ON OPTIMAL STRAIN PATHS IN LINEAR VISCOELASTICITY*

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Abstract. For a viscoelastic material the work $W(e)$ needed to produce a given strain e_0 in a given time T depends on the strain path $e(t)$, $0 \le t \le T$, connecting the unstrained state with e_0. We here ask the question: Of all strain paths of this type, is there one which is optimal,[1] that is, one which renders W a minimum? In answer to this question we show that:

(i) There is no *smooth* optimal strain path.

(ii) There exists a unique optimal path in $L_2(0, T)$; this path is smooth on the open interval $(0, T)$, but suffers jump discontinuities[2] at the end points 0 and T (i.e., $e(0^+) \neq 0$, $e(T^-) \neq e_0$).

(iii) For a Maxwell material the optimal path is linear on $(0, T)$.

1. Nonexistence of smooth optimal paths. For a one-dimensional linear viscoelastic material, which has been unstrained at all times prior to $t = 0$, the stress $s(t)$ at time t is determined by the strain history $e(\tau)$, $0 \le \tau \le t$, through the constitutive relation (cf., e.g., Gurtin and Sternberg [1])

$$s(t) = \int_0^t G(t - \tau)\dot{e}(\tau)d\tau \tag{1}$$

with $G(\tau)$, $0 \le \tau < \infty$, the *relaxation function.*

Given a value $e_0 \neq 0$ of the strain and a fixed time $T > 0$, the work needed to produce e_0 in the time T depends on the strain path $e(t)$, $0 \le t \le T$, from the unstrained state to e_0. In fact, this work is given by

$$W(e) = \int_0^T s(t)\dot{e}(t)dt, \tag{2}$$

or equivalently by

$$W(e) = \int_0^T \int_0^t G(t - \tau)\dot{e}(\tau)\dot{e}(t)d\tau dt. \tag{3}$$

Of course, in writing (1) and (2) (and hence (3)) it is tacit that the strain path e be smooth, and we therefore begin our search for a minimizer within the class

$$S = \{e \in C^1[0, T]: e(0) = 0, e(T) = e_0\}.$$

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[1]A formal construction of the optimal strain path was given by Breuer [2] for the special case in which the relaxation function is the sum of exponentials. We became aware of Breuer's work only after we had completed the analysis presented here.

[2]Cf. Leitmann [4], who obtains continuous optimal paths by adding certain constraints.

An absolute minimizer for W within this class will be called a *smooth optimal strain path*.

In order to state our results concisely, we assume, once and for all, that $G \in C^3[0, T]$, and that, for all $\tau \in [0, T]$,

$$G(\tau) > 0, \; \dot{G}(\tau) < 0, \; \ddot{G}(\tau) \geq 0. \tag{4}$$

THEOREM. There does not exist a smooth optimal strain path.

We begin by proving the following

LEMMA. Let $f \in C[0, T]$. Then

$$2\dot{G}(0) \int_0^T f(t)^2 dt + \int_0^T \int_0^T \ddot{G}(|t - \tau|) f(\tau) f(t) d\tau dt =$$

$$- \frac{1}{2} \int_0^T \int_0^T \ddot{G}(|t - \tau|)[f(t) - f(\tau)]^2 d\tau dt + \int_0^T [\dot{G}(t) + \dot{G}(T - t)] f(t)^2 dt, \tag{5}$$ [3]

and if

$$2\dot{G}(0) f(t) + \int_0^T \ddot{G}(|t - \tau|) f(\tau) d\tau = 0 \tag{6}$$

for all $t \in [0, T]$, then $f \equiv 0$.

