CONFORMAL TRANSFORMATIONS OF THREE TYPES OF EDGE NOTCHES*

By

CHIH-BING LING

Virginia Polytechnic Institute and State University, Blacksburg

1. Synopsis and introduction. This paper presents the mapping functions for the curves of three types of edge notches in the upper half of the complex z plane onto a portion of the circumference of a unit circle in a complex ζ plane. The three types of edge notches under consideration are V, U and keyhole notches.

As shown in Fig. 1(i), the curve of the V notch $ADCGA'$ is contained in the upper half of the z plane at the edge $y = 0$. It encloses the origin 0 and is symmetrical with respect to the y axis. It consists of two equal line segments AD and GA' connected smoothly by a circular arc DCG at the closed end. The line segments incline symmetrically with the normal to the x axis at an angle ψ. The open end is divergent when ψ is positive and consequently the arc is a minor circular arc. At the opening AA', $z = \mp 1$, respectively. At the crown of the notch C, $z = i\gamma$, where γ is the depth ratio of the notch given by

$$\gamma = OC/OA' = (1 + \lambda) \tan\left(\frac{\pi}{4} - \frac{\psi}{2}\right), \tag{1}$$

where λ is the length of either line segment.

The U notch as shown in Fig. 1(ii) may be considered as a particular case of the V notch in which $\psi = O$. In this case, the two line segments of the curve are parallel to each other and intersect the edge $y = O$ normally. They are connected smoothly at the closed end by a semicircular arc.

A keyhole notch $ADCGA'$ as shown in Fig. 1(iii) consists of two equal line segments AD and GA' on the sides of a wedge $AO'A'$ and a circular arc DCG with its center at $0'$. The circular arc intersects both line segments normally. The notch has a vertical axis of symmetry. Let 2ψ be the subtending angle at $0'$. Also let λ be the radius of the circular arc and $0'$ be at unit distance from the origin 0. A typical example of such a notch is the cut-out of a metallic pencil-cap, which contains a wedge-shaped slot and a circular hole at the tip of the slot.

2. The V notch. To find the mapping function of the V notch, suppose that the angle ψ is positive and the two line segments of the curve of notch in the upper half of the z plane are extended in both directions as shown in Fig. 2(i). When extended upward from D and G, they intersect at a point H on the y axis in the upper half plane with an interior angle 2ψ. On the other hand, when extended downward from A and A' to the lower half plane, they may be regarded as being two symmetrical circular arcs of equal

* Received October 2, 1978; revised version received November 13, 1978.

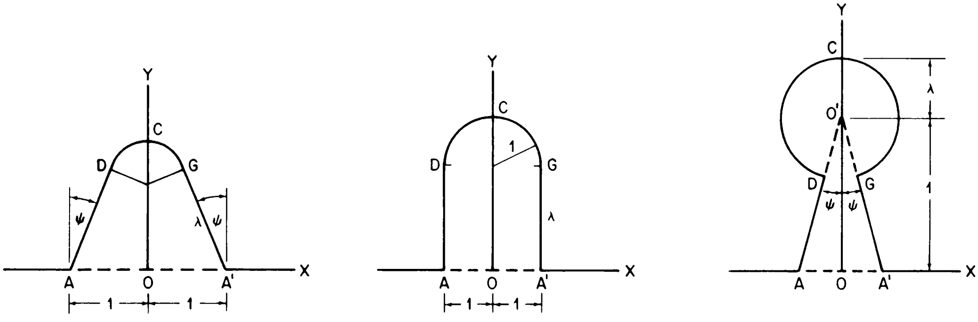
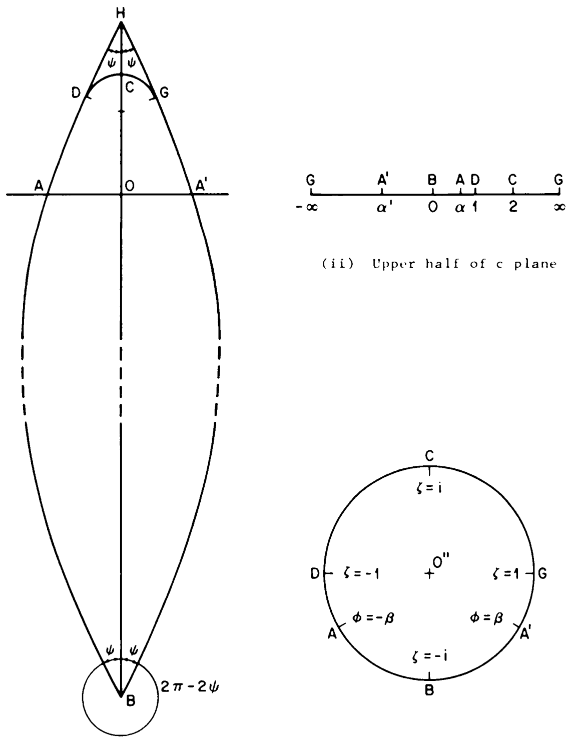


