\eta^2). \tag{3.11}$$

The behavior of series of the above type is best discussed with reference to (3.5) and (3.6) since the notation there is in line with Titchmarsh [4]. Set

$$\phi(\eta, \lambda) = \frac{\Gamma\left(\frac{\nu+1}{2} - \frac{\lambda}{4}\right)}{2\Lambda^{1/2}\eta^{1/2}\Gamma(\nu+1)} \{W_{\lambda/4,\nu/2}(\Lambda^2)M_{\lambda/4,\nu/2}(\eta^2) - M_{\lambda/4,\nu/2}(\Lambda^2)W_{\lambda/4,\nu/2}(\eta^2)\} \tag{3.12}$$

and

$$\psi(\eta, \lambda) = \frac{\eta^{-1/2}M_{\lambda/4,\nu/2}(\eta^2)}{\Lambda^{-1/2}M_{\lambda/4,\nu/2}(\Lambda^2)}. \tag{3.13}$$

Then the appropriate Green's function is

$$\Phi(\eta, \lambda) = \phi(\eta, \lambda) \int_0^\eta \psi(s, \lambda) f(s)\, ds + \psi(\eta, \lambda) \int_\eta^\Lambda \phi(s, \lambda) f(s)\, ds. \tag{3.14}$$

With $\lambda = s^2$, $s = \sigma + i\tau$, then on the quarter square

$$\Lambda\sigma = n\pi + \frac{\pi\nu}{2} + \frac{\pi}{4}, \qquad \Lambda\tau = n\pi + \frac{\pi\nu}{2} + \frac{\pi}{4}$$

we have

$$M_{\lambda/4,\nu/2}(\eta^2) = 2^\nu\Gamma(1+\nu)\left(\frac{2\eta}{\pi}\right)^{1/2} s^{-\nu-1/2} \cos\left(\eta s - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + O\{|s|^{-\nu-3/2} \exp(\eta\,|\tau|)\} \tag{3.15}$$

when n is large, $0 < \delta \le \eta \le \Lambda$, valid, according to Slater [5] for

$$-\pi/2 \le \arg(s) \le \pi/2.$$

Then

$$\psi(\eta, \lambda) = \frac{\cos\left(\eta s - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)}{\cos\left(\Lambda s - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)} + O[|s|^{-1} \exp\{(\eta - \Lambda)\,|\tau|\}], \tag{3.16}$$

$$\phi(\eta, \lambda) = -\frac{\sin s(\Lambda - \eta)}{s} + O[|s|^{-2} \exp\{(\Lambda - \eta)\,|\tau|\}]. \tag{3.17}$$

If $\Phi_1(\eta, \lambda)$ denotes the part of (3.14) involving the integral from 0 to δ, and $\Phi_2(\eta, \lambda)$ the remainder, then, since

$$\int_0^\delta p^{-1/2}M_{\lambda/4,\nu/2}(p^2)f(p)\, dp = O[\varepsilon\,|s|^{-\nu-1/2} \exp(\delta\,|\tau|)] \tag{3.18}$$

following Titchmarsh [4] and Slater [5, Eq. (4.4.29)], it follows that

$$\Phi_1(\eta, \lambda) = O[\varepsilon|s|^{-1} \exp\{(\delta - \eta)\,|\tau|\}] = O(\varepsilon), \qquad \eta \ge \delta. \tag{3.19}$$

The asymptotic estimates can now be substituted in $\Phi_2(\eta, \lambda)$ and the remaining steps to establish convergence of the series (3.5) are identical with [4].

For the interval (Λ, ∞) similar steps indicate that the corresponding expressions are

$$f(\eta) = \sum_{n=1}^{\infty} A_n \eta^{-1/2} W_{\lambda_n/4, \nu/2}(\eta^2),$$

$$A_n = \frac{\Lambda}{W_n'(\Lambda^2) W_n^*(\Lambda^2)} \int_{\Lambda}^{\infty} p^{-1/2} W_{\lambda_n/4, \nu/2}(p^2) f(p)\, dp,$$

with notation analogous to (3.5), (3.6).

