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HOMOGENIZATION AND RANDOM EVOLUTIONS: APPLICATIONS TO THE MECHANICS OF COMPOSITE MATERIALS*

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Abstract. The technique of homogenization is used to derive the effective properties of laminated composites. A new probabilistic justification for homogenization using the concept of random evolutions is provided and indicates that the effective properties of deterministic periodic composite and those of a randomly perturbed periodic composite are the same.

1. Introduction. When studying the behavior of an heterogeneous material one often attempts to replace it by an homogeneous material, the behavior of which approximates in some way that of the original heterogeneous one. The properties of this "equivalent" homogeneous material are then termed effective (or bulk) properties of the heterogeneous material.

This idea, which is very old and can be traced as far back as to Poisson [1], has been entertained by many authors, among whom we mention Maxwell [2], Rayleigh [3], and de Vries [4]. For a more extensive survey, we refer to the article by Babuška [5]. Recently, with the ever-increasing use of composite materials in engineering applications, the question of effective properties has attracted much attention from mechanicians (see [5] again for reference to these works).

The technique of homogenization, which allows one to define effective properties of periodically heterogeneous materials, was introduced by Babuška [5, 6 and the references therein] and Sanchez-Palencia [7]. Lions and co-workers, using the notion of *G*-convergence introduced by DeGiorgi and Spagnolo [8 and the references therein], have recently provided a functional analytic justification for the technique of homogenization (see for example [9, 10] and the references therein). See also [11] for a recent extension of the ideas underlying homogenization. Effective properties of materials with random heterogeneities have also been defined by various authors. Let us mention here as typical the works of Hashin et al. [12], Kröner [13] and Beran [14].

In this article we specialize in Sec. 2 the results of homogenization to the case of a laminated periodic composite material. We consider propagation of waves normal to the layering and obtain the homogenized wave equation (which specifies the effective properties of the composite) by using a multiple scale expansion in Part B and functional analytic arguments in Part C. Part A gives a brief description of homogenization in the present context.

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In Sec. 3 we consider the random laminated composite. The results, based on the concept of random evolutions [15], are twofold: on the one hand, they provide a new probabilistic interpretation and justification for homogenization. On the other hand, they provide a means of obtaining effective properties for random composites. Of particular significance is the fact that they show that the effective properties of the random composite which is periodic in the average are the same as those of the deterministic average periodic composite. Thus our results indicate that the periodicity assumption which underlies homogenization is not crucial in determining effective properties, since effective properties for the periodic composite and the randomly perturbed periodic composite are the same.

2. Effective properties of laminated composites via homogenization.

A. *The homogenization problem.* The differential equations governing one-dimensional wave propagation in a linear elastic medium are

$$\partial\sigma/\partial x = \rho(\partial v/\partial t), \qquad \partial v/\partial x = \partial e/\partial t, \qquad \sigma = \eta e. \tag{2.1}$$

Here, x is the direction of propagation, $\sigma(x, t)$, $e(x, t)$, and $v(x, t)$ are the normal stress, strain, and velocity respectively, while ρ and η are, respectively, the density and elastic stiffness of the elastic medium.

If the medium is made up of a periodic layering of sheets normal to the direction of propagation, each of linear elastic material with constant properties (ρ and η), the density and stiffness for this laminated composite are periodic functions of x:

$$\rho(x + p) = \rho(x), \qquad \eta(x + p) = \eta(x), \tag{2.2}$$

where p is the period, i.e. the thickness of the basic cell for the composite. The problem of homogenization consists in investigating what becomes of Eqs. (2.1) as $p \to 0$.

More precisely, let us rewrite (2.1) as

$$\eta^{-1}\frac{\partial^2\sigma}{\partial t^2} = \frac{\partial}{\partial x}\left[\rho^{-1}\frac{\partial\sigma}{\partial x}\right] \tag{2.3}$$

and let us introduce a new length scale

$$y = x/\varepsilon \tag{2.4}$$

where $\varepsilon > 0$ is a parameter, so that (2.3) can be written as

$$\eta^{-1}(y)\,\frac{\partial^2\sigma^\varepsilon(x, y, t)}{\partial t^2} = \frac{\partial}{\partial x}\left[\rho^{-1}(y)\,\frac{\partial\sigma^\varepsilon(x, y, t)}{\partial x}\right]. \tag{2.5}$$

The problem of homogenization then consists in studying Eq. (2.5) as $\varepsilon \to 0$. In (2.5) the ε superscript indicates the dependence of σ on ε.

