PLANE-STRAIN PROBLEM OF TWO COPLANAR CRACKS IN AN INITIALLY STRESSED NEO-HOOKEAN ELASTIC LAYER*

By

RANJIT S. DHALIWAL AND BRIJ MOHAN SINGH

University of Calgary, Alberta, Canada

Abstract. We consider the problem of determining the stress intensity factors and the crack energy in an infinitely long strip of an initially stressed neo-Hookean elastic material containing two coplanar Griffith cracks. We assume that the cracks are opened by a constant internal pressure and that the edges of the strip are either rigidly fixed or stress-free. By the use of Fourier transforms we reduce the problem to solving a set of triple integral equations with cosine kernel and a weight function. These equations are reduced to a Fredholm integral equation of the second kind by using finite Hilbert transform technique. Analytical expressions up to the order δ^{-10} are derived for the stress intensity factors and the crack energy, where 2δ denotes the width of the strip and δ is much greater than 1. Numerical values of the stress intensity factors and the crack energy are graphed to display the effect of initial stress.

1. Introduction. The theory of cracks in a two-dimensional elastic medium was first developed by Griffith [1]. Sneddon and Elliot [2] solved the problem of finding the distribution of stress in the neighborhood of a Griffith crack that is subject to an internal pressure, by considering the corresponding boundary-value problem for a semi-infinite two-dimensional medium. Recently, Willmore [3] and Tranter [4] solved the problem of determining the distribution of stress when two coplanar cracks are opened by an internal pressure in an infinite elastic aelotropic and isotropic medium, respectively. Lowengrub and Srivastava [5] considered the problem of an infinitely long strip containing two coplanar Griffith cracks, and they employed the finite Hilbert transform technique to solve the problem.

The incremental deformation theory concerns the infinitesimal deformation of a solid with a known initial finite deformation. The basic equations of such incremental deformation theory have been derived by Trefftz [6], Biot [7, 8], Neuber [9], Green, Rivlin and Shield [10] and Green and Zerna [11]. Neuber used his theory to solve the buckling problems of sandwich plates [12, 13] and a spherical shell [14] with nonlinear stress-strain law. Biot [15, 16, 17] investigated the effect of initial stress on surface buckling, internal buckling and elastic waves and to the single-crack problem with initial stress [18]. The theory and many applications were presented in a monograph [19] by Biot. Kurashige [20, 21, 22] used Biot's [7] theory to solve the problems of a penny-shaped crack in an infinite medium, a line crack in a thin infinite strip and a slipless indentation problem of an infinite circular cylinder.

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This paper applies Biot's theory [7, 8, 18] to the problem of determining the distribution of stress in an infinitely long strip containing two coplanar Griffith cracks. The problem in the title is considered for an initially stressed neo-Hookean elastic material. In Sec. 2, we give the basic equations and a Fourier transform solution of the equilibrium equations and obtain expressions for the components of displacement and stress. In Sec. 3, we give boundary conditions for two cases: (A) the edges of the strip are fixed, (B) the edges of the strip are stress-free; and derive the appropriate triple integral equations. In Sec. 4, the triple integral equations are reduced to a single Fredholm integral equation of the second kind. The iterative solution of the integral equation is obtained in Sec. 5 for $\delta \gg 1$ up to the order δ^{-10} when half the width of the strip is δ times the distance of the far end of the crack from the origin. The analytical expressions up to the order δ^{-10} are obtained for the stress intensity factors and the crack energy in Sec. 6. The numerical values of the stress intensity factors and the crack energy are graphed to demonstrate the effect of initial stress.

2. Basic equations of incremental deformation theory and their solution. In rectangular cartesian coordinates x_i and time t, the equations of motion associated with incremental deformation theory of elasticity are

$$s_{ij,j} + S_{jk}\omega_{ik,j} + S_{ik}\omega_{jk,j} - e_{jk}S_{ik,j} = \rho\ddot{u}_i\,, \tag{1}$$

where the usual summation convention over repeated indices is applied, $\ddot{u}_i = \partial^2 u_i/\partial t^2$, ρ = density in the initial state, u_i = incremental infinitesimal displacement components, e_{ij} = incremental strain tensor, s_{ij} = incremental stress tensor referred to axes which are incrementally displaced with the medium, and S_{ij} = initial stress tensor, corresponding to initial finite deformation, referred to x_i. The last three terms on the left-hand side of Eq. (1) are due to the effect of initial stress.

Incremental strains and rotations may be written in terms of the incremental infinitesimal displacements by the following relations (similar to the classical theory of elasticity):

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad \omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}). \tag{2}$$

If the material is a neo-Hookean solid, elastic potential per unit volume is expressed in the form [6]

$$W = \tfrac{1}{2}\mu_0(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \qquad \lambda_1\lambda_2\lambda_3 = 1, \tag{3}$$

from which the stress-strain relations are obtained as

$$\{S_{11} - S_{22},\ \ S_{22} - S_{33},\ \ S_{33} - S_{11}\} = \mu_0\{\lambda_1^2 - \lambda_2^2,\ \ \lambda_2^2 - \lambda_3^2,\ \ \lambda_3^2 - \lambda_1^2\}, \tag{4}$$

where λ_i is the extension ratio in the x_i-direction and μ_0 is the shear modulus in an unstrained state.

The total differentiation of Eq. (4) and consideration of incremental shear deformation give the following incremental stress-strain relations [6]:

$$\{s_{11} - s_{22}\,,\ s_{22} - s_{33}\,,\ s_{33} - s_{11}\} = \mu_0\{\lambda_1^2 e_{11} - \lambda_2^2 e_{22}\,,\ \lambda_2^2 e_{22} - \lambda_3^2 e_{33}\,,\ \lambda_3^2 e_{33} - \lambda_1^2 e_{11}\}, \tag{5}$$

$$\{s_{12}\,,\ s_{23}\,,\ s_{31}\} = \mu_0\{(\lambda_1^2 + \lambda_2^2)e_{12}\,,\ (\lambda_2^2 + \lambda_3^2)e_{23}\,,\ (\lambda_3^2 + \lambda_1^2)e_{31}\}. \tag{6}$$

If we consider plane strain perpendicular to the z-axis in a rectangular cartesian coordinate system (x, y, z) and assume that the initial finite deformation produces a normal stress S_{xx} which is uniform throughout the neo-Hookean elastic solid then, we have

$$S_{xx} = \mu_0(\lambda_x^2 - \lambda_y^2) = -P, \tag{7}$$

$$\{s_{xx} - s, s_{yy} - s, s_{xy}\} = \mu_0(\lambda_x^2 + \lambda_y^2)\{e_{xx}, e_{yy}, e_{xy}\}, \tag{8}$$

where P is constant and

$$s = \tfrac{1}{2}(s_{xx} + s_{yy}). \tag{9}$$

The two-dimensional equations of equilibrium for the incremental stress field are

$$\frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y} + S_{xx}\frac{\partial \omega}{\partial y} = 0, \quad \frac{\partial s_{xy}}{\partial x} + \frac{\partial s_{yy}}{\partial y} + S_{xx}\frac{\partial \omega}{\partial x} = 0, \tag{10}$$

where $\omega = \omega_{xy}$ is an incremental rotation. Introducing a scalar incremental displacement function $\phi(x, y)$ by the relations

$$u_x = -\partial\phi/\partial y, \qquad u_y = \partial\phi/\partial x, \tag{11}$$

we find that the condition of incompressibility

$$e_{xx} + e_{yy} = 0 \tag{12}$$

is satisfied.