4. The ablation problem. Let I denote the interval $(0, R(t))$ or $(R(t), \infty)$ depending on whether we deal with the interior or exterior problem, and T denote the terminal time, being finite or infinite according as the boundary moves in or out. The problem

$$u_{rr} + \frac{2\nu + 1}{r} u_r - u_t = 0, \quad \text{on} \quad I \times (0, T), \tag{4.1}$$

$$u(r, 0) = f(r), \quad \text{on} \quad I, \qquad u(R(t), t) = g(t), \qquad 0 < t < T, \tag{4.2}$$

is referred to as the ablation problem by Langford [1], and Bluman [2] in the case $g(t) = 0$. $R(t)$ has the form described in sec. 2. In general we assume a condition of the form

$$u_r(0, t) = 0, \qquad 0 < t < T, \tag{4.3}$$

on the fixed boundary. Obviously, other boundary conditions are possible.

Evidently the expansions of Sec. 3 supply a basis for the solutions of problems of this type and extend the cases considered in [1] and [2]. For example, in the case

$$\delta = 0, \quad g(t) = 0, \quad R(t) = A + Bt, \quad A > 0, \tag{4.4}$$

we have, in the notation of Sec. 2,

$$u(r, t) = (A + Bt)^{-(\nu+1)} \exp\left\{-\frac{Br^2}{4(A + Bt)}\right\} w(\xi, \tau), \tag{4.5}$$

$$w(\xi, \tau) = \xi^{-\nu} \sum_{n=1}^{\infty} C_n J_\nu(\xi\lambda_n) \exp(-\lambda_n^2 \tau), \tag{4.6}$$

$$\xi = \frac{r}{R(t)}, \quad \tau = \frac{1}{AB} - \frac{1}{B(A + Bt)}, \tag{4.7}$$

and the λ_n are the zeros of $J_\nu(\lambda)$.

The coefficients C_n are given by

$$C_n = \frac{2A^{\nu+1}}{J_{\nu+1}^2(\lambda_n)} \int_0^1 p^{\nu+1} \exp\left(\frac{ABp^2}{4}\right) f(Ap) J_\nu(p\lambda_n) dp. \tag{4.8}$$

The second part of the problem arises if a variable condition is imposed on the moving boundary or if the differential equation is inhomogeneous with time-dependent forcing term, a case which arises if an internal reaction takes place. Duhamel's integral allows us to write the solution in the form

$$w(\xi, \tau) = \int_0^{\tau} G(p) \frac{\partial}{\partial \tau} U(\xi, \tau - p) dp, \tag{4.9}$$

where $F(r) = 0$, and $G(t)$ is the transformed boundary condition.

For the case above, (4.4), we have

$$U(\xi, \tau) = 1 - 2\xi^{-\nu} \sum_{n=1}^{\infty} \frac{J_\nu(\xi\lambda_n) \exp(-\lambda_n^2\tau)}{\lambda_n J_{\nu+1}(\lambda_n)}, \tag{4.10}$$

so that

$$u(r, t) = 2(A + Bt)^{-1} r^{-\nu} \exp\left\{\frac{-Br^2}{4(A + Bt)}\right\} \sum_{n=1}^{\infty} \frac{\lambda_n Q_n(t) J_\nu\{\lambda_n r/(A + Bt)\}}{J_{\nu+1}(\lambda_n)}, \tag{4.11}$$

the coefficients being given by

$$Q_n(t) = \exp\left\{\frac{\lambda_n^2}{B(A + Bt)}\right\} \int_0^t (A + Bp)^{\nu-1} g(p) \exp\left\{\frac{B(A + Bp)}{4} - \frac{\lambda_n^2}{B(A + Bp)}\right\} dp. \tag{4.12}$$