Remark 2.1. We could have just as well rewritten (2.1) in terms of strains as

$$\frac{\partial^2 e}{\partial t^2} = \frac{\partial}{\partial x}\left[\rho^{-1}\frac{\partial}{\partial x}(\eta e)\right], \tag{2.6}$$

instead of (2.3). It turns out that (2.3) is easier to work with than (2.6) while leading to the same results.

Remark 2.2. As $\varepsilon \to 0$, the periodic variations of ρ and η in (2.5) become more and more frequent, so that the study of (2.5) as $\varepsilon \to 0$ does indeed provide us with information about (2.1) as $p \to 0$.

B. Homogenization via multiple scale expansion. Noting that, in view of (2.4), the operator $\partial/\partial x$ applied to a function of both x and y becomes $(\partial/\partial x) + \varepsilon^{-1}(\partial/\partial y)$, we can rewrite (2.5) as

$$\eta^{-1}(y)\frac{\partial^2\sigma^\varepsilon}{\partial t^2} = \rho^{-1}(y)\frac{\partial^2\sigma^\varepsilon}{\partial x^2} + \varepsilon^{-1}\rho^{-1}(y)\frac{\partial^2\sigma^\varepsilon}{\partial x\partial y} + \varepsilon^{-1}\frac{\partial}{\partial y}\left[\rho^{-1}(y)\frac{\partial\sigma^\varepsilon}{\partial x}\right] + \varepsilon^{-2}\frac{\partial}{\partial y}\left[\rho^{-1}(y)\frac{\partial\sigma^\varepsilon}{\partial y}\right]. \tag{2.7}$$

Upon expanding σ^ε in powers of ε as

$$\sigma^\varepsilon(x, y, t) = \sigma_0(x, y, t) + \varepsilon\sigma_1(x, y, t) + \varepsilon^2\sigma_2(x, y, t) + \cdots,$$

where σ_i, $i = 0, 1, 2, \cdots$ are periodic of period p in y, and substituting in (2.7), one obtains

$$\eta^{-1}(y)\frac{\partial^2}{\partial t^2}(\sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \cdots) = [\varepsilon^{-2}A_0 + \varepsilon^{-1}A_1 + A_2](\sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \cdots), \tag{2.8}$$

where

$$A_0 = \frac{\partial}{\partial y}\left[\rho^{-1}(y)\frac{\partial}{\partial y}\right], \qquad A_1 = \frac{\partial}{\partial y}\left[\rho^{-1}(y)\frac{\partial}{\partial x}\right] + \rho^{-1}(y)\frac{\partial^2}{\partial x\partial y}, \tag{2.9}$$

$$A_2 = \rho^{-1}(y)\frac{\partial^2}{\partial x^2}.$$

Identifying terms in (2.8) involving equal powers of ε leads to the following equations for σ_i, $i = 0, 1, 2, \cdots$:

$$A_0\sigma_0 = 0, \tag{2.10}$$

$$A_0\sigma_1 + A_1\sigma_0 = 0, \tag{2.11}$$

$$A_0\sigma_2 + A_1\sigma_1 + A_2\sigma_0 = \eta^{-1}(y)(\partial^2\sigma_0/\partial t^2). \tag{2.12}$$

Upon recalling the y-periodicity of η, ρ, and σ_0, the solution of (2.10) is easily found to be of the form

$$\sigma_0 = \bar{\sigma}(x, t) \tag{2.13}$$

where $\bar{\sigma}$ is some arbitrary function of x and t.

By substituting (2.13) in (2.11) and taking (2.9) into account, the solution of (2.11) is obtained in the form

$$\sigma_1 = f(y)(\partial\bar{\sigma}/\partial x) + \tilde{\sigma}(x, t) \tag{2.14}$$

where $f(y)$ is the y-periodic solution of

$$A_0 f = -\frac{\partial}{\partial y}[\rho^{-1}(y)] \tag{2.15}$$

and $\tilde{\sigma}$ is some arbitrary function of x and t.