From Eqs. (2), (7), (8) and (10) we find that the equations of equilibrium may be written in terms of two unknown functions s and ϕ:

$$\frac{\partial s}{\partial x} - \frac{1}{2}\{\mu_0(\lambda_x^2 + \lambda_y^2) - P\}\frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = 0,$$

$$\frac{\partial s}{\partial y} + \frac{1}{2}\{\mu_0(\lambda_x^2 + \lambda_y^2) + P\}\frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = 0. \tag{13}$$

Eliminating s between the above two equations, we find that ϕ must satisfy the equation [19]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(k^2\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = 0, \tag{14}$$

where

$$k^2 = \frac{\mu_0(\lambda_x^2 + \lambda_y^2) + P}{\mu_0(\lambda_x^2 + \lambda_y^2) - P} = \left(\frac{\lambda_y}{\lambda_x}\right)^2. \tag{15}$$

Taking the Fourier transform of Eq. (14) with respect to x, we find that ϕ satisfies the equation

$$\left(\frac{d^2}{dy^2} - \xi^2\right)\left(\frac{d^2}{dy^2} - k^2\xi^2\right)\bar{\phi} = 0, \tag{16}$$

whose solution may be taken as

$$\bar{\phi}(\xi, y) = A(\xi)\sinh(\xi y) + B(\xi)\sinh(k\xi y) + C(\xi)\cosh(\xi y) + D(\xi)\cosh(k\xi y) \tag{17}$$

where

$$\bar{\phi}(\xi, y) = \left(\frac{2}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}\phi(x, y)\exp(ix\xi)\,dx. \tag{18}$$

Now taking the Fourier transform of Eq. (13) with respect to x and substituting for $\bar{\phi}$ from Eq. (17), we find that

$$\bar{s}(\xi, y) = -i\mu_0\lambda_y{}^2k(k^2 - 1)\xi^2[D(\xi)\sinh(k\xi y) + B(\xi)\cosh(k\xi y)]. \tag{19}$$

From Eqs. (2), (8), (9) and (11), we obtain

$$\bar{u}_x = -\partial\bar{\phi}/\partial y, \qquad \bar{u}_y = -i\xi\bar{\phi}, \qquad \bar{s} = \tfrac{1}{2}(\bar{s}_{xx} + \bar{s}_{yy}),$$

$$\{\bar{s}_{xx} - \bar{s}, \bar{s}_{xy}\} = \mu_0(\lambda_x{}^2 + \lambda_y{}^2)\{i\xi\,\partial\bar{\phi}/\partial y, -\tfrac{1}{2}(\partial^2\bar{\phi}/\partial y^2 + \xi^2\bar{\phi})\}. \tag{20}$$

When there is symmetry about the line $x = 0$, the expressions for the required stress and displacement components may be written as

$$u_y(x, y) = F_c[\xi\{A\sinh(\xi y) + B\sinh(k\xi y) + C\cosh(\xi y) + D\cosh(k\xi y)\}; \xi \to x],$$

$$s_{yy}(x, y) = \mu_0\lambda_y{}^2F_c[\xi^2\{(1 + k^2)A\cosh(\xi y) + 2kB\cosh(k\xi y) + (1 + k^2)C\sinh(\xi y) + 2kD\sinh(k\xi y)\}; \xi \to x],$$

$$s_{xy}(x, y) = -\tfrac{1}{2}\mu_0(\lambda_x{}^2 + \lambda_y{}^2)F_s[\xi^2\{2A\sinh(\xi y) + (1 + k^2)B\sinh(k\xi y) + 2C\cosh(\xi y) + (1 + k^2)D\cosh(k\xi y)\}; \xi \to x] \tag{21}$$

where

$$F_s[f(\xi, y); \xi \to x] = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty f(\xi, y)\sin(\xi x)\,d\xi,$$

$$F_c[f(\xi, y); \xi \to x] = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty f(\xi, y)\cos(\xi x)\,d\xi.$$

3. Statement and solution of the problem. We consider an infinite elastic strip of breadth 2δ given by $-\delta \leq y \leq \delta$, $-\infty < x < \infty$ and initially deformed in a manner given by Eq. (7). If a pair of coplanar cracks develops in the middle line of the strip perpendicular to the y-axis and also symmetrical about it, we may consider the problem of a strip $0 \leq y \leq \delta$, $0 < x < \infty$ if both the edges $y = \delta, -\delta$ are given as either stress-free or fixed. We assume that the cracks are opened by an internal pressure $p(x)$ which is an even function of x.

CASE A: For this case, assume that the edges of the strip are bonded to a rigid material such that the displacement u_y and the shear stress s_{xy} vanish at the edges of the strip. The boundary conditions may be taken as

$$u_y(x, \delta) = 0, \qquad s_{xy}(x, \delta) = 0, \qquad s_{xy}(x, 0) = 0, \qquad 0 < |x| < \infty, \tag{22}$$

$$\begin{aligned} s_{yy}(x, 0) &= -p(x), && \kappa \leqq |x| \leqq 1, \\ u_y(x, 0) &= 0, && 0 < |x| < \kappa, \quad |x| > 1. \end{aligned} \tag{23}$$

The boundary conditions (22) are satisfied if we take

$$\{A, B, C\} = \tfrac{1}{2}(1 + k^2)D(\xi)\{\coth(\xi\delta), \frac{-2}{1 + k^2}\coth(k\xi\delta), -1\}. \tag{24}$$

Now if we define a new unknown function $G(\xi)$ in terms of $D(\xi)$ by the relation

$$G(\xi) = \tfrac{1}{2}(1 - k^2)\xi D(\xi), \tag{25}$$

we find that

$$u_y(x, 0) = F_c[G(\xi);\ \xi \to x], \tag{26}$$

$$s_{yy}(x, 0) = -\frac{1}{\gamma} F_c[\xi G(\xi)\{1 + H_1(\xi\delta)\};\qquad \xi \to x], \tag{27}$$

where

$$-\frac{1}{\gamma} = \mu_0 \lambda_y^2 \frac{(1 + k^2)^2 - 4k}{1 - k^2}, \tag{28}$$

$$H_1(x) = \frac{-4k\{1 - \coth(k\xi\delta)\} + (1 + k^2)^2\{1 - \coth(\xi\delta)\}}{(1 + k^2)^2 - 4k}. \tag{29}$$

The remaining boundary conditions (23) will be satisfied provided $G(\xi)$ satisfies the set of triple integral equations:

$$F_c[G(\xi);\ \xi \to x] = 0, \qquad x \in L_1, L_3, \tag{30}$$

$$F_c[\xi G(\xi)\{1 + H_1(\xi\delta)\};\ \xi \to x] = \gamma p(x), \qquad x \in L_2, \tag{31}$$

where

$$L_1 = \{x \mid 0 < x < \kappa\}, \quad L_2 = \{x \mid \kappa \le x \le 1\}, \quad L_3 = \{x \mid x > 1\}. \tag{32}$$