For a simple polynomial

$$g(t) = (A + Bt)^{\alpha}, \tag{4.13}$$

$Q_n(t)$ takes the form

$$Q_n(t) = \exp\left\{\frac{\lambda_n^2}{B(A + Bt)}\right\} \int_{1/B(A+Bt)}^{1/AB} (Bq)^{-\nu-\alpha-1} \exp\left(\frac{1}{4q} - q\lambda_n^2\right) dq, \tag{4.14}$$

so that if $\alpha = 0$, (4.11) reduces to the expected Bessel expansion on noting that

$$\int_\alpha^\beta q^{-\nu-1} \exp\left(\pm \frac{1}{4q} \mp \lambda_n^2 q\right) dq = \frac{1}{\lambda_n J_{\nu+1}(\lambda_n)} \left[z^{-\nu-1} \exp(\mp z\lambda_n^2) \int_0^1 p^{\nu+1} \exp\left(\pm \frac{p^2}{4z}\right) J_\nu(p\lambda_n) dp\right]_{z=\alpha}^{\beta} \tag{4.15}$$

If $\delta > 0$, U is given by

$$U(\xi, \tau) = \exp\left(\frac{\delta(1 - \xi^2)}{4}\right) \left[\frac{\phi_0^\nu\left(\frac{\delta\xi^2}{2}\right)}{\phi_0^\nu\left(\frac{\delta}{2}\right)} - \sum_{n=1}^{\infty} \frac{\phi_n^\nu\left(\frac{\delta\xi^2}{2}\right) \exp\left(-\frac{\delta\lambda_n\tau}{2}\right)}{\lambda_n \theta_n^*\left(\frac{\delta}{2}\right)}\right] \tag{4.16}$$

If we now take

$$R(t) = (A + Bt)^{1/2}, \quad A > 0, \quad B > 0,$$

$$\tau = \left(\frac{1}{B}\right) \ln\left(1 + \frac{Bt}{A}\right), \quad t > 0, \quad \delta = B/2, \tag{4.17}$$

then the solution in the parabolic case is

$$u(r, t) = \frac{B}{4} \exp\left\{\frac{B}{4}\left(1 - \frac{r^2}{A + Bt}\right)\right\} \sum_{n=1}^{\infty} \frac{Q_n(t)\phi_n^\nu\left(\frac{B\xi^2}{4}\right)}{\theta_n^*\left(\frac{B}{4}\right)}, \tag{4.18}$$

where

$$Q_n(t) = (A + Bt)^{-((\nu+1)/2)-(\lambda_n/4)} \int_0^t (A + Bs)^{((\nu-1)/2)+(\lambda_n/4)} g(s)ds. \tag{4.19}$$

In the hyperbolic case, we have

$$R(t) = \{\kappa(t+t_1)(t+t_2)\}^{1/2}, \qquad \kappa > 0, \qquad t_2 > t_1 > 0,$$

$$\tau = \frac{1}{2\delta} \ln \left\{ \frac{t_2(t + t_1)}{t_1(t + t_2)} \right\}, \quad \delta = \frac{\kappa}{2}(t_2 - t_1), \tag{4.20}$$

and it follows that

$$u(r, t) = \frac{\delta}{2\kappa} \{(t + t_1)(t + t_2)\}^{-(\nu+1/2)} \exp\left\{\frac{\delta}{4} - \frac{r^2}{4(t + t_1)}\right\} \sum_{n=1}^{\infty} \frac{Q_n(t)\phi_n{}^\nu\left(\frac{\delta\xi^2}{2}\right)}{\theta_n{}^*\left(\frac{\delta}{2}\right)}. \tag{4.21}$$

The coefficients differ slightly from the previous case and are given by

$$Q_n(t) = \left(\frac{t + t_1}{t + t_2}\right)^{-(\lambda_n/4)} \int_0^t (s + t_1)^{((\nu-1)/2)+(\lambda_n/4)}(s + t_2)^{((\nu-1)/2)-(\lambda_n/4)} g(s) \exp\left\{\frac{\kappa}{8}(2s + t_1 + t_2)\right\} ds. \tag{4.22}$$