Integrating (2.12) over one period, one easily sees that it has a y-periodic solution iff

$$\int_y^{y+p}\left[\eta^{-1}(y)\frac{\partial^2\sigma_0}{\partial t^2} - A_1\sigma_1 - A_2\sigma_0\right]dy = 0$$

or, taking (2.13, 14) into account, iff

$$(\overline{\eta^{-1}})\frac{\partial^2\bar{\sigma}}{\partial t^2} = \frac{1}{p}\int_y^{y+p}\left[\rho^{-1} + \rho^{-1}\frac{\partial f}{\partial y}\right]\frac{\partial^2\bar{\sigma}}{\partial x^2}\,dy \tag{2.16}$$

where

$$(\overline{\eta^{-1}}) = \frac{1}{p}\int_y^{y+p} \eta^{-1}(y)\, dy \tag{2.17}$$

is the average of η^{-1} over one period. Now, recalling (2.15) and the y-periodicity of f, one easily finds

$$\partial f/\partial y = \rho/\bar{\rho} - 1 \tag{2.18}$$

where

$$\bar{\rho} = \frac{1}{p}\int_y^{y+p} \rho\ dy \tag{2.19}$$

is the average of ρ over one period. Substitution of (2.18) into (2.16) yields the homogenized equation

$$(\partial^2\bar{\sigma}/\partial t^2) = (\bar{\eta}/\bar{\rho})(\partial^2\bar{\sigma}/\partial x^2) \tag{2.20}$$

where we have set

$$\bar{\eta} = (\overline{\eta^{-1}})^{-1}\,. \tag{2.21}$$

A comparison of (2.20) with (2.3) (in the case of constant η and ρ, i.e. $\partial^2\sigma/\partial t^2 = (\eta/\rho)(\partial^2\sigma/\partial x^2)$ indicates that the effective properties of the laminated composites are specified by $\bar{\eta}$ and $\bar{\rho}$.

In the case of a composite made up of a periodic layering of sheets of two different materials with properties η_1, ρ_1 and η_2, ρ_2 and thicknesses a_1 and a_2, respectively, formulae (2.17), (2.19) and (2.21) yield

$$\bar{\eta} = \frac{(a_1 + a_2)\,\eta_1\,\eta_2}{a_1\,\eta_1 + a_2\,\eta_2}\,, \tag{2.22}$$

$$\bar{\rho} = \frac{a_1\,\rho_1 + a_2\,\rho_2}{a_1 + a_2}\,, \tag{2.23}$$

from which the effective speed of propagation $\bar{c}$ is obtained from $\bar{c}^2 = \bar{\eta}/\bar{\rho}$ as

$$\bar{c} = (a_1 + a_2)c_1c_2\ \{[(a_1/\rho_1)c_2{}^2 + (a_2/\rho_2)c_1{}^2]\,(a_1\rho_1 + a_2\rho_2)\}^{-1/2} \tag{2.24}$$

where $c_i^2 = \eta_i/\rho_i$, $i = 1, 2$. Thus, we recover formula (2) in [16].

C. *Homogenization via energy methods.* We rederive the results of Part B by using functional analytic arguments. This new derivation will actually serve as a justification for the asymptotic expansion of Part B.

For the sake of simplicity, we consider the homogeneous initial boundary-value problem for the inhomogeneous version of Eq. (2.3), i.e.

$$\eta^{-1}\,\frac{\partial^2\sigma}{\sigma t^2} = \frac{\partial}{\partial x}\left[\rho^{-1}\,\frac{\partial\sigma}{\partial x}\right] + g(x, t) \tag{2.25}$$

in $(x, t) \in (0, X) \times (O, T) = \Omega\times (O, T) = Q$, where X and T are finite, together with the boundary conditions

$$\sigma(0, t) = \sigma(X, t) = 0, \quad t \in (0, T) \tag{2.26}$$

and the initial conditions

$$\sigma(x, 0) = (\partial\sigma/\partial t)(x, 0) = 0, \; x \in \Omega. \tag{2.27}$$

Remark 2.3. Of course, problem (2.25–27) could be recast in an equivalent inhomogeneous initial boundary-value problem for the homogeneous equation. The auxiliary conditions as well as the inhomogeneous part of the equation play no role in the homogenization problem. As a matter of fact, we could have considered, in Part B, Eq. (2.25) instead of (2.3), the only modification being the addition of g to the right-hand side of Eq. (2.12), and we would have obtained the homogenized equation

$$(\partial^2\bar{\sigma}/\partial t^2) = (\bar{\eta}/\bar{\rho})(\partial^2\bar{\sigma}/\partial x^2) + g$$

instead of (2.20).