CASE B: For this second case, we assume that the edges of the strip are stress-free; then we have the boundary conditions (23) and the boundary conditions (22) are replaced by

$$s_{yy}(x, \delta) = 0, \qquad s_{xy}(x, \delta) = 0, \qquad s_{xy}(x, 0) = 0, \qquad 0 < |x| < \infty. \tag{33}$$

The boundary conditions (33) will be satisfied if we take

$$A = \tfrac{1}{2}(1 + k^2)D[4k\{1 - \cosh(\xi\delta)\cosh(k\xi\delta)\} + (1 + k^2)^2 \sinh(\xi\delta)\sinh(k\xi\delta)][\Delta(\xi\delta)]^{-1},$$

$$B = D[(1 + k^2)^2\{1 - \cosh(\xi\delta)\cosh(k\xi\delta)\} + 4k \sinh(\xi\delta)\sinh(k\xi\delta)][\Delta(\xi\delta)]^{-1},$$

$$C = -\tfrac{1}{2}(1 + k^2)D, \tag{34}$$

where

$$\Delta(x) = (1 + k^2)^2 \sinh(kx)\cosh(x) - 4k \sinh(x)\cosh(kx). \tag{35}$$

Now, from Eqs. (34), (25) and (21), we find that

$$u_y(x, 0) = F_c[G(\xi);\ \xi \to x],$$

$$s_{yy}(x, 0) = -\frac{1}{\gamma} F_c[\xi G(\xi)\{1 + H_2(\xi\delta)\};\ \xi \to x], \tag{37}$$

where $G(\xi)$ and γ are given by Eqs. (25) and (28), respectively, and

$$\begin{aligned} H_2(x) = {} & [\Delta(x)\{(1 + k^2)^2 - 4k\}]^{-1}[8k(1 + k^2)^2 \\ & - (1 + k^2)^2 \exp(-x)\{4k \cosh(kx) + (1 + k^2)^2 \sinh(kx)\} \\ & - 4k \exp(-kx)\{(1 + k^2)^2 \cosh(x) + 4k \sinh(x)\}]. \end{aligned} \tag{38}$$

The boundary conditions (23) now lead to the following triple integral equations for the determination of $G(\xi)$:

$$F_c[G(\xi); \xi \to x] = 0, \qquad x \in L_1, L_3, \tag{39}$$

$$F_c[\xi G(\xi)\{1 + H_2(\xi\delta)\}; \xi \to x] = \gamma p(x), \qquad x \in L_2. \tag{40}$$

In the following sections, $H(\xi\delta)$ may be taken to mean $H_1(\xi\delta)$ (or $H_2(\xi\delta)$) for the problem of case A (or case B).

4. Solution of the triple integral equations. The solution of the set of triple integral equations

$$F_c[G(\xi); \xi \to x] = 0, \qquad x \in L_1, L_3, \tag{41}$$

$$F_c[\xi G(\xi)\{1 + H(\xi\delta)\}; \xi \to x] = \gamma p(x), \qquad x \in L_2, \tag{42}$$

as given in Srivastava and Lowengrub [5], is

$$G(\xi) = \left(\frac{\pi}{2}\right)^{1/2} \xi^{-1} \int_\kappa^1 h(t^2) \sin(\xi t)\, dt, \tag{43}$$

where $h(t^2)$ is the solution of the Fredholm integral equation of the second kind

$$h(x^2) + \int_\kappa^1 h(t^2) K_1(x^2, t)\, dt = M(x^2), \qquad x \in L_2, \tag{44}$$

satisfying the condition

$$\int_\kappa^1 h(t^2)\, dt = 0, \tag{45}$$

and

$$K_1(x^2, t) = -\frac{4}{\pi^2}\left(\frac{x^2 - \kappa^2}{1 - x^2}\right)^{1/2} \int_\kappa^1 \left(\frac{1 - y^2}{y^2 - \kappa^2}\right)^{1/2} \frac{y K_2(y, t)}{y^2 - x^2}\, dy, \tag{46}$$

with

$$K_2(y, t) = \int_0^\infty H(\xi\delta) \cos(\xi y) \sin(\xi t)\, d\xi, \tag{47}$$

and

$$M(x^2) = -\frac{4\gamma}{\pi^2}\left(\frac{x^2 - \kappa^2}{1 - x^2}\right)^{1/2} \int_\kappa^1 \left(\frac{1 - y^2}{y^2 - \kappa^2}\right)^{1/2} \frac{y p(y)}{y^2 - x^2}\, dy + C'\{(x^2 - \kappa^2)(1 - x^2)\}^{-1/2}, \tag{48}$$

where C' is an arbitrary constant to be determined from condition (45). Now integrating (44) with respect to x from κ to 1 and using (45), we find that

$$C' = \frac{1}{F} \int_\kappa^1 h(t^2) \left\{\int_\kappa^1 K_1(x^2, t)\, dx\right\} dt + \frac{4\gamma}{\pi^2} \int_\kappa^1 W(x^2)\, dx, \tag{49}$$

where

$$W(x^2) = \left(\frac{x^2 - \kappa^2}{1 - x^2}\right)^{1/2} \int_\kappa^1 \left(\frac{1 - y^2}{y^2 - \kappa^2}\right)^{1/2} \frac{y p(y)}{y^2 - x^2}\, dy \tag{50}$$

and $F = F(\kappa', \pi/2)$ is an elliptic integral of the first kind with $\kappa'^2 = 1 - \kappa^2$.

Hence from (44), (48) and (49), we find that h must satisfy the integral equation

$$h(x^2) + \int_\kappa^1 h(t^2)K(x^2, t)\, dt = P(x^2), \qquad x \in L_2\,, \tag{51}$$

where

$$K(x^2, t) = K_1(x^2, t) - \frac{1}{F}\{(x^2 - \kappa^2)(1 - x^2)\}^{-1/2}\int_\kappa^1 K_1(x^2, t)\, dx, \tag{52}$$

$$P(x^2) = \frac{4\gamma}{\pi^2}\left[W(x^2) - \frac{1}{F}\{(x^2 - \kappa^2)(1 - x^2)\}^{-1/2}\int_\kappa^1 W(x^2)\, dx\right] \cdot \tag{53}$$