There does not appear to be any significant simplification in these formulae even when $g(s)$ is simple. For example, if we set $g(s) = s$ in Eqs. (4.17) and (4.18) we find that $u(r, t)$ is given as the sum of two terms $u_1(r, t)$ and $u_2(r, t)$ where

$$u_1(r, t) = \left(t + \frac{r^2 - A}{4(\nu + 1)}\right) \Big/ \left(1 + \frac{B}{4(\nu + 1)}\right)$$

together with the correction term

$$u_2(r, t) = \frac{A}{4B}\left(1 + \frac{Bt}{A}\right)^{-(\nu+1)/2} \exp\left\{\frac{B}{4}\left(1 - \frac{r^2}{4(A + Bt)}\right)\right\}$$

$$\cdot \sum_{n=1}^{\infty} \frac{(A + Bt)^{-\lambda_n/4}\phi_n{}^\nu\left(\frac{B\xi^2}{4}\right)}{\left(\frac{\lambda_n}{4} + \frac{\nu + 3}{2}\right)\left(\frac{\lambda_n}{4} + \frac{\nu + 1}{2}\right)\theta_n{}^*\left(\frac{B}{4}\right)}. \tag{4.23}$$

Similar results follow for other forms of $R(t)$.

5. The control problem.

Langford [1] has considered the two-phase problem

$$u_{rr} + \frac{2\nu + 1}{r} u_r - \kappa u_t = 0, \qquad 0 < r < R(t), \tag{5.1}$$

$$u_{rr}{}^* + \frac{2\nu + 1}{r} u_r{}^* - \kappa^* u_t{}^* = 0, \qquad R(t) < r < A, \tag{5.2}$$

subject to the conditions

$$u_r(0, t) = 0, \qquad 0 < t < A/B,$$
$$u(r, 0) = f(r), \qquad 0 < r < A, \tag{5.3}$$
$$u(R(t), t) = u^*(R(t), t) = 0, \qquad 0 < t < A/B,$$
$$u_r(R(t), t) - u_r^*(R(t), t) = \kappa \dot{R}(t), \qquad 0 < t < A/B, \tag{5.4}$$

in the case $R(t) = (A - Bt)^{1/2}$, $A > 0$, $B > 0$, supposing it possible to control the moving boundary by supplying suitable values of u^*, u_r^* on the outer face $r = A$. The technique used is to split u^* into two parts

$$u^* = u_1{}^* + \bar{u},$$

and choose $u_1{}^*$ to satisfy (5.2), (5.3), together with (5.4) when the right-hand side is set equal to zero. Then we require $\bar{u}$ to satisfy (5.2); (5.3) and (5.4) then give

$$\bar{u}(R(t), t) = 0, \qquad \bar{u}_r(R(t), t) = -k\dot{R}(t). \tag{5.5}$$

The constant κ^* is suppressed in the following for simplicity.

In the notation of Sec. 2, for the general form of $R(t)$ considered there, the corresponding problem for $\bar{w}(\xi, \tau)$ becomes

$$\bar{w}_{\xi\xi} + \frac{2\nu + 1}{\xi}\bar{w}_\xi - \frac{\delta^2\xi^2}{4}\bar{w} - \bar{w}_\tau = 0, \qquad \xi > 1, \qquad \tau > 0, \tag{5.6}$$

$$\left.\begin{aligned} &\bar{w}(1, \xi) = 0, && \tau > 0, \\ &\bar{w}_\xi(1, \tau) = -kR^{\nu+2}\dot{R}\exp(R\dot{R}/4), && \tau > 0. \end{aligned}\right\} \tag{5.7}$$

In the parabolic case

$$R(t) = (A - Bt)^{1/2}, \qquad \delta = B/2,$$
$$\tau = -\ln\left(1 - \frac{Bt}{A}\right), \qquad 0 < \tau, \tag{5.8}$$
$$\bar{u} = (A - Bt)^{-(\nu+1)/2}\exp\left\{\frac{Br^2}{8(A - Bt)}\right\}\bar{w},$$

we have

$$\bar{w}(1, \tau) = 0, \qquad \bar{w}_\xi(1, \tau) = \delta k A^{(\nu+1)/2}\exp\left\{-\frac{\delta}{4} - \delta(\nu + 1)\tau\right\}, \tag{5.9}$$

and the required solution reduces to the limiting case of the Whittaker equation (2.6) with $\chi = \delta(\nu + 1)$. Then

$$\bar{w}(\xi, \tau) = \delta k A^{(\nu+1)/2}\exp\left\{-\frac{\delta}{2} - \delta(\nu + 1)\tau - \frac{\delta\xi^2}{4}\right\}\int_1^\xi p^{-(2\nu+1)}\exp\left(\frac{\delta p^2}{2}\right)dp \tag{5.10}$$

and as a result

$$\bar{u}(r, t) = -\frac{kB}{2}\exp\left(-\frac{B}{4}\right)\int_1^{r/\sqrt{A-Bt}} p^{-(2\nu+1)}\exp\left(\frac{Bp^2}{4}\right)dp. \tag{5.11}$$

