OBTAINING ANALYTIC FUNCTIONS AND CONJUGATE HARMONIC FUNCTIONS*

By RAIMOND A. STRUBLE (*North Carolina State University*)

E. V. Laitone [2] has called our attention to a useful (apparently not so well advertised) procedure for obtaining an analytic function from one of its conjugate harmonics. The idea is simply this: if $f(z) = f(x + iy) = \phi(x, y) + i\psi(x, y)$ is analytic in a neighborhood of some point $z_0 = x_0 + iy_0$, then it can be recovered from either of its conjugate harmonics through the integration formulas,

$$f(z) = \int \left[\frac{\partial \phi}{\partial x}(z - iy_0, y_0) - i\frac{\partial \phi}{\partial y}(z - iy_0, y_0)\right] dz = \int \left[\frac{\partial \psi}{\partial y}(z - iy_0, y_0) + i\frac{\partial \psi}{\partial x}(z - iy_0, y_0)\right] dz \tag{1}$$

in the "variable" z. These formulas are readily obtained from the Cauchy-Riemann equations once it is observed that

$$f(z) = \phi(z - iy_0, y_0) + i\psi(z - iy_0, y_0) \tag{2}$$

holds for all z in some neighborhood of z_0.

Actually, Laitone derived (2) only for $y_0 = 0$, using the conjugate complex function $\bar{f}(\bar{z})$. An alternative (and perhaps more direct and illuminating) derivation can be given using Taylor series. For if

$$\phi(u + x_0, y_0) = \sum a_n u^n, \psi(u + x_0, y_0) = \sum b_n u^n \tag{3}$$

are the Taylor series expansions (in the first variable) of ϕ and ψ about x_0, then

$$a_n = \frac{1}{n!}\frac{\partial^n \phi}{\partial x^n}(x_0, y_0), b_n = \frac{1}{n!}\frac{\partial^n \psi}{\partial x^n}(x_0, y_0)$$

hold for all n. But $d^n f/dz^n = \partial^n\phi/\partial x^n + i\partial^n\psi/\partial x^n$, so that the Taylor series expansion of f about $z_0 = x_0 + iy_0$ gives

$$f(z) = \sum \frac{1}{n!}\frac{d^n f}{dz^n}(z_0)[z - z_0]^n = \sum (a_n + ib_n)[z - (x_0 + iy_0)]^n = \phi(z - iy_0, y_0) + i\psi(z - iy_0, y_0)$$

where we have used (3) with $u = z - (x_0 + iy_0)$.

Using analogous Taylor series and $d^n f/dz^n = (-i)^n\partial^n\phi/\partial y^n + (-i)^{n-1}\partial^n\psi/\partial y^n$, we similarly obtain an analogous identity

$$f(z) = \phi(x_0, -iz + ix_0) + i\psi(x_0, -iz + ix_0). \tag{4}$$

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This alternative derivation might help to explain some of the symbolism used in (1) (the permanence of the Cauchy-Riemann equations under analytic continuation etc.) and so help (along with Laitone's examples) to rescue the idea from near oblivion.

Even more helpful is the Taylor series (in two variables)

$$\phi(h + x_0, k + y_0) = \sum \frac{1}{n!}\left(h\frac{\partial}{\partial x_0} + k\frac{\partial}{\partial y_0}\right)^n \phi(x_0, y_0).$$

For with $h = u/2$ and $k = -iu/2$, we obtain the identity

$$2\phi\left(\frac{u}{2} + x_0, \frac{-iu}{2} + y_0\right) = \sum \frac{2}{n!}\left(\frac{u}{2}\right)^n \left(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial y_0}\right)^n \phi(x_0, y_0). \tag{5}$$

But from the Cauchy-Riemann equations we have

$$(\partial/\partial x_0 - i\partial/\partial y_0)\phi(x_0, y_0) = \partial\phi(x_0, y_0)/\partial x_0 + i\partial\psi(x_0, y_0)/\partial x_0 = df(z_0)/dz_0.$$