5. Iterative solution of the integral equation. If we consider the case $\delta \gg 1$, then by substituting $\delta\xi = \zeta$ and expanding $\cos(\zeta y/\delta)$ and $\sin(\zeta t/\delta)$ in series, we may write (47) in the form

$$K_2(y, t) = \sum_{n=0}^{\infty} \frac{I_n}{\delta^{2n+2}} M_n(t, y), \tag{54}$$

where

$$M_n(t, y) = \tfrac{1}{2}[(t + y)^{2n+1} + (t - y)^{2n+1}]$$

and

$$I_n = \frac{(-1)^n}{(2n + 1)!}\int_0^\infty H(\zeta)\zeta^{2n+1}\, d\zeta.$$

Now from (54), (46) and (52), we find that

$$K(x^2, t) = \frac{t}{8\pi X}\sum_{n=1}^{\infty} \frac{\Lambda_n(x^2, t^2)}{\delta^{2n}}\,, \tag{55}$$

where

$$\begin{gathered} X = \{(x^2 - \kappa^2)(1 - x^2)\}^{1/2}, \qquad \Lambda_1 = I_0A_0(x^2), \\ \Lambda_2 = I_1[A_0(x^2)t^2 + 8A_1(x^2)], \qquad \Lambda_3 = I_2[A_0(x^2)t^4 + 10A_1(x^2)t^2 + A_2(x^2)], \\ \Lambda_4 = I_3[A_0(x^2)t^6 + 21A_1(x^2)t^4 + 7A_2(x^2)t^2 + A_3(x^2)], \end{gathered} \tag{56}$$

with

$$\begin{gathered} A_0(x^2) = 16(x^2 - E/F), \qquad A_1(x^2) = 8(2x^4 + \alpha_0x^2 + \alpha_1), \\ A_2(x^2) = 10(8x^6 + 4\alpha_0x^4 - \kappa'^4x^2 + \alpha_2), \\ A_3(x^2) = 7(16x^8 + 8\alpha_0x^6 - 2\kappa'^4x^4 + \alpha_0\kappa'^4x^2 + \alpha_3); \end{gathered} \tag{57}$$

$$\begin{gathered} \alpha_0 = -(1 + \kappa^2), \qquad \alpha_1 = \kappa'^2E/F - 2I_1', \\ \alpha_2 = \kappa'^2(1 + 3\kappa^2)E/F + 4\kappa'^2I_1' - 8I_2', \\ \alpha_3 = \kappa'^2(1 + 2\kappa^2 + 5\kappa^4)E/F + 2\kappa'^2(1 + 3\kappa^2)I_1' + 8\kappa'^2I_2' - 16I_3', \qquad \kappa'^2 = 1 - \kappa^2, \end{gathered} \tag{58}$$

and

$$I_n' = \frac{1}{F}\int_\kappa^1 \left(\frac{x^2 - \kappa^2}{1 - x^2}\right)^{1/2} x^{2n}\, dx, \tag{59}$$

from which we find that

$$I_0' = -\kappa^2 + E/F, \qquad I_1' = \tfrac{1}{3}[-\kappa^2 + (2 - \kappa^2)E/F],$$
$$I_2' = \tfrac{1}{15}[\kappa^2(\kappa^2 - 4) + (8 - 3\kappa^2 - 2\kappa^4)E/F],$$
$$I_3' = \tfrac{1}{105}[\kappa^2(4\kappa^4 + 5\kappa^2 - 21) + (48 - 16\kappa^2 - 9\kappa^4 - 8\kappa^6)E/F], \tag{60}$$

where F and E are elliptic integrals of the first and second kind, respectively, defined by

$$F = F\left(\kappa', \frac{\pi}{2}\right) = \int_\kappa^1 \frac{dx}{X}, \qquad E = E\left(\kappa', \frac{\pi}{2}\right) = \int_\kappa^1 \frac{x^2\,dx}{X}. \tag{61}$$

For $p(x) = p_0$, we find from (50) that

$$W(x^2) = \frac{\pi}{2} p_0 \left(\frac{x^2 - \kappa^2}{1 - x^2}\right)^{1/2}, \tag{62}$$

and hence from (53) we have

$$p(x^2) = \frac{p_0 \gamma}{8\pi X} A_0(x^2). \tag{63}$$

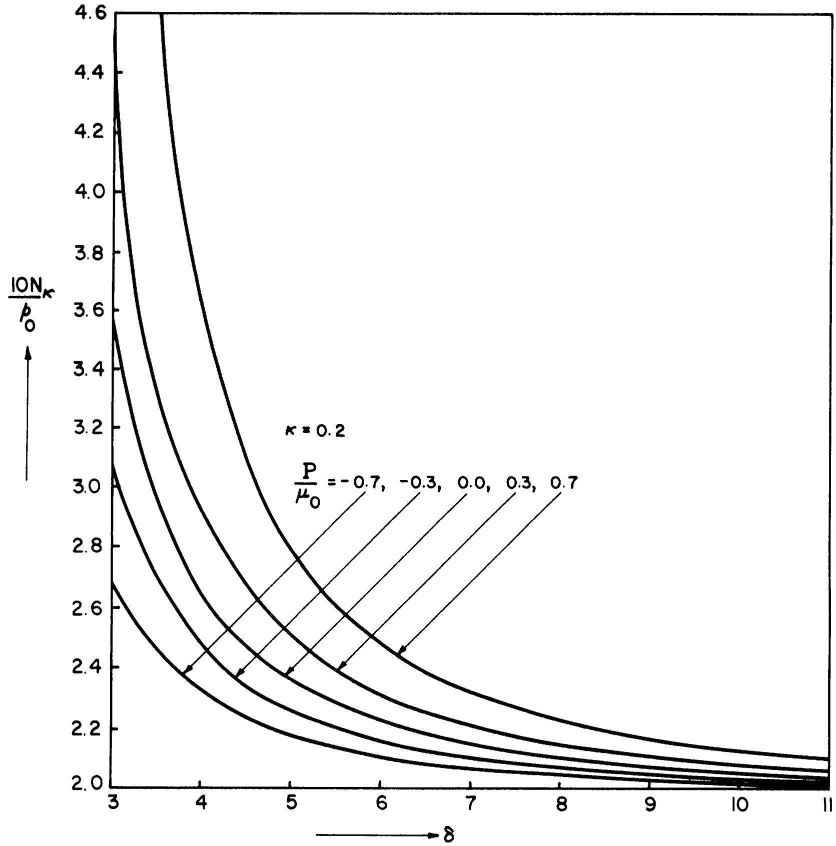

FIG. 1. Values of $10\, N_\kappa/p_0$ against half the strip width δ for various values of P/μ_0 and $\kappa = 0.2$.

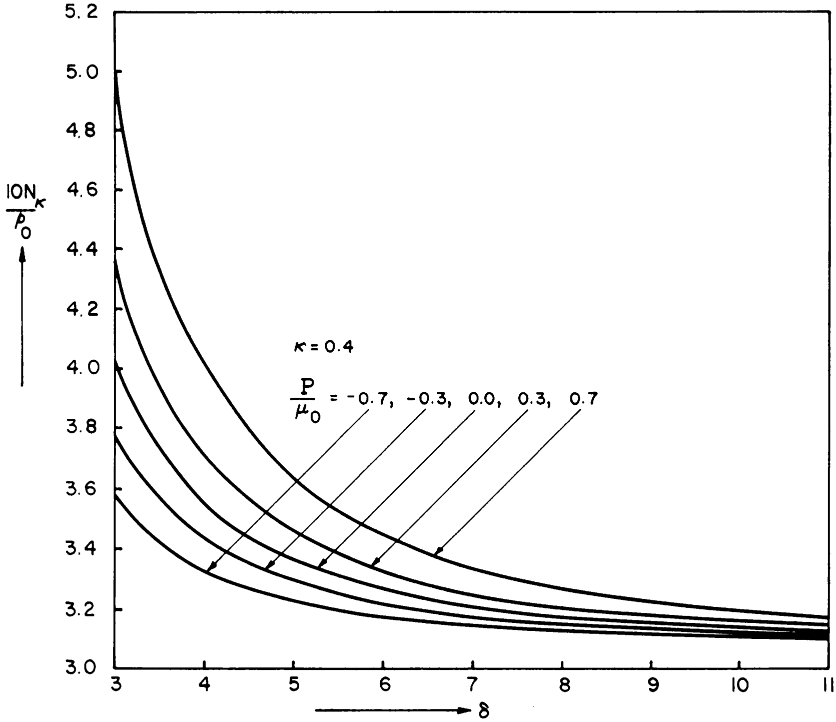

FIG. 2. Values of 10 N_κ/p_0 against half the strip width δ for various values of P/μ_0 and $\kappa = 0.4$.