FIG. 1. Edge notches in upper half of the z plane.

but indefinitely large radii such that they first diverge, then converge, and finally intersect on the y axis at a point B where $y = -\infty$. Together with the parts in the upper half plane, the curves $HDAB$ and $HGA'B$ form a lenticular region. By symmetry, the interior angle at B is also equal to 2ψ. Consequently, the exterior angle at B is equal to $2\pi - 2\psi$. The closed curve formed by the curve of notch $ADCGA'$ in the upper half plane and the extension $A'BA$ in the lower half plane may be regarded as a curvilinear triangle bounding the exte-



(i) Exterior region of curvilinear triangle in z plane
(iii) Exterior region of unit circle |ζ| = 1 in ζ plane

FIG. 2. Transformations of V notch.

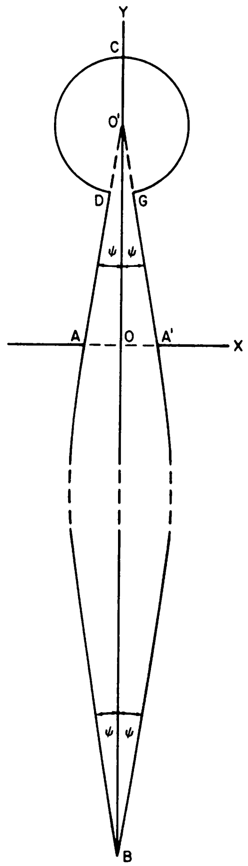


FIG. 3. Keyhole notch and curvilinear triangle in z plane.

rior region, whose angles at the vertices D, G and B are π, π and $2\pi - 2\psi$ radians, respectively.

It is known that, by conformal transformation, the sides of a curvilinear triangle in the z plane can be transformed onto the real axis of a complex c plane and its exterior region can be transformed onto the upper half of that plane [1]. The transformatioon gives z as a function of two independent solutions of the hypergeometric equation, say W_1 and W_2 of the new complex variable c, in the form:

$$z = \frac{a_1 W_1(c) + a_2 W_2(c)}{b_1 W_1(c) + b_2 W_2(c)}, \tag{2}$$

where a_1, a_2, b_1 and b_2 are real or complex constants, with $a_1 b_2 \neq a_2 b_1$. In the present case, the three parameters involved in the hypergeometric equation are found as

$$\alpha_1 = \frac{3}{2} - \delta, \qquad \beta_1 = \frac{1}{2} - \delta, \qquad \gamma_1 = 3 - 2\delta, \tag{3}$$

where $\delta = \psi/\pi$, a positive constant between 0 and $\frac{1}{2}$ when the open end of the notch is divergent. Now,

$$W_1(c) = {}_2F_1\left(\frac{1}{2} - \delta, \frac{3}{2} - \delta; 3 - 2\delta; c\right),$$

$$W_2(c) = c^{-(2-2\delta)}(1 - c){}_2F_1(\delta + \tfrac{1}{2}, \delta - \tfrac{1}{2}; 2; 1 - c) \tag{4}$$

are two independent solutions of the hypergeometric equation, where ${}_2F_1$ is the hypergeometric function. Let the homologues of the three vertices D, G and B be at $c = 1$, $\pm\infty$ and 0, respectively. Then the ratios of the constants involved in (2) can be determined and the equation becomes

$$z = -1 + i\lambda\exp(-i\psi) + 2p(1 - \lambda \sin \psi)\,(W_2(c)/W_1(c)), \tag{5}$$