Clearly,

$$- \frac{1}{2} \int_0^T \int_0^T \ddot{G}(|t - \tau|)[f(t) - f(\tau)]^2 d\tau dt = \int_0^T \int_0^T \ddot{G}(|t - \tau|) f(\tau) f(t) d\tau dt$$

$$- \int_0^T \int_0^T \ddot{G}(|t - \tau|) f(\tau)^2 d\tau dt. \tag{7}$$

Further,

$$\int_0^T \ddot{G}(|t - \tau|) dt = \int_0^\tau \ddot{G}(\tau - t) dt + \int_\tau^T \ddot{G}(t - \tau) dt = -2\dot{G}(0) + \dot{G}(\tau) + \dot{G}(T - \tau),$$

and hence

$$\int_0^T \int_0^T \ddot{G}(|t - \tau|) f(\tau)^2 d\tau dt = - 2\dot{G}(0) \int_0^T f(t)^2 dt + \int_0^T [\dot{G}(t) + \dot{G}(T - t)] f(t)^2 dt. \tag{8}$$

Eqs. (7) and (8) imply (5). Assume next that (6) holds. If we multiply this equation by $f(t)$ and integrate from $t = 0$ to $t = T$, we arrive at the conclusion that (5) equals zero; in view of (4), this in turn implies that $f \equiv 0$.

To prove the theorem assume that a smooth optimal strain path, e, exists. Then $W(e) \leq W(g)$ for all $g \in S$, and hence the variation

$$\delta W(e)[\beta] = \frac{d}{d\alpha} W(e + \alpha\beta)\big|_{\alpha=0}$$

vanishes for every $\beta \in C^1[0, T]$ with $\beta(0) = \beta(T) = 0$. Let β be such a function. Then a

[3] Londen [3], Eq. (2.19).

simple calculation shows that

$$\delta W(e)[\beta] = -\int_0^T \beta(t)\left\{\int_0^t \dot{G}(t-\tau)\dot{e}(\tau)d\tau - \int_t^T \dot{G}(\tau - t)\dot{e}(\tau)d\tau\right\}dt,$$

and, since β is arbitrary, the optimal path e must necessarily satisfy

$$\int_0^t \dot{G}(t-\tau)\dot{e}(\tau)d\tau - \int_t^T \dot{G}(\tau - t)\dot{e}(\tau)d\tau = 0.$$

If we differentiate this expression with respect to t, we arrive at

$$2\dot{G}(0)\dot{e}(t) + \int_0^T \ddot{G}(|t-\tau|)\dot{e}(\tau)d\tau = 0;$$

hence we may conclude from the second part of the lemma (with $f = \dot{e}$) that e is constant. But $e(0) = 0$ and $e(T) = e_0 \neq 0$, and we have a contradiction. Thus there does not exist a smooth optimal strain path.

Remark. We have actually established the nonexistence of a stationary point for W over S.

2. Existence in L_2. It is clear from the preceding section that to find an optimal strain path we must enlarge our class of admissible paths. With this in mind, we use the end conditions

$$e(0) = 0, \qquad e(T) = e_0 \tag{9}$$

to derive an alternative expression for $W(e)$ which does not require differentiation of e. We begin by using $(9)_1$ to rewrite (1) in the form

$$s(t) = G(0)e(t) + \int_0^t \dot{G}(t-\tau)e(\tau)d\tau.$$

Thus (2) and $(9)_2$ imply that

$$W(e) = \frac{1}{2}G(0)e_0^2 + \int_0^T \dot{e}(t)\int_0^t \dot{G}(t-\tau)e(\tau)d\tau dt,$$

and if we integrate the second term by parts, we conclude, with the aid of (9), that

$$W(e) = \frac{1}{2}G(0)e_0^2 + e_0\int_0^T \dot{G}(T-t)e(t)dt - \dot{G}(0)\int_0^T e(t)^2 dt$$
$$-\frac{1}{2}\int_0^T\int_0^T \ddot{G}(|t-\tau|)e(\tau)e(t)d\tau dt. \tag{10}$$

In view of the above derivation, $W(e)$ defined by (10) agrees with $W(e)$ defined by (3) on the class S of *smooth* paths. But what is more important, (10) is well-defined on any path e in $L_2(0, T)$. On such paths (10) has an immediate interpretation. Indeed, it is not difficult to verify that the map $e \to W(e)$ defined by (10) is continuous on $L_2(0, T)$. Consider an arbitrary strain path e in $L_2(0, T)$. Then, since S is dense in $L_2(0, T)$, there exists a sequence $\{e_n\}$ of paths in S such that $e_n \to e$ in $L_2(0, T)$, and hence such that $W(e_n) \to W(e)$. Thus the work done on any $L_2(0, T)$ strain path e is simply the limit of the work done on any sequence of smooth strain paths which satisfy the end conditions (9) and have e as their limit.