The linear case

$$R(t) = A - Bt, \qquad \delta = 0, \qquad \tau = \frac{1}{B(A - Bt)} - \frac{1}{AB},$$

leads to the more complicated expression

$$\bar{w}(\xi, \tau) = -\frac{\pi k}{2}\left(\frac{B}{2}\right)^{-(\nu+1)} \xi^{-\nu} \int_0^\infty \lambda^{(\nu+1)/2} J_{\nu+1}(\sqrt{\lambda}) K(\sqrt{\lambda}, \xi) \exp\left(-\lambda\left(\tau + \frac{1}{AB}\right)\right) d\lambda, \tag{5.12}$$

where

$$K(\sqrt{\lambda}, \xi) = J_\nu(\sqrt{\lambda}, \xi) Y_\nu(\sqrt{\lambda}) - Y_\nu(\sqrt{\lambda}, \xi) J_\nu(\sqrt{\lambda}). \tag{5.13}$$

The expression (5.12) does not appear to simplify in general. However in the two-dimensional case, $\nu = -1/2$, we have

$$\bar{u}(r, t) = k[\exp\{B(r - (A - Bt))\} - 1]. \tag{5.14}$$

In the elliptic case

$$R(t) = (\kappa(t + t_1)(t_2 - t))^{1/2}, \qquad \kappa > 0, \qquad t_1 > t_2 > 0, \tag{5.15}$$

a formal computation gives

$$\bar{u}(r, t) = U(r, t) \sum_{n=0}^{\infty} (-1)^n A_n \left(\frac{t_2 - t}{t + t_1}\right)^n Q_n\left(\frac{r}{R(t)}\right), \tag{5.16}$$

with

$$U(r, t) = -(\delta k) \exp\left(-\frac{\delta}{2} - \frac{r^2}{4(t + t_1)}\right)\left(\frac{t_1 + t_2}{t + t_1}\right)^{(\nu+1)/2} \tag{5.17}$$

where

$$A_0 = 1, \qquad A_n = L_n^{\nu+1}\left(\frac{\delta}{2}\right) + L_{n-1}^{\nu+1}\left(\frac{\delta}{2}\right),$$

$$Q_0(\xi) = \int_1^\xi p^{-2\nu-1} \exp\left(\frac{\delta p^2}{2}\right) dp,$$

$$\begin{aligned} 2\nu Q_n(\xi) = {} & F\left(-n, \nu + 1, \frac{\delta\xi^2}{2}\right) F\left(-n - \nu, 1 - \nu, \frac{\delta}{2}\right) \\ & - F\left(-n, \nu + 1, \frac{\delta}{2}\right) F\left(-n - \nu, 1 - \nu, \frac{\delta\xi^2}{2}\right), \qquad \nu \neq 0, 1, 2, \cdots \end{aligned}$$

and $L_n^\alpha(z)$ is the Laguerre polynomial. For integral values of ν it is necessary to replace $F(-n - \nu, 1 - \nu, z)$ by the appropriate logarithmic solution of the confluent hypergeometric equation. For $\nu = 0$, we can obtain the correct form of $Q_n(\xi)$, $n > 0$, by taking the limit as $\nu \to 0$.

The solutions above hold for $R(t) > 0$. As $R(t) \to 0$, or more generally as t approaches its upper limit, the expressions become unbounded. This implies that, with the exception of (5.14), the process cannot be forced to follow the path $r = R(t)$ as far as $r = 0$, and must be halted before this stage.