where

$$p = \lim_{c\to\pm\infty} (W_1(c)/W_2(c)). \tag{6}$$

To evaluate p, we use the transformations [2]

$$W_1(c) = \left(1 - \frac{c}{2}\right)^{-(1-2\delta)/2} {}_2F_1\left(\frac{1}{4} - \frac{\delta}{2}, \frac{3}{4} - \frac{\delta}{2}; 2 - \delta; \frac{c^2}{(2 - c)^2}\right),$$

$$W_2(c) = c^{-(3-2\delta)/2}(1 - c){}_2F_1\left(\frac{3}{2} - \delta, \delta - \frac{1}{2}; 2; \frac{c - 1}{c}\right). \tag{7}$$

When $c \to \pm\infty$, we find by taking the principal value

$$p = -\frac{\exp(\pi i(1 - 2\delta)/2)}{2^{-(1-2\delta)/2}} \frac{{}_2F_1\left(\frac{1}{4} - \frac{\delta}{2}, \frac{3}{4} - \frac{\delta}{2}; 2 - \delta; 1\right)}{{}_2F_1\left(\frac{3}{2} - \delta, \delta - \frac{1}{2}; 2; 1\right)}$$

$$= -2^{(1-2\delta)/2} i\exp(-\pi i\delta) \frac{\Gamma(2 - \delta)\Gamma\left(\frac{1}{2} + \delta\right)\Gamma\left(\frac{5}{2} - \delta\right)}{\Gamma\left(\frac{7}{4} - \frac{\delta}{2}\right)\Gamma\left(\frac{5}{4} - \frac{\delta}{2}\right)}$$

$$= -\frac{i}{\pi^{1/2}} 2^{2-2\delta} \exp(-i\psi)\,\Gamma(2 - \delta)\,\Gamma\left(\frac{1}{2} + \delta\right). \tag{8}$$

Here, the second relation is obtained by Gauss theorem and the last by the duplication formula for gamma functions. Thus, Eq. (5) is fully determined.

Furthermore, we use a different transformation

$$W_1(c) = (1 - c)^{-(1-2\delta)/4} {}_2F_1\left(\frac{1}{4} - \frac{\delta}{2}, \frac{5}{4} - \frac{\delta}{2}; 2 - \delta; \frac{c^2}{4c - 4}\right). \tag{9}$$

Here, $c^2/(4c - 4) = 1$ when $c = 2$. We find, by taking the principal value,

$$W_1(2) = \exp(\pi i(1 - 2\delta)/4) \frac{\pi^{1/2}\,\Gamma(2 - \delta)}{\Gamma\left(\frac{7}{4} - \frac{\delta}{2}\right)\Gamma\left(\frac{3}{4} - \frac{\delta}{2}\right)}. \tag{10}$$

Again, when $c = 2$, we find directly from the second expression in (4) with the aid of Kummer's theorem,

$$W_2(2) = -2^{-(2-2\delta)}\,{}_2F_1(\delta + \tfrac{1}{2}, \delta - \tfrac{1}{2}; 2; -1) = -2^{-(5-2\delta)/2} \frac{\pi^{1/2}}{\Gamma\left(\frac{3}{4} + \frac{\delta}{2}\right)\Gamma\left(\frac{7}{4} - \frac{\delta}{2}\right)} \cdot \tag{11}$$

With these values, we find by virtue of the properties of gamma functions that when $c = 2$, the value of z in (5) is

$$z = i(1 + \lambda) \tan\left(\frac{\pi}{4} - \frac{\psi}{2}\right) = i\gamma. \tag{12}$$

Hence, at the crown of notch C, $c = 2$.

The next is to transform conformally the upper half of the c plane onto the exterior region of a unit circle in the complex ζ plane with its center at the origin $0'$. At B let $\zeta = -i$, at G let $\zeta = 1$, and at D let $\zeta = -1$. Then the mapping function is

$$c = i\,\frac{\zeta + 1}{\zeta - 1} + 1. \tag{13}$$

Note that at C, $\zeta = i$. Define a pair of polar coordinates (ρ, ϕ) in the ζ plane by

$$\zeta = i\rho \exp(-i\phi). \tag{14}$$

The transformations are shown in Fig. 2. The resulting mapping function is obtained by substituting (13) into (5) as a function of ζ in the form:

$$z = \Omega(\zeta). \tag{15}$$

On the unit circle $\rho = 1$ in the ζ plane, at A denote $\phi = -\beta$ and at A' denote $\phi = \beta$. Thus the curve of the V notch in the upper half of the z plane is transformed onto a portion of the circumference of the unit circle $\rho = 1$ in the ζ plane from $\phi = -\beta$ to $\phi = \beta$. Furthermore, the value of β is given by

$$\beta = \frac{\pi}{2} + 2\tan^{-1}(1 - \alpha), \quad \text{or}$$

$$\beta = \frac{\pi}{2} + 2\cot^{-1}(1 - \alpha'), \tag{16}$$

where α is the positive real root between 0 and 1, and α' the negative real root between $-\infty$ and 0, of the following equations of c, respectively:

$$(1 - \lambda \sin\psi)\Gamma(2 - \delta)\Gamma\left(\frac{1}{2} + \delta\right) \frac{2^{3-2\delta}}{\pi^{1/2}} \frac{W_2(c)}{W_1(c)} = \begin{cases} \lambda, \\ \lambda + 2i\exp(i\psi). \end{cases} \tag{17}$$

The roots α and α' are connected by

$$(1 - \alpha)(1 - \alpha') = 1. \tag{18}$$

The homologues of the points on the curve of the V notch are shown in Table 1.

3. The U notch. In the case of a U notch, the two line segments of the curve of notch are parallel to each other so that $\psi = 0$ or $\delta = 0$. By a similar extension of the line seg-

TABLE 1.
Homologues of Points on Curve of V notch.

Point	z	c	ζ	$\phi(\rho = 1)$
A	-1	α	$ie^{i\beta}$	$-\beta$
D	$-1 + i\lambda e^{-i\psi}$	1	-1	$-\pi/2$
C	$i(1+\lambda)\tan\left(\frac{\pi}{4} - \frac{\psi}{2}\right)$	2	i	0
G	$1 + \lambda e^{i\psi}$	$\pm\infty$	1	$\pi/2$
A′	1	α'	$ie^{-i\beta}$	β

ments, the resulting curvilinear triangle bounding the exterior region has the angles at the vertices D, G and B equal to π, π and 2π radians, respectively. The three parameters involved in the hypergeometric equation are therefore

$$\alpha_1 = \frac{3}{2}, \qquad \beta_1 = \frac{1}{2}, \qquad \gamma_1 = 3. \tag{19}$$

Consequently, the two independent solutions W_1 and W_2 of the hypergeometric equation are

$$\begin{aligned} W_1(c) &= {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; 3; c\right), \\ W_2(c) &= c^{-2}(1-c)_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 2; 1-c\right). \end{aligned} \tag{20}$$

Likewise, let the homologues of the three vertices D, G and B be at $c = 1$, $\pm\infty$ and 0, respectively. It is found that the mapping function to transform the exterior region of the curvilinear triangle in the z plane onto the upper half of c plane is

$$z = -1 + i\lambda - 8i(W_2(c)/W_1(c)). \tag{21}$$

It appears that in this particular case the functions W_1 and W_2 can be expressed in terms of the complete elliptic integrals K and E. From the following known relations:

$$K(k) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \quad E(k) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; k^2\right), \tag{22}$$

it can be shown from the contiguous relations of hypergeometric functions that

$$4\frac{W_2(c)}{W_1(c)} = \frac{(2-c)E' - cK'}{(2-c)E - 2(1-c)K'}, \tag{23}$$

in which the square of the modulus of the complete elliptic integrals is c. In the expression, the customary notations for the complete elliptic integrals are used. It follows therefore that the mapping function is

$$z = -1 + i\lambda - 2i\frac{(2-c)E' - cK'}{(2-c)E - 2(1-c)K}. \tag{24}$$

It may be noted that the four complete elliptic integrals are not all independent but are connected by the Legendre relation [3]:

$$KE' + K'E - KK' = \pi/2. \tag{25}$$

Similarly, by further applying the mapping function in (13), the curve of the U notch is transformed onto a portion of the circumference of the unit circle in the ζ plane from $\phi = -\beta$ to $\phi = \beta$. Again, β is given by (16) and α and α' are now the roots of the following equations, respectively:

$$2\,\frac{(2-c)E' - cK'}{(2-c)E - 2(1-c)K} = \begin{cases} \lambda, \\ \lambda + 2i. \end{cases} \tag{26}$$