Hence $(\partial/\partial x_0 - i\partial/\partial y_0)^2\phi(x_0, y_0) = (\partial/\partial x_0 - i\partial/\partial y_0)df(z_0)/dz_0 = 2d^2f(z_0)/dz_0^2$, and so by induction, $(\partial/\partial x_0 - i\partial/\partial y_0)^n\phi(x_0, y_0) = 2^{n-1}d^nf(z_0)/dz_0^n$ holds for all $n \geq 1$. The series in (5) then becomes

$$\phi(x_0, y_0) - i\psi(x_0, y_0) + \sum \frac{1}{n!}(d^nf(z_0)/dz_0^n)\, u^n$$

and with $u = z - z_0$ we obtain what may be the simplest of all recovery formulas,

$$f(z) = 2\phi\left(\frac{z - z_0}{2} + x_0, \frac{-i(z - z_0)}{2} + y_0\right) - \phi(x_0, y_0) + i\psi(x_0, y_0).$$

$$\left(= 2i\psi\left(\frac{z - z_0}{2} + x_0, \frac{-i(z - z_0)}{2} + y_0\right) - i\psi(x_0, y_0) + \phi(x_0, y_0)\right) \tag{6}$$

The simplicity in (6), of course, lies in the fact that no integrations are required to obtain f from ϕ (or ψ). Once (6) is anticipated, it can be verified directly. For the left-hand and right-hand members are clearly equal when $z = z_0$, and upon using the Cauchy-Riemann equations, their derivatives can be seen to be equal. In fact, $(z - z_0)/2 + x_0 = (z + \bar{z}_0)/2$, $-i(z - z_0)/2 + y_0 = (z - \bar{z}_0)/2i$, and

$$\begin{aligned}\frac{d}{dz}2\phi\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right) &= \frac{\partial\phi}{\partial x}\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right) - i\frac{\partial\phi}{\partial y}\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right)\\ &= \frac{\partial\phi}{\partial x}\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right) + i\frac{\partial\psi}{\partial x}\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right)\\ &= \frac{df}{dz}(X + iY) \qquad \text{with} \qquad X = \frac{z + \bar{z}_0}{2}, Y = \frac{z - \bar{z}_0}{2i}\\ &= \frac{df}{dz}(z).\end{aligned}$$

The circumstance that $X + iY$ becomes simply z when $X = (z + \bar{z}_0)/2$ and $Y = (z - \bar{z}_0)/2i$ plays a crucial role here, of course. Similar verifications of (2) and of (4) can be made.

Eq. (6), for $z_0 = 0$, has been derived (somewhat heuristically) in [1] and [3] and shows that a real analytic function ϕ of two variables is harmonic, say near the origin, *iff*

$$\phi(x, y) = 2\,\mathrm{Re}\,\phi\left(\frac{x + iy}{2}, \frac{y - ix}{2}\right) - \phi(0, 0) \tag{7}$$

holds there. Of considerable interest also is the companion equation giving the conjugate harmonic function,

$$\psi(x, y) = 2\,\mathrm{Im}\,\phi\left(\frac{x+iy}{2}, \frac{y-ix}{2}\right) + \psi(0, 0). \tag{8}$$

Thus each stream function ψ can be expressed algebraically (without differentiations or integrations) in terms of the corresponding potential function ϕ, and vice versa.

Perhaps the identities (2), (4), (6) and (8) (along with these elementary derivations) should be better known than they appear to be.

REFERENCES

[1] L. V. Ahlfors, *Complex analysis,* McGraw-Hill Book Company, New York, 1966, p. 27

[2] E. V. Laitone, *Relation of the conjugate harmonic functions to f(z), Amer. Math. Monthly* **84,** 281–283 (1977)

[3] L. M. Milne-Thomson, *Theoretical hydrodynamics,* The Macmillan Company, New York, 4th ed., 1962, 5th ed., 1967