Since $\delta >> 1$, $|K(x^2, t)| < \sigma$ where $\sigma < 1$, the solution to the integral equation (51) may be taken in the form

$$h(x^2) = \sum_{n=0}^{\infty} \frac{h_n(x^2)}{\delta^{2n}} . \tag{64}$$

Now substituting for $K(x^2, t)$ and h respectively from (55) and (64) in (51) and equating the various powers of δ from both sides, we obtain:

$$h_0(x^2) = P(x^2),$$
$$h_n(x^2) = -\frac{1}{8\pi X}\sum_{m=1}^{n}\int_{\kappa}^{1} t\,\Lambda_m(x^2, t^2)h_{n-m}(t^2)\,dt, \qquad n = 1, 2, 3, \cdots . \tag{65}$$

By carrying out the above iteration process up to h_4, we find that

$$h(x^2) = \frac{p_0\gamma}{4\pi X}(\beta_0 + \beta_1 x^2 + \beta_2 x^4 + \beta_3 x^6 + \beta_4 x^8) + (\delta^{-10}), \tag{66}$$

where

$$\beta_0 = -\frac{8E}{F} + \frac{4E}{F} I_0 b_0 \delta^{-2} + c_0\delta^{-4} + d_0\delta^{-6} + e_0\delta^{-8},$$
$$\beta_1 = 8 - 4I_0 b_0\delta^{-2} + c_1\delta^{-4} + d_1\delta^{-6} + e_1\delta^{-8},$$
$$\beta_2 = c_2\delta^{-4} + d_2\delta^{-6} + e_2\delta^{-8}, \qquad \beta_3 = d_3\delta^{-6} + e_3\delta^{-8}, \qquad \beta_4 = e_4\delta^{-8}, \tag{67}$$

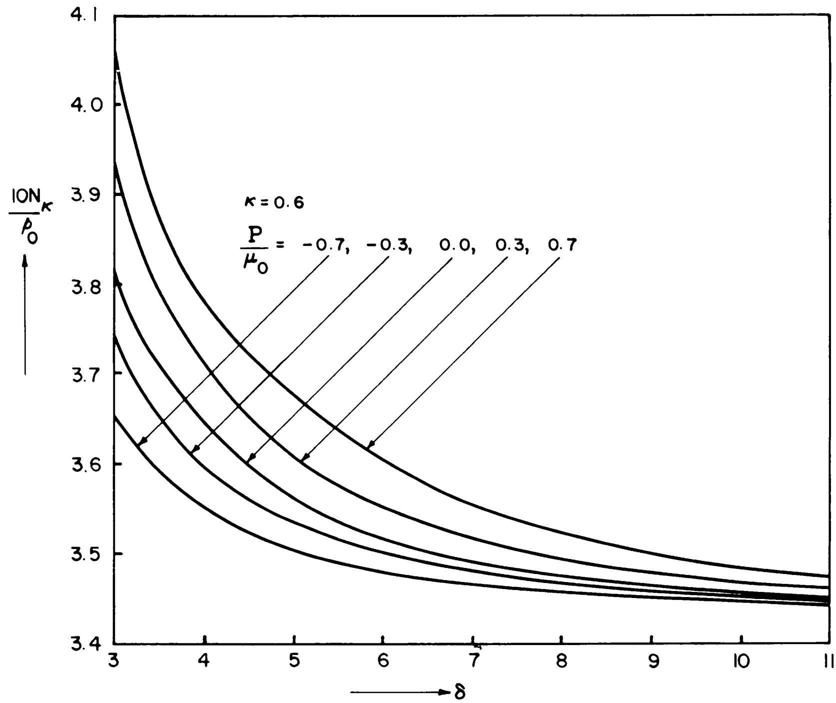

FIG. 3. Values of 10 N_κ/p_0 against half the strip width δ for various values of P/μ_0 and $\kappa = 0.6$.

with

$$b_0 = 4(s_1 - s_0 E/F)/\pi, \qquad b_1 = 16(s_2 - s_1 E/F)/\pi,$$

$$b_2 = 32(s_3 - s_2 E/F)/\pi, \qquad b_3 = 64(s_4 - s_3 E/F)/\pi,$$

$$b_4 = 32(c_0 s_1 + c_1 s_2 + c_2 s_3)/\pi, \qquad b_5 = 16(c_0 s_0 + c_1 s_1 + c_2 s_2)/\pi,$$

$$c_0 = -2I_0{}^2 b_0{}^2 E/F + I_1 b_1 E/F - 6I_1 b_0 \alpha_1 ,$$

$$c_1 = 2I_0{}^2 b_0{}^2 - I_1 b_1 - 6I_1 b_0 \alpha_0 , \qquad c_2 = -12 I_1 b_0 ,$$

$$d_0 = \frac{E}{2F}(I_2 b_2 - I_0 I_1 b_1{}^2) - 5I_2 b_1 \alpha_1 - \tfrac{5}{2} I_2 b_0 \alpha_2 + \tfrac{1}{3} I_0 I_1 b_0 b_1 \alpha_1 + \frac{1}{8}\frac{E}{F} b_5 ,$$

$$d_1 = -\tfrac{1}{2} I_2(b_2 + 10 b_1 \alpha_0 - 35\kappa'^4 b_0) + \tfrac{1}{2} I_0 I_1 b_1(b_1 + 6 b_0 \alpha_0) - \tfrac{1}{8} I_0 b_5 ,$$

$$d_2 = -10 I_2(b_1 + b_0 \alpha_0) + 6 I_0 I_1 b_0 b_1 , \qquad d_3 = -20 I_2 b_0 ,$$

$$e_0 = \tfrac{1}{8} I_3(2b_3 E/F - 42 b_2 \alpha_1 - 35 b_1 \alpha_2 - 14 b_3 \alpha_3) - \tfrac{1}{4} I_0 I_2 b_1 (b_2 E/F$$

$$- 10 b_1 \alpha_1 - 5 b_0 \alpha_2) + \frac{2I_0 E}{\pi F}(d_0 s_0 + d_1 s_1 + d_2 s_2 + d_3 s_3)$$