Likewise, α and α' are connected by (18). Also, at the crown of notch C, $c = 2$ and $z = i\gamma$, where the depth ratio γ is now simply

$$\gamma = 1 + \lambda. \tag{27}$$

4. The keyhole notch. In a similar manner, suppose that the two line segments of the curve of the keyhole notch are extended beyond the open ends A and A' to the lower half of the z plane. The resulting curves $O'DAB$ and $O'GA'B$ may likewise be considered as being two symmetrical circular arcs, each with an equal but indefinitely large radius, intersecting at a point B on the y axis where $y = -\infty$ so as to form a lenticular region. By symmetry, the interior angle at B is also 2ψ. The exterior angle is thus $2\pi - 2\psi$. The closed curve formed by the curve of notch $ADCGA'$ and the extension $A'BA$ in the lower half plane becomes a curvilinear triangle bounding the exterior region with the angles at the vertices D, G and B equal to $\pi/2$, $\pi/2$ and $2\pi - 2\psi$ radians, respectively.

Corresponding to these angles at the vertices of the curvilinear triangle, the three parameters involved in the hypergeometric equation are found as

$$\alpha_1 = 1 - \delta, \qquad \beta_1 = \frac{3}{2} - \delta, \qquad \gamma_1 = 3 - 2\delta, \tag{28}$$

where, as before, $\delta = \psi/\pi$ $(0 \le \delta < \frac{1}{2})$. Now,

$$W_1(c) = {}_2F_1\left(1 - \delta, \frac{3}{2} - \delta; 3 - 2\delta; c\right),$$

$$W_2(c) = c^{-(2-2\delta)}(1-c)^{1/2}{}_2F_1(\delta, \delta - \tfrac{1}{2}; \tfrac{3}{2}; 1 - c) \tag{29}$$

are two independent solutions of the hypergeometric equation. Let the homologues of the three vertices D, G and B of the curvilinear triangle be likewise on the real axis of the c plane at $c = 1, \pm\infty$ and o, respectively. Then the equation of transformation becomes

$$z = i(1 - \lambda \exp(-i\psi)) + 2p\lambda \sin\psi\, W_2(c)/W_1(c), \tag{30}$$

where

$$p = \lim_{c\to\pm\infty} (W_1(c)/W_2(c)). \tag{31}$$

With the aid of the following transformations [2]:

$$W_1(c) = \left(1 - \frac{c}{2}\right)^{-1+\delta} {}_2F_1\left(\frac{1}{2} - \frac{\delta}{2}, 1 - \frac{\delta}{2}; 2 - \delta; \frac{c^2}{(2-c)^2}\right),$$

$$W_2(c) = (1-c)^{1/2}\, c^{-(3-2\delta)/2}\, {}_2F_1\left(\frac{3}{2} - \delta, \delta - \frac{1}{2}; \frac{3}{2}; \frac{c-1}{c}\right), \tag{32}$$

we have, when $c \to \pm\infty$, by taking the principal value,

$$p = \frac{2^{1-\delta}\exp(\pi i(1-\delta))}{\exp(\pi i/2)} \frac{{}_2F_1\left(\frac{1}{2}-\frac{\delta}{2}, 1-\frac{\delta}{2}; 2-\delta; 1\right)}{{}_2F_1\left(\frac{3}{2}-\delta, \delta-\frac{1}{2}; \frac{3}{2}; 1\right)}. \tag{33}$$

By Gauss' theorem and further by the properties of gamma functions, we find

$$p = i(1-\delta)\,2^{3-2\delta}\exp(-i\psi)\csc\psi. \tag{34}$$

Hence, the mapping function which transforms conformally the exterior of the curvilinear triangle onto the upper half of the c plane is

$$z = i(1-\lambda\exp(-i\psi)) + 2^{4-2\delta}(1-\delta)\,i\lambda\exp(-i\psi)\,W_2(c)/W_1(c). \tag{35}$$