6. Gibson problems. Gibson [3] considered the problem

$$u_{rr} + \frac{2\nu + 1}{r} u_r - u_t = F(t), \quad 0 < r < R(t), \quad 0 < t, \tag{6.1}$$

$$\begin{aligned} u(R(t), t) &= u_0, \quad 0 < t, \\ u_r(0, t) &= 0, \qquad 0 < t, \end{aligned} \tag{6.2}$$

when $\nu = 1/2$ and $R(t)$ has the form Gt or $G\sqrt{t}$, for G constant. Clearly this reduces to the type of problem considered here with $\bar{u} = u + u_0 - g(t)$, and $g'(t) = F(t)$.

Consider the solutions developed in Secs. 3 and 4. If $\delta = 0$, we have $R(t) = A + Bt$, so that (4.10) and (4.11) give, when $A \to 0$,

$$u(r, t) = 2(Bt)^{-1} r^{-\nu} \exp\left(-\frac{r^2}{4t}\right) \sum_{n=1}^{\infty} \frac{Q_n(t)\lambda_n J_\nu\left(\frac{\lambda_n r}{Bt}\right)}{J_{\nu+1}(\lambda_n)} \tag{6.3}$$

where

$$Q_n(t) = \exp\left\{\frac{\lambda_n^2}{B^2 t}\right\} \int_0^t (Bq)^{\nu-1} \exp\left\{\frac{B^2 q}{4} - \frac{\lambda_n^2}{B^2 q}\right\} g(q) dq. \tag{6.4}$$

If in (6.4) we set $p = 1/B^2 q$, $g(t) = Gt^\alpha$, and note that

$$\frac{I_\nu(\xi\sqrt{p})}{I_\nu(\sqrt{p})} = 2 \sum_{n=1}^{\infty} \frac{\lambda_n J_\nu(\lambda_n \xi)}{(\lambda_n^2 + p) J_{\nu+1}(\lambda_n)}, \quad 0 < \xi < 1, \tag{6.5}$$

$$2^{\nu+\alpha} \int_0^\infty \tau^{\nu+\alpha/2} I_{\nu+\alpha}(\sqrt{\tau}) \exp(-p\tau) d\tau = p^{-\nu-\alpha-1} \exp\left(\frac{1}{4p}\right), \quad \mathscr{R}(\nu + \alpha) > -1, \tag{6.6}$$

then

$$u(r, t) = \bar{G} t^{-1} r^{-\nu} \exp\left(-\frac{r^2}{4t}\right) \int_0^\infty \tau^{(\nu+\alpha)/2} \frac{I_{\nu+\alpha}(\sqrt{\tau}) I_\nu\left(\frac{r\sqrt{\tau}}{Bt}\right)}{I_\nu(\sqrt{\tau})} \exp\left(-\frac{\tau}{B^2 t}\right) d\tau, \tag{6.7}$$

in agreement with [3], when $\bar{G} = 2^{\nu+\alpha} B^{-\nu-2-2\alpha} G$, and $\nu = 0, \frac{1}{2}$.

Again, if $\delta > 0$, then with the notation of Sec. 4, $w(\xi, \tau)$ is given by (4.9) and $U(\xi, \tau)$ by (4.16). In the parabolic case

$$\begin{aligned} R(t) &= (A + Bt)^{1/2}, \quad \tau = B^{-1} \ln\left(1 + \frac{Bt}{A}\right), \quad B > 0, \\ u(r, t) &= (A + Bt)^{-(\nu+1)/2} \exp\left\{-\frac{Br^2}{8(A + Bt)}\right\} w(\xi, \tau), \end{aligned} \tag{6.8}$$

we obtain, on substituting the appropriate terms in $w(\xi, \tau)$, and converting the integral to one from 0 to t, on letting $A \to 0$,

$$u(r, t) = \frac{1}{4} \exp\left(\frac{B}{4}\right) t^{-(\nu+1)/2} \exp\left(-\frac{r^2}{4t}\right) \sum_{n=1}^{\infty} Q_n(t) \phi_n^\nu\left(\frac{B\xi^2}{4}\right) \theta_n^{\cdot -1}\left(\frac{B}{4}\right), \tag{6.9}$$

with

$$Q_n(t) = t^{-\lambda_n/4} \int_0^t s^{((\nu-1)/2)+(\lambda_n/4)} g(s) ds. \tag{6.10}$$