$$+ \tfrac{1}{16} I_1(b_4 E/F - 3 b_5 \alpha_1),$$

$$e_1 = -\frac{1}{8\pi} I_3(2b_3 + 42 b_2 \alpha_0 - 245\kappa'^4 b_1 + 14\kappa'^4 b_0 \alpha_0)$$

$$+ \tfrac{1}{4} I_0 I_2 b_1(b_2 + 10b_1\alpha_0 - 35\kappa'^4 b_0) - \frac{2}{\pi} I_0(d_0 s_0 + d_1 s_1 + d_2 s_2 + d_3 s_3)$$

$$- \tfrac{1}{16} I_1(b_4 + 3b_5\alpha_0),$$

$$e_2 = -\tfrac{7}{2} I_3(3b_2 + 5b_1\alpha_0 - \kappa'^4 b_0) + 5I_0 I_2 b_1(b_1 + b_0\alpha_0) - \tfrac{3}{8} I_1 b_5\,,$$

$$e_3 = -\frac{7}{\pi} I_3(5b_1 + 2b_0\alpha_0) + 10 I_0 I_2 b_0 b_1\,, \qquad e_4 = -28 I_3 b_0\,, \tag{68}$$

and

$$s_n = \int_\kappa^1 \frac{x^{2n+1}}{X}\,dx, \qquad n = 0, 1, 2, \cdots, \tag{69}$$

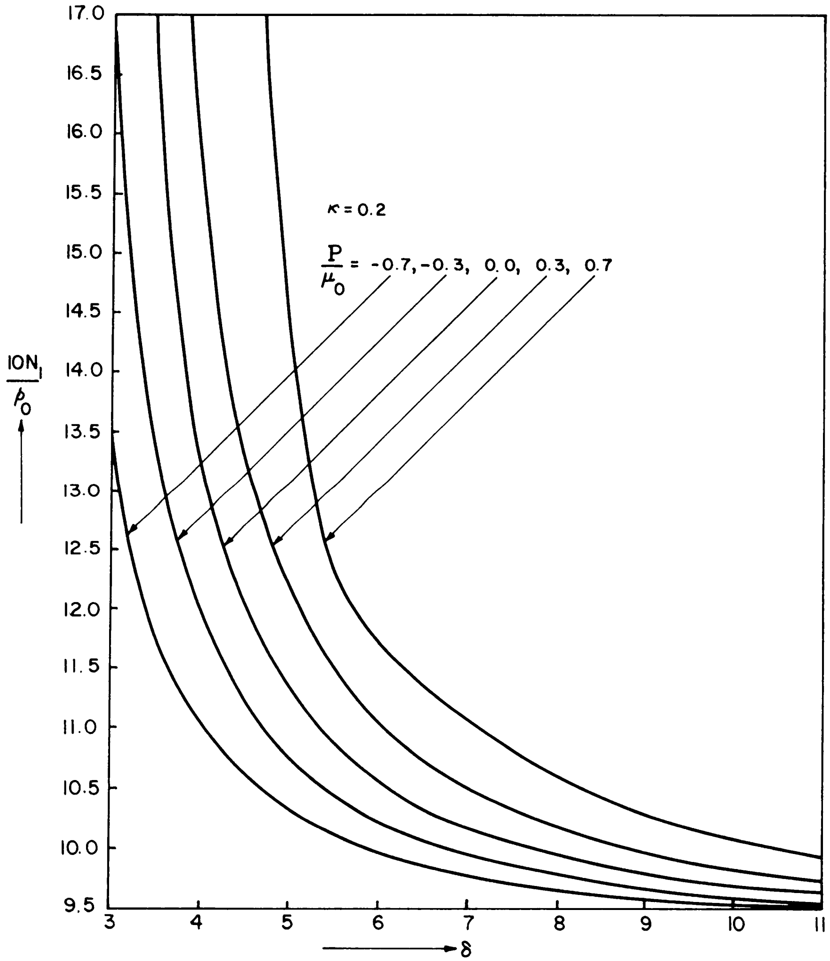

FIG. 4. Values of 10 N_1/p_0 against half the strip width δ for various values of P/μ_0 and $\kappa = 0.2$.

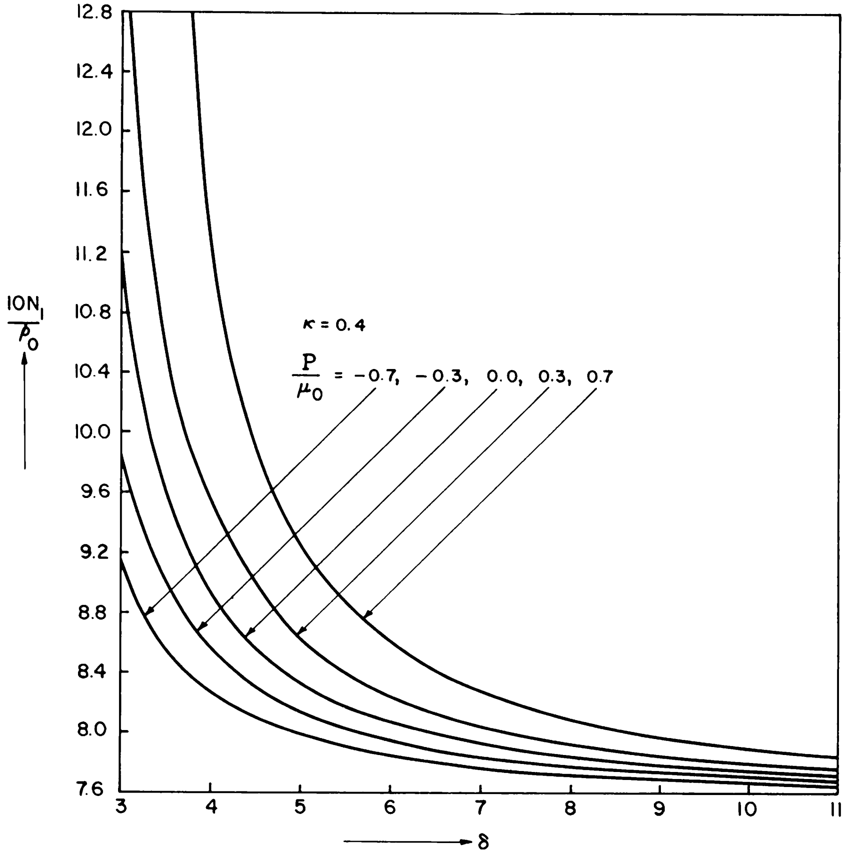

FIG. 5. Values of 10 N_1/p_0 against half the strip width δ for various values of P/μ_0 and $\kappa = 0.4$.

so that

$$s_0 = \pi/2, \qquad s_1 = \pi(1 + \kappa^2)/4, \qquad s_2 = \pi(3 + 2\kappa^2 + 3k^4)/16,$$

$$s_3 = \pi(5 + 3\kappa^2 + 3\kappa^4 + 5\kappa^6)/32, \qquad s_4 = \pi(35 + 20\kappa^2 + 18\kappa^4 + 20\kappa^6 + 35\kappa^8)/256 \tag{70}$$