The imhomogeneous version of (2.5) is

$$\eta^{-1}(x/\varepsilon)(\partial^2\sigma^\varepsilon/\partial t^2) - A^\varepsilon\sigma = g \tag{2.28}$$

where

$$A^\varepsilon = -\frac{\partial}{\partial x}\left[\rho^{-1}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x}\right]\cdot$$

Introducing the bilinear form associated to A^ε on $H_0{}^1(\Omega) = \{u \in H^1(\Omega)$: u satisfies (2.26)$\}$ where H^1 is the usual Sobolev space of square integrable functions with square-integrable (distributional) first-order derivatives,

$$a^\varepsilon(u, v) = \int_\Omega \rho^{-1}\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx,$$

we can rewrite problem (2.28, 2.26) in the equivalent variational formulation

$$(\eta^{-1}(\partial^2\sigma^\varepsilon/\partial t^2), v) + a^\varepsilon(\sigma^\varepsilon, v) = (g, v), \; v \in H_0{}^1(\Omega), \tag{2.29}$$

where η^{-1}, $\rho^{-1} \in L^\infty(R)$ and are strictly positive and $g \in L^2(0, T; L^2(\Omega))$. (For more details on the notation, see for example [10].)

Taking $v = \partial\sigma^\varepsilon/\partial t$ in (2.29), integrating from 0 to T and using Gronwall's inequality (cf. [17, p. 19]), we obtain the energy inequality

$$\|\sigma^\varepsilon(t)\|^2 + |\partial\sigma^\varepsilon(t)/\partial t|^2 \leq C$$

where $\|\;\|$ and $|\;|$ denote the norm in $V = H_0{}^1(\Omega)$ and $H = L^2(\Omega)$ respectively, and C is a constant, from which there follows that $\|\sigma^\varepsilon\|_{L^2(0,T;V)}$ and $\|\partial\sigma^\varepsilon/\partial t\|_{L^2(0,T;H)}$ are bounded as well as $\|\partial^2\sigma/\partial t^2\|_{L^2(0,T;V')}$ in view of (2.28) (V' = dual of V). Thus, we can extract a subsequence (again denoted σ^ε) of σ^ε such that $\sigma^\varepsilon \to \bar{\sigma}$, $\partial\sigma^\varepsilon/\partial t \to \partial\bar{\sigma}/\partial t$, $\partial^2\sigma^\varepsilon/\partial t^2 \to \partial^2\bar{\sigma}/\partial t^2$ weakly in $L^2(0, T; V)$, $L^2(0, T; H)$, $L^2(0, T; V')$ respectively.

Let

$$\rho_\varepsilon{}^{-1} = \rho^{-1}(x/\varepsilon), \quad \eta_\varepsilon{}^{-1} = \eta^{-1}(x/\varepsilon), \quad \xi^\varepsilon = \rho_\varepsilon{}^{-1}(\partial\sigma^\varepsilon/\partial x). \tag{2.30}$$

Then (2.28) can be written as

$$\eta_\varepsilon{}^{-1}(\partial^2\sigma^\varepsilon/\partial t^2) - (\partial\xi^\varepsilon/\partial x) = g. \tag{2.31}$$

Since $\|\xi^\varepsilon\|_{L^2(0,T;H)}$ is also bounded (as can be seen from the last of (2.30)), we can also extract a subsequence again denoted ξ^ε such that $\xi^\varepsilon \to \xi$ weakly in $L(0, T; H)$. Thus, we can take the limit as $\varepsilon \to 0$ in (2.31) to obtain

$$(\overline{\eta^{-1}})(\partial^2\bar{\sigma}/\partial t^2) - (\partial\xi/\partial x) = g. \tag{2.32}$$

Eq. (2.32) is the homogenized equation corresponding to (2.28). All that remains is to find ξ. The details being lengthy, only a sketch of the determination of ξ will be given here. (The method is similar to that of [10] for the case of a parabolic equation and the reader is referred to that article for more details.)