Thus if $g(t) = Gt^\alpha$ we have, on using (3.11),

$$u(r, t) = Gt^\alpha \exp\left(\frac{B}{4}\right) \frac{F\left(\nu + 1 + \alpha, \nu + 1, \frac{r^2}{4t}\right)}{F\left(\nu + 1 + \alpha, \nu + 1, \frac{B}{4}\right)} \cdot \exp\left(\frac{-r^2}{4t}\right). \tag{6.11}$$

In the hyperbolic case

$$R(t) = (\kappa(t + t_1)(t + t_2))^{1/2}, \quad t_2 > t_1 > 0, \quad \kappa > 0,$$

$$\tau = \frac{1}{2\delta} \ln \frac{t_2(t + t_1)}{t_1(t + t_2)}, \tag{6.12}$$

$$u = \{(t + t_1)(t + t_2)\}^{-(\nu+1)/2} \exp\left[-\frac{r^2}{8}\left\{\frac{1}{(t + t_1)} + \frac{1}{(t + t_2)}\right\}\right] w,$$

and a similar procedure on letting $t_1 \to 0$ gives

$$u(r, t) = \frac{\delta}{2\kappa} \{t(t + t_2)\}^{-(\nu+1)/2} \exp\left\{-\frac{r^2}{4t} + \frac{\delta}{4}\right\} \sum_{n=1}^{\infty} \frac{Q_n(t)\phi_n{}^\nu\left(\frac{\delta\xi^2}{2}\right)}{\theta_n{}^*\left(\frac{\delta}{2}\right)}, \tag{6.13}$$

where

$$Q_n(t) = \left(\frac{t}{t + t_2}\right)^{-\lambda_n/4} \int_0^t s^{((\nu-1)/2)+(\lambda_n/4)}(s + t_2)^{((\nu-1)/2)-(\lambda_n/4)} \exp \frac{\kappa}{8}(2s + t_2)\, g(s) ds. \tag{6.14}$$

If $g(t) = Gt^\alpha$, then on using the expansion in terms of the Laguerre polynomials

$$(1 + u)^{\beta+1} \exp(-xu) = \sum_{n=0}^{\infty} L_n{}^\beta(x) \left(\frac{u}{1 + u}\right)^n,$$

we obtain

$$Q_n(t) = Gt_2{}^{\nu+\alpha} \left(\frac{t}{t + t_2}\right)^{((\nu+1)/2)+\alpha} \exp\frac{\delta}{4} \sum_{n=0}^{\infty} \frac{\left(\frac{t}{t+t_2}\right)^n L_n{}^{\nu+\alpha}\left(-\frac{\delta}{2}\right)}{\left(\frac{\nu + 1}{2} + \lambda_n + n + \alpha\right)}. \tag{6.15}$$

Now again using (3.11) in the resultant double sum we arrive at

$$u(r, t) = Gt^\alpha \left(\frac{t_2}{t + t_2}\right)^{\alpha+\nu+1} \exp\left(-\frac{r^2}{4t} + \frac{\delta}{2}\right) \sum_{m=0}^{\infty} \left(\frac{t}{t + t_2}\right)^m L_m{}^{\nu+\alpha}\left(-\frac{\delta}{2}\right)$$

$$\frac{F\left(\nu + 1 + \alpha + m, \nu + 1, \frac{\delta\xi^2}{2}\right)}{F\left(\nu + 1 + \alpha + m, \nu + 1, \frac{\delta}{2}\right)}.$$

REFERENCES

[1] D. Langford, *Pseudo-similarity solutions of the one-dimensional diffusion equation with applications to the phase change problem*, Quart. Appl. Math. **25**, 45–52 (1967)
[2] G. W. Bluman, *Applications of the general similarity solution of the heat equation to boundary value problems*, Quart. Appl. Math. **31**, 403–415 (1974)
[3] R. E. Gibson, *A heat conduction problem involving a specified moving boundary*, Quart. Appl. Math. **16**, 426–430 (1958)
[4] E. C. Titchmarsh, *Eigenfunction expansions, Part I*, Oxford University Press, 1962
[5] L. J. Slater, *Confluent hypergeometric functions*, Cambridge University Press, 1960