6. The stress intensity factors and the crack energy. The stress intensity factors N_κ, N_1 at the ends of the crack, which are of interest to people working in fracture mechanics, are defined by

$$N_\kappa = \lim_{x \to \kappa} \{(\kappa - x)^{1/2}[s_{yy}(x, 0)]\} \tag{71}$$

and

$$N_1 = \lim_{x \to 1} \{(x - 1)^{1/2}[s_{yy}(x, 0)]\}. \tag{72}$$

From (27) or (37) and (43), we find that

$$s_{yy}(x, 0) = -\frac{1}{\gamma} \int_\kappa^1 \frac{th(t^2)}{t^2 - x^2}\, dt - \frac{1}{\gamma} \int_\kappa^1 h(t^2) K_2(x, t)\, dt. \tag{73}$$

It is easily seen that the second integral in (73) does not contribute to the singular part of $s_{yy}(x, 0)$ and for the first integral we find

$$\int_{\kappa}^{1} \frac{th(t^2)}{t^2 - x^2}\, dt = \frac{p_0\gamma}{8} \begin{cases} R(x^2)/X_1 + N(x^2) + O(\delta^{-10}), & 0 < x < \kappa, \\ -R(x^2)/X_2 + N(x^2) + O(\delta^{-10}), & x > 1, \end{cases} \tag{74}$$

where

$$R(x^2) = \sum_{n=0}^{4} \beta_n x^{2n},$$

$$N(x^2) = \left[\frac{2}{\pi} s_2 \sum_{n=1}^{4} \beta_n x^{2n-2} + s_1 \sum_{n=2}^{4} \beta_n x^{2n-4} + s_2(\beta_3 + \beta_4 x^2) + s_3\beta_4\right], \tag{75}$$

$$X_1 = \{(\kappa^2 - x^2)(1 - x^2)\}^{1/2}, \qquad X_2 = \{(x^2 - \kappa^2)(x^2 - 1)\}^{1/2}.$$

Hence the stress intensity factors N_κ and N_1 as estimated from (71) and (72) are given by

$$N_\kappa = \frac{-p_0}{8[2\kappa(1 - \kappa^2)]^{1/2}} \sum_{n=0}^{4} \beta_n \kappa^{2n} + O(\delta^{-10}), \tag{76}$$

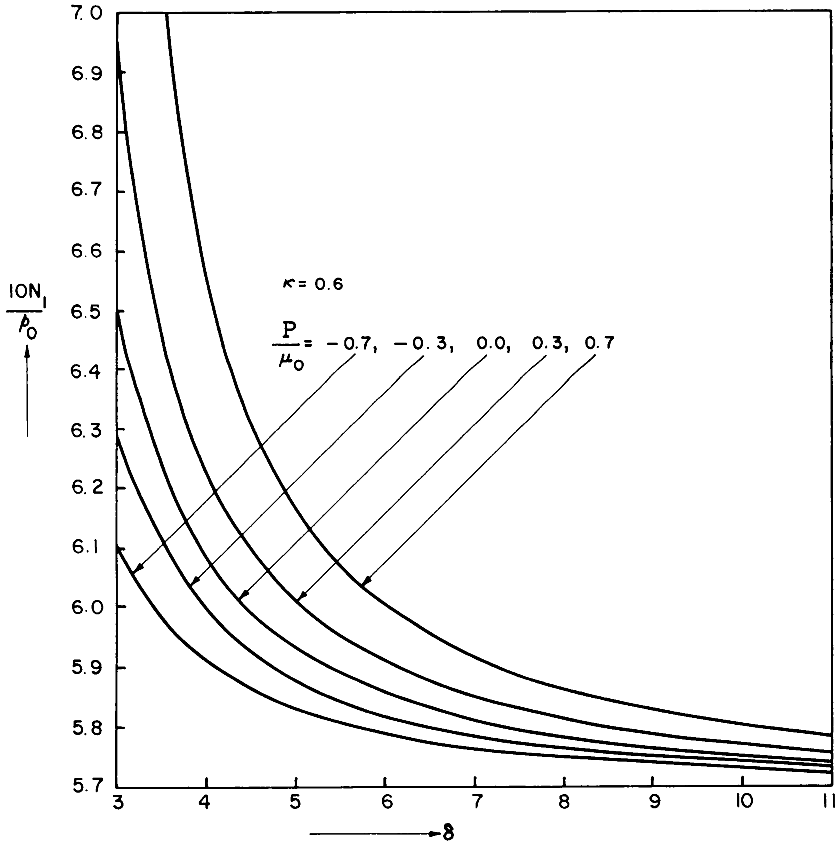

FIG. 6. Values of 10 N_1/p_0 against half the strip width δ for various values of P/μ_0 and $\kappa = 0.6$.

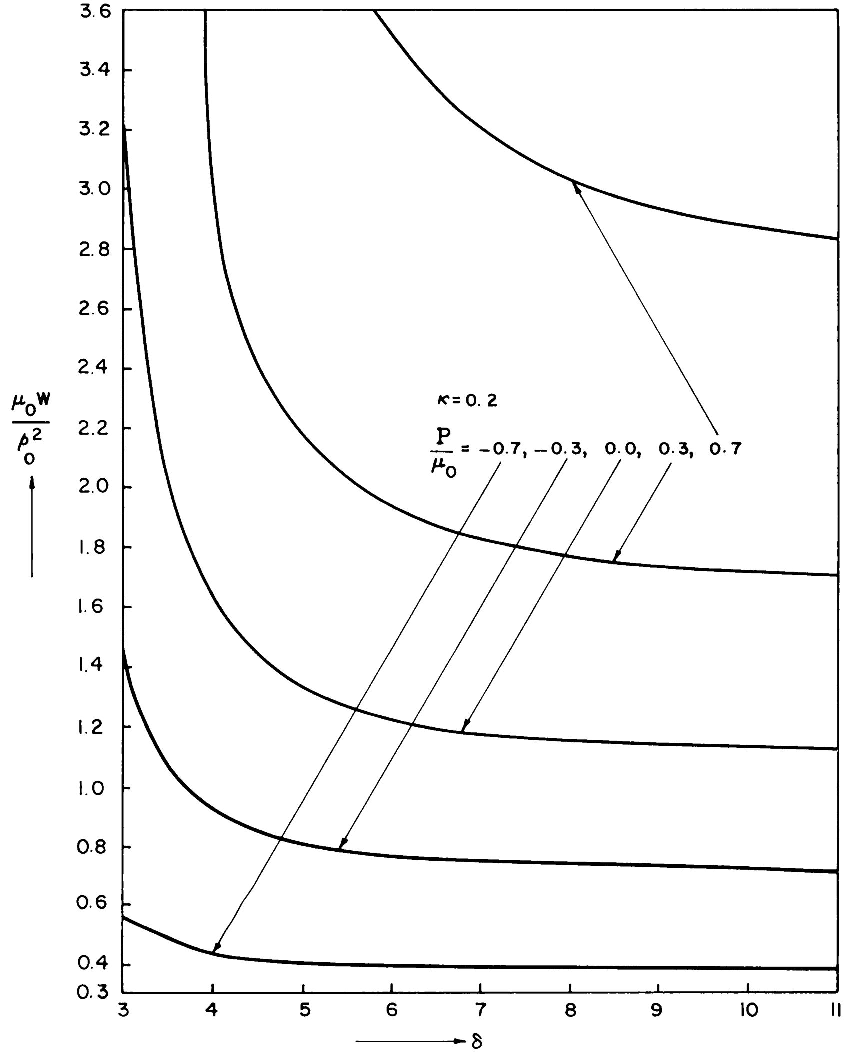

FIG. 7. Values of $\mu_0 W/p_0^2$ against half the strip width δ for various values of P/μ_0 and $\kappa = 0.2$.