An $L_2(0, T)$ function e which minimizes (10) over $L_2(0, T)$ will be called an *optimal strain path in* L_2.

THEOREM. There exists a unique optimal strain path e in L_2. Moreover, $e \in C^1(0, T)$ and has limits $e(0^+)$, $\dot{e}(0^+)$, $e(T^-)$, $\dot{e}(T^-)$, but $e(0^+) \neq 0$, $e(T^-) \neq e_0$.

We begin the proof by writing (10) in the form

$$W(e) = A + L(e) + Q(e, e)$$

with

$$A = \frac{1}{2} G(0)e_0^2, \qquad L(e) = e_0 \int_0^T \dot{G}(T - t)e(t)dt,$$
$$Q(e, \beta) = -\dot{G}(0) \int_0^T e(t)\beta(t)dt - \frac{1}{2}\int_0^T \int_0^T \ddot{G}(|t - \tau|)e(\tau)\beta(t)d\tau dt, \tag{11}$$

so that L is linear, while Q is bilinear and symmetric. Thus for $e, \beta \in L_2(0,T)$,

$$W(e + \beta) - W(e) = Q(\beta,\beta) + 2Q(e,\beta) + L(\beta).$$

Further, by $(11)_3$, (5), and (4), $Q(\beta,\beta) \geq 0$ for all $\beta \in C[0, T]$ and hence (by continuity) for all $\beta \in L_2(0, T)$. Thus a necessary and sufficient condition for e in $L_2(0, T)$ to be optimal is that for all $\beta \in L_2[0, T]$,

$$2Q(e, \beta) + L(\beta) = 0,$$

or equivalently,

$$\int_0^T \beta(t)\left\{e_0\dot{G}(T - t) - 2\dot{G}(0)e(t) - \int_0^T \ddot{G}(|t - \tau|)e(\tau)d\tau\right\} dt = 0;$$

and this in turn holds if and only if e satisfies the Euler equation

$$e_0\dot{G}(T - t) = 2\dot{G}(0)e(t) + \int_0^T \ddot{G}(|t - \tau|)e(\tau)d\tau \tag{12}$$

at almost every $t \in (0, T)$.

Eq. (12) is a Fredholm integral equation of the second kind; by the Fredholm alternative (12) will have a unique solution in $L_2(0, T)$ if the homogeneous equation ((12) with $e_0 = 0$) has only the zero solution. Thus the existence, uniqueness, and smoothness of the optimal solution are immediate consequences of the second part of the lemma and the fact that, since $G \in C^3[0, T]$ and $\dot{G}(0) \neq 0$, any solution of (12) will belong to $C^1(0, T)$ and have limits $e(0^+)$, $\dot{e}(0^+)$, $e(T^-)$, and $\dot{e}(T^-)$.

Henceforth let e denote the optimal path. In view of the above remarks we may assume, without loss in generality, that $e \in C^1[0, T]$. Define f on $[0, T]$ by

$$f(t) = e_0 - e(T - t).$$

Then a simple calculation shows that f also satisfies the Euler equation (12). Thus, since the optimal path is unique, $f(t) \equiv e(t)$; hence

$$e(t) = e_0 - e(T - t). \tag{13}$$

Assume that either $e(0) = 0$ *or* $e(T) = e_0$. Then, by (13), $e(0) = 0$ *and* $e(T) = e_0$, so that

$e \in S$. Thus e is a smooth optimal strain path, which contradicts the theorem established in the previous section. Therefore $e(0) \neq 0$, $e(T) \neq e_0$, and the proof is complete.

We *conjecture* that the optimal path is monotone.

Remark. Let e be the optimal strain path in L_2. Then, since S is dense in L_2,

$$W(e) = \inf_{g \in S} W(g)$$

(cf. the argument given in the paragraph following (10)); that is, $W(e)$ is the greatest lower bound for the work in smooth processes consistent with the end conditions.