Let us define as in (2.15) f as the y-periodic solution of

$$A_0 f = -\frac{d}{dy}\left[\rho^{-1}(y)\,\frac{dy}{dy}\right] = -\frac{d}{dy}\,(\rho^{-1}(y)) \tag{2.33}$$

and let

$$w_\varepsilon(x) = \varepsilon w(x/\varepsilon) = x + \varepsilon f(x/\varepsilon)$$

so that

$$A^\varepsilon w_\varepsilon = 0. \tag{2.34}$$

Let $\phi \in D(Q)$, the space of C^∞ functions with compact support in Q, take the scalar product of (2.28) with ϕw_ε and subtract the scalar product of (2.34) with $\phi\sigma^\varepsilon$, simplify and integrate between 0 and T to get after further simplifications

$$-\int_Q \eta_\varepsilon^{-1}\,\frac{\partial\sigma^\varepsilon}{\partial t}\,w_\varepsilon\,\frac{\partial\phi}{\partial t}\,dxdt + \int_Q \xi^\varepsilon\,\frac{\partial\phi}{\partial x}\,w_\varepsilon dxdt \;\; -\int_Q \rho_\varepsilon^{-1}\,\frac{dw_\varepsilon}{dy}\,\frac{\partial\phi}{\partial x}\,\sigma^\varepsilon dxdt = \int_0^T (g, \phi w_\varepsilon)dt. \tag{2.35}$$

In view of the weak convergence of σ^ε and its derivatives and of ξ^ε and the strong convergence of w_ε to x in $L^2(Q)$, the first two terms in (2.35) are seen to converge respectively to

$$-\int_Q (\overline{\eta^{-1}})\,\frac{\partial\sigma}{\partial t}\,x\,\frac{\partial\phi}{\partial t}\,dxdt \quad \text{and} \quad \int_Q \xi\,\frac{\partial\phi}{\partial x}\,x\,dxdt.$$

Also, since $\rho_\varepsilon^{-1}(dw_\varepsilon/dy) \to (\overline{\rho^{-1}(y)(\partial w/\partial y)})$ weakly in $L(Q)$ (where the overbar indicates average over one period) and upon evaluating the right-hand side of (2.35) using (2.32), one obtains from (2.35) in the limit and after further simplifications

$$\left(\overline{\rho^{-1}(y)\,\frac{\partial w}{\partial y}}\right)\int_Q \phi\,\frac{\partial\bar{\sigma}}{\partial x}\,dxdt = \int_Q \xi\phi dxdt. \tag{2.36}$$

Since (2.36) was obtained for arbitrary $\phi \in D(Q)$ there follows

$$(\overline{\rho^{-1}(y)(\partial w/\partial y)})(\partial\bar{\sigma}/\partial x) = \xi. \tag{2.37}$$

Substituting (2.37) in (2.31), we obtain, in view of (2.32) and the definition of w,

$$(\overline{\eta^{-1}})\,\frac{\partial^2\bar{\sigma}}{\partial x^2} = \frac{1}{p}\int_y^{y+p} \rho^{-1}(y)\left(1 + \frac{df}{dy}\right)\frac{\partial^2\bar{\sigma}}{\partial x^2}\,dy + g$$

which, upon using (2.18) as well as (2.19), (2.21), yields the homogenized equation (cf. Remark 2.3)

$$\partial^2 \bar{\sigma}/\partial x^2 = (\bar{\eta}/\bar{\rho})(\partial^2 \bar{\sigma}/\partial x^2) + g.$$

Thus, we recover the results of Part B: the effective properties of a laminated composite are $\bar{\rho}$ and $\bar{\eta}$ as defined by (2.19) and (2.21) respectively.

3. Effective properties of laminated composites via random evolutions. In this section, we provide a new probabilistic justification of homogenization. Another probabilistic interpretation has already been mentioned for the case where the periodic coefficients of the equation to be homogenized (ρ and η in the context of the present article) are smooth enough [18]. Unfortunately, this is not the case for laminated composites. Our interpretation does not require such smoothness assumptions and furthermore, it is of interest in its own right as it allows one to define effective properties of laminated composites with randomly imperfect periodic structures. For the sake of simplicity, we will restrict the analysis to a two-layer laminated composite.

Consider then a laminated composite made up of alternating layers of reinforcing (subscript 1) and matrix (subscript 2) material. The layers are assumed to be homogeneous with known deterministic properties (ρ_i, η_i, $i = 1, 2$) but with randomly varying thicknesses with averages (i.e. expected values) a_i, $i = 1, 2$.