$$N_1 = \frac{p_0}{8[2(1-\kappa^2)]^{1/2}} \sum_{n=0}^{4} \beta_n + O(\delta^{-10}). \tag{77}$$

Substituting for $G(\xi)$ from (43) in (26) or (36) and interchanging the orders of integration, we obtain:

$$u_y(x, 0) = \frac{\pi}{2} \int_x^1 h(t^2)\, dt, \qquad \kappa < x < 1. \tag{78}$$

The energy to open the crack is defined by

$$W = \int_\kappa^1 p(x) u_y(x, 0)\, dx = \frac{\pi}{2} p_0 \int_\kappa^1 t h(t^2)\, dt, \tag{79}$$

since

$$p(x) = p_0, \qquad \int_\kappa^1 h(t^2)\,dt = 0.$$

Substituting for $H(t^2)$ from (66) in (79), we obtain

$$W = \gamma p_0^2 \sum_{n=0}^{4} \beta_n s_n + O(\delta^{-10}).$$

The numerical values of the stress intensity factors and the crack energy are graphed against $\delta = 3, 4, 5, 6, 7, 8, 9, 10, 11$ for $P/\mu_0 = -0.7, -0.3, 0, 0.3, 0.7$ and $\kappa = 0.2, 0.4, 0.6$ in Figs. 1–9, for case A when the edges of the strip are fixed. From Eq. (7) and the relation $\lambda_x = 1/\sqrt{\lambda_y}$, we find that λ_y is given by the equation

$$\lambda_y^3 - \frac{P}{\mu_0}\lambda_y - 1 = 0, \tag{80}$$

for various values of P/μ_0 and then k is given by $k = (\lambda_y)^{3/2}$.

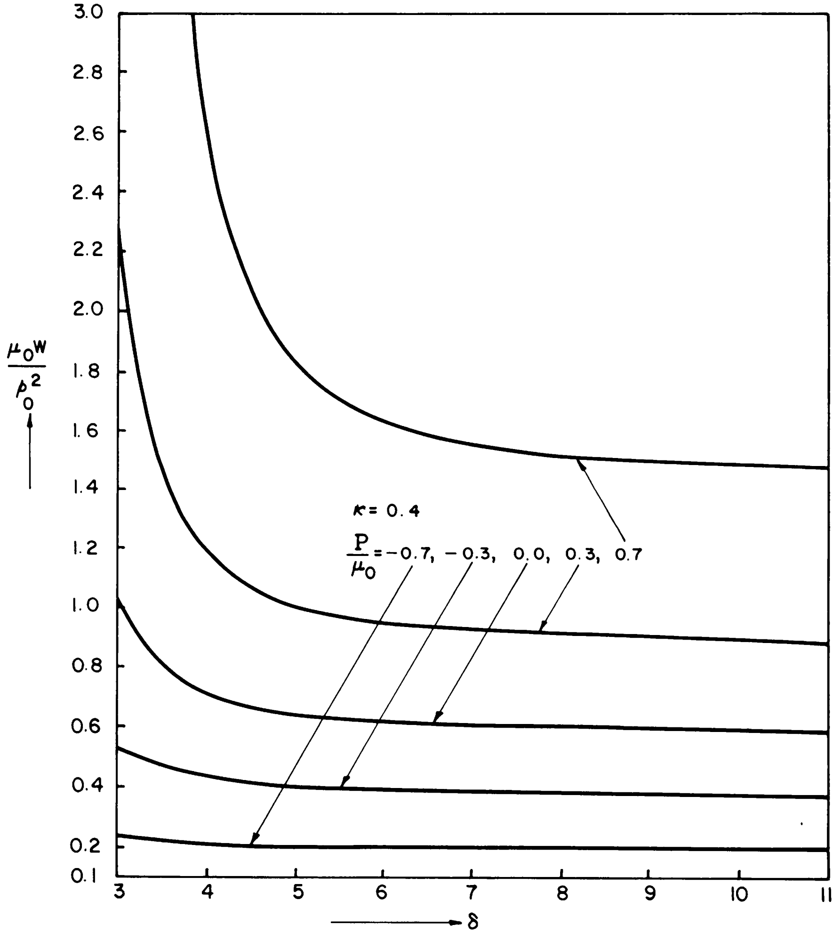

FIG. 8. Values of $\mu_0 W/p_0^2$ against half the strip width δ for various values of P/μ_0 and $\kappa = 0.4$.

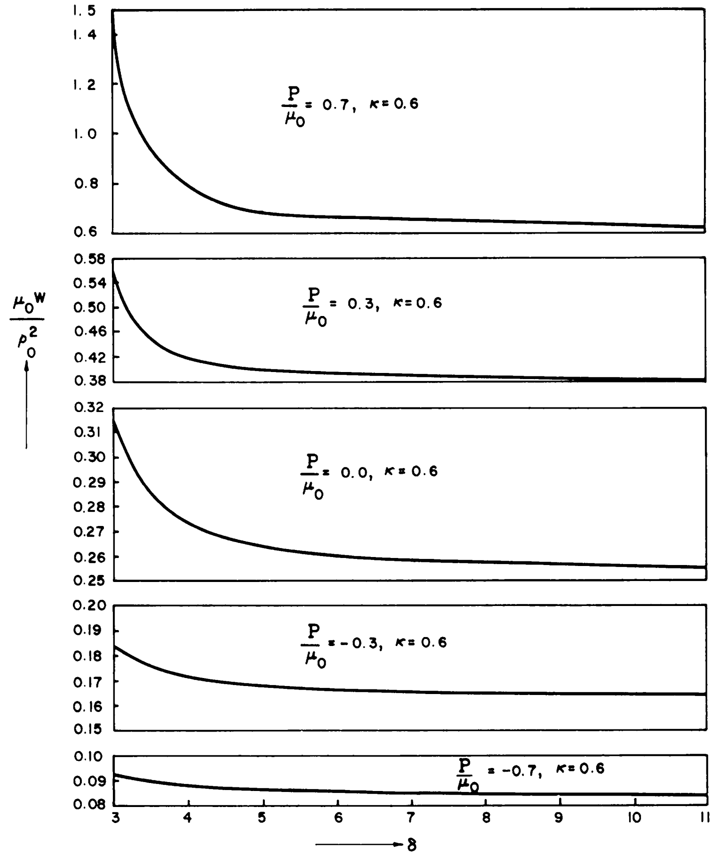

FIG. 9. Values of $\mu_0 W/p_0^2$ against half the strip width δ for various values of P/μ_0 and κ = 0.6.

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