3. Optimal path for a Maxwell material. A Maxwell material is characterized by a relaxation function of the form

$$G(t) = G_\infty + (G_0 - G_\infty)\exp(-t/\lambda) \tag{14}$$

with $G_\infty \geq 0$, $G_0 - G_\infty > 0$, and $\lambda > 0$. For this choice of relaxation function the Euler equation is easily solved. Indeed, we simply differentiate (12) twice with respect to t and conclude that $\ddot{e}(t) \equiv 0$, and hence that $e(t) = c_0 + c_1 t$. We then evaluate the constants c_0 and c_1 using the conditions

$$e(T/2) = e_0/2, \qquad e_0\dot{G}(T) = 2\dot{G}(0)e(0^+) + \int_0^T \ddot{G}(t)e(t)dt,$$

which follow from (13) and (12). The resulting solution is (cf. Breuer [2], Eq. (15))

$$e(t) = e_0(1 + t/\lambda)/(2 + T/\lambda). \tag{15}$$

Thus e is linear on $(0, T)$ and suffers jump discontinuities of amount $e_0/(2 + T/\lambda)$ at $t = 0$ and $t = T$. Further, the least work — that is, $W(e)$ for e defined by (15) — is equal to (cf. Breuer [2], Eq. (27))

$$W(e) = \frac{1}{2} e_0^2\left[G_0 + \frac{G_\infty - G_0}{1 + 2\lambda/T}\right].$$

Note that $W(e) \to \frac{1}{2} G_0 e_0^2$ or $\frac{1}{2} G_\infty e_0^2$ according as $T/\lambda \to 0$ or ∞.

4. Kelvin materials. A Kelvin material is defined by a constitutive equation of the form

$$s(t) = Ee(t) + \mu\dot{e}(t)$$

with $\mu > 0$. Because of the presence of the viscous term, $\mu\dot{e}(t)$, this type of material is not a special case of the materials studied in the previous sections. By (2),

$$W(e) = \frac{1}{2} Ee_0^2 + \mu \int_0^T \dot{e}(t)^2 dt,$$

and a simple analysis establishes the existence and uniqueness of a *smooth* optimal strain path. (The viscous term precludes the possibility of jump discontinuities in strain.) Moreover, the optimal path has the simple form

$$e(t) = e_0 t/T. \tag{16}$$

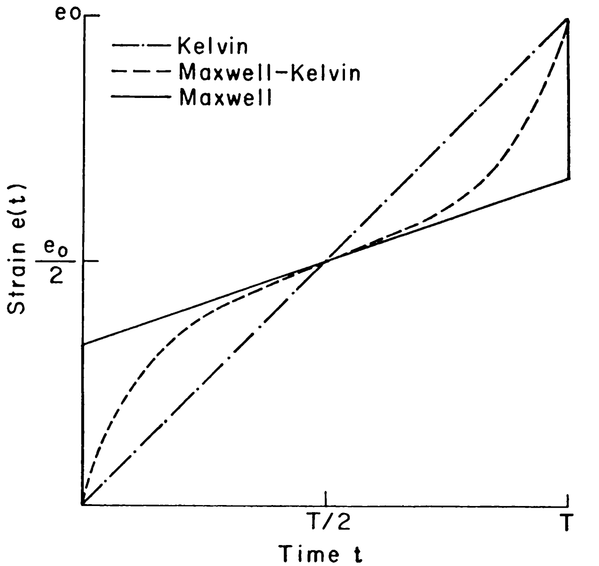

FIG. 1. Optimal strain paths for Maxwell, Kelvin, and Maxwell-Kelvin materials.

The *same* observations apply to the more general constitutive relation

$$s(t) = \mu\dot{e}(t) + \int_0^t G(t - \tau)\dot{e}(\tau)d\tau, \tag{17}$$

although the optimal path will generally not be of the simple form (16). A thorough study of (17), however, is beyond the scope of this paper.

Fig. 1 compares the optimal strain paths for:

(i) a Kelvin material;

(ii) a Maxwell material with $\lambda/T = 1$;

(iii) a Maxwell-Kelvin material of the form (17), (14) with $\lambda/T = 1$, $\mu/(G_0 - G_\infty)T = .01$.

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