Upon considering steady oscillatory waves of circular frequency ω, the reduced wave equation governing the propagation of waves perpendicular to the layering can be written as

$$dV/dx = A_i V, \quad x \text{ in layer } i, \quad i = 1, 2 \tag{3.1}$$

with

$$V = \begin{bmatrix} U \\ \Sigma \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & \eta_i^{-1} \\ -\rho_i \omega^2 & 0 \end{bmatrix} \tag{3.2}$$

where U and Σ are the displacement and stress amplitudes respectively.

Eq. (3.1) can be written as a random evolution as

$$dV/dx = A(Z(x))V \tag{3.3}$$

where $Z(x)$ is a Markov chain with state space $\{1, 2\}$ and $A(Z = i) = A_i$. (For more details about the above model, the reader is referred to [19] where it was originally proposed as a model for wave propagation in randomly imperfect periodic structures and was studied in the context of Floquet theory.)

Thus, the wave propagates in a reinforcement layer over a random distance α_1 with expected value a_1 until the Markov chain Z jumps to state 2, at which point it propagates in a matrix layer over a random distance α_2 with expected value a_2 until the chain Z jumps back to state 1 and the process is repeated.

The Markov chain $Z(x)$ governing the switching between different layers will be taken to be a generalized telegraph process with infinitesimal matrix

$$Q = \begin{bmatrix} -a_1^{-1} & a_1^{-1} \\ a_2^{-1} & -a_2^{-1} \end{bmatrix}. \tag{3.4}$$

Thus, the switching from reinforcement to matrix layers occurs at distances which are distributed according to a Poisson process with intensity a_1^{-1} while a Poission process with intensity a_2^{-1} governs the switching from matrix to reinforcement layers.

If V_i denotes the conditional expectation of V given that propagation started in a layer i, then a straightforward application of Theorem 2 in [15] yields

$$dV_i/dx = A_i V_i + \sum_{j=1}^{2} q_{ij} V_j . \tag{3.5}$$

To obtain the effective properties of such a random laminated composite, we use the same idea as that underlying homogenization: we speed up the switching process. To this end, we replace the matrix Q by ϵQ and let $\epsilon \to \infty$.

Remark 3.1. In the above, effective properties refer to the properties of a deterministic homogeneous material in which wave propagation approximates in some way the expected wave propagation in the random composite.

Remark 3.2. When Q is replaced by ϵQ, a_i is replaced by $\epsilon^{-1} a_i$, $i = 1, 2$ so that as $\epsilon \to \infty$, the average thicknesses $\to 0$.

With Q replaced by ϵQ, Eq. (3.5) becomes

$$dV_i/dx = A_i V_i + \epsilon \sum_{j=i}^{2} q_{ij} V_j . \tag{3.6}$$

To study the behavior of (3.6) as $\epsilon \to \infty$, we use Theorem 2.1 of [20] (or more generally Theorem 2.1 of [21]) to obtain that as $\epsilon \to \infty$ the solution to (3.6) converges to the solution of

$$d\bar{V}/dx = \bar{A}\bar{V} \tag{3.7}$$

where $\bar{A}$ is the expected value of $A(Z)$ with respect to the ergodic measure for the Markov chain Z.

The ergodic measure for the Markov chain Z with infinitesimal matrix (3.4) is easily shown to be defined by $P\ \{Z = i\} = a_i/(a_1 + a_2)$, $i = 1, 2$ so that

$$\bar{A} = (a_1 A_1 + a_2 A_2)/(a_1 + a_2)$$

or, making use of the second equation in (3.2),

$$\bar{A} = \begin{bmatrix} 0 & \bar{\eta}' \\ -\bar{\rho}\omega^2 & 0 \end{bmatrix}, \tag{3.8}$$

where $\bar{\eta}$ and $\bar{\rho}$ are as defined in (2.22) and (2.23) respectively. A comparison of (3.8) with the second equation in (3.2) indicates that the effective properties of the random laminated composite are specified by $\bar{\eta}$ and $\bar{\rho}$ and thus coincide with the effective properties of the periodic composite considered at the end of Sec. 2, Part B and obtained by homogenization. Thus, the results of this section provide a probabilistic derivation of the results of Sec. 2. Furthermore, they also indicate that the effective properties of a random composite which in the average (with respect to the ergodic measure) is periodic are the same as those of the deterministic average periodic composite as obtained by homogenization. Thus, whenever effective properties are adequate to describe the behavior of a laminated composite (e.g., propagation of large wavelength waves in a composite made up of thin layers) the effects of randomness can be disregarded, at least insofar as expected values are concerned.

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