To find the value of z at $c = 2$, we make use of the transformation

$$W_1(c) = (1-c)^{-(1-\delta)/2}\,{}_2F_1\left(\frac{1}{2}-\frac{\delta}{2}, 1-\frac{\delta}{2}; 2-\delta; \frac{c^2}{4c-4}\right). \tag{36}$$

Here, $c^2/(4c-4) = 1$ when $c = 2$. Similarly, by Gauss' theorem we find

$$W_1(2) = 2^{1-\delta}\,i\exp(-i\psi/2). \tag{37}$$

Also, when $c = 2$, we have directly from the second equation in (29)

$$W_2(2) = 2^{-2+2\delta}i\,{}_2F_1\left(\delta, \delta-\frac{1}{2}; \frac{3}{2}; -1\right). \tag{38}$$

Again, by Kummer's theorem, we find

$$W_2(2) = 2^{-(2-\delta)}\,i\cos\frac{\psi}{2}/(1-\delta). \tag{39}$$

Thence, one is led to

$$z = i(1+\lambda), \tag{40}$$

which is the point C or the crown of the notch. Hence, at C, $c = 2$.

Furthermore, suppose that the points A and A' at the open end are transformed onto $c = \alpha$ and $c = \alpha'$, respectively, on the real axis of the c plane. Then, α is the positive real root between o and 1, and α' the negative real root between $-\infty$ and o, of the following equations of c, respectively:

$$2^{4-2\delta}(1-\delta)\frac{W_2(c)}{W_1(c)} = \begin{cases} \lambda - \sec\psi, \\ \lambda - \exp(2i\psi)\sec\psi. \end{cases} \tag{41}$$

α and α' are connected by (18).

Next, by applying the mapping function in (13), the upper half of the c plane is further transformed conformally onto the exterior of a unit circle in the ζ plane with its center at the origin. The curve of the keyhole notch is transformed onto a portion of the circumference of the unit circle in the ζ plane from $\phi = -\beta$ to $\phi = \beta$. Also, β is given by (16) in terms of α or α'. The homologues of the points on the curve of the keyhole notch are shown in Table 2.

TABLE 2
Homologues of Points on Curve of Keyhole Notch.

Point	z	c	ζ	$\phi(\rho = 1)$
A	$-\tan\psi$	α	$ie^{i\beta}$	$-\beta$
D	$i(1 - \lambda e^{-i\psi})$	1	-1	$-\pi/2$
C	$i(1 + \lambda)$	2	i	0
G	$i(1 - \lambda e^{i\psi})$	$\pm\infty$	1	$\pi/2$
A′	$\tan\psi$	α'	$ie^{-i\beta}$	β

5. Concluding remarks. In conclusion, some remarks will be made on the numerical aspects of the transformations. The root α (or α') may be solved from (17), (26), or (41) by using the Newton-Raphson method. The hypergeometric function ${}_2F_1$ may be evaluated by direct summation from the series expression of the function itself or of the transformed function of better convergence. The complete elliptic integrals can be evaluated by Gauss' arithmetic-geometric mean method even when the modulus is complex. A description of the method was given by Cambi [4].

Moreover, it may be mentioned that an essentially similar mapping function in terms of complete elliptic integrals as in (24) was obtained previously by Daymond and Hodgkinson [5] for a particular airfoil. Davy and Lanston [6] applied a similar mapping function to solve a certain magnetic problem. The present mapping functions may find immediate applications to problems of edge notches with the specified geometric configurations.

References

[1] A. R. Forsyth, *Theory of functions of a complex variable,* Vol. II, Ch. XX, third edition, Dover reprint, 1965

[2] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and theorems for special functions of mathematical physics,* third edition, Springer-Verlag, 1966

[3] P. F. Byrd and M. D. Friedman, *Handbook of elliptic integrals for engineers and physicists,* Springer-Verlag, 1954

[4] E. Cambi, *Complete elliptic integrals of complex Legendrian modulus,* J. Math. and Phys. **26,** 234–245 (1948)

[5] S. D. Daymond and J. Hodgkinson, *A type of aerofoil,* Quart. J. Math., Oxford **10,** 135–144 (1939)

[6] N. Davy and N. H. Landston, *The external magnetic field of a single thick semi-infinite parallel plate terminated by a convex semi-circular cylinder,* Quart. J. Mech. Appl. Math. **6,** 115–121 (1953)