QUARTERLY OF APPLIED MATHEMATICS
JANUARY 1979

—NOTES—

NONLINEAR STABILITY OF SURFACE WAVES IN ELECTROHYDRODYNAMICS*

BY BHIMSEN K. SHIVAMOGGI (*University of Colorado, Boulder*)

Introduction. Early studies of electrostatic effects on the motion of fluids were made by Rayleigh [2] who considered the effect of surface charges on the vibration of spherical drops. Michael [1] considered the effects of electrostatic forces on the stability of wave motion at the surface of highly conducting fluids, and developed an analysis using the Poincare-Lighthill-Kuo method. Michael's results for the linear problem showed that the electrostatic forces can have destabilizing effects on the wave motion at the surface. However, Michael's nonlinear analysis remains valid only for wavenumbers away from the linear cut-off value, and breaks down for wavenumbers near the latter. The purpose of this paper is to treat the latter case. The analysis is restricted to long waves for analytical convenience. The results show that the electrostatic forces continue to have a destabilizing effect on the wave motion at the surface in the nonlinear case.

The boundary-value problem. Consider wave motions at the surface of an incompressible, inviscid, and conducting fluid of infinite depth with a conducting plate maintained at a potential V_0' at a distance above the surface (see Fig. 1). Initially, the surface is taken to be disturbed according to a simple sinusoidal standing wave with an amplitude a' and a wavelength λ'. Nondimensionalize the various physical quantities with respect to a reference length $\lambda'/2\pi$, a time $(\lambda'/2\pi g')^{1/2}$, and an electrostatic potential V_0' (the primes denote the dimensional quantities), and make a change of variable

$$\tau = \sigma t. \tag{1}$$

Then one has the following boundary-value problem,

$$z > \eta: \quad \nabla^2\phi = 0, \tag{2}$$

$$z < \eta: \quad \nabla^2\Omega = 0, \tag{3}$$

$$z = \eta: \quad \Omega_z - \Omega_x\eta_x - \sigma\eta_\tau = 0, \tag{4}$$

$$\phi = 0, \tag{5}$$

$$\beta k^2 \frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}} - \eta + \frac{\alpha k^3}{8\pi}(\phi_x\eta_x - \phi_z)^2 - \tfrac{1}{2}(\Omega_x^2 + \Omega_z^2) - \sigma\eta_\tau = \text{const.}, \tag{6}$$

* Received March 2, 1978. Helpful discussion with Dr. M. S. Uberoi is gratefully acknowledged. The author is thankful to the referee for his valuable criticism.

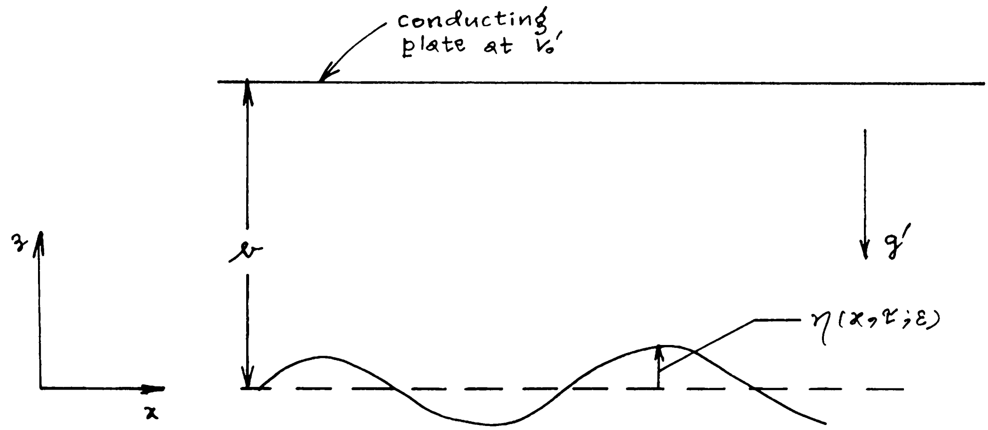

FIG. 1.

$$z = b: \qquad \phi = 1, \tag{7}$$

$$z \Rightarrow -\infty: \qquad \Omega \Rightarrow 0, \tag{8}$$

$$\tau = 0: \qquad \eta = \epsilon \cos x, \tag{9}$$

where ϕ denotes the electrostatic potential above the fluid, Ω the velocity potential inside the fluid, the disturbed shape of the free surface is given by $z = \eta(x, \tau; \epsilon)$, and

$$\epsilon = a' (2\pi/\lambda'), \qquad k = 2\pi/\lambda',$$

$$\alpha = V_0'^2/\rho' g', \qquad \beta = T'/\rho' g';$$

T' denotes the surface tension of the fluid, g' the acceleration due to gravity, and ρ' the mass density of the fluid.

Nonlinear analysis for wavenumbers near the linear cut-off value. Seek solutions to (2)–(9) of the form, for wavenumbers near the linear cut-off value k_c,

$$\phi(x, z, \tau; \epsilon) = \sum_{n=0}^{\infty} \epsilon^n \phi_n(x, z, \tau) \tag{10}$$

$$\Omega(x, z, \tau; \epsilon) = \sum_{n=1}^{\infty} \epsilon^n \Omega_n (x, z, \tau) \tag{11}$$

$$\eta(x, \tau; \epsilon) = \sum_{n=1}^{\infty} \epsilon^n \eta_n (x, \tau) \tag{12}$$

$$\sigma(k; \epsilon) = \sum_{n=1}^{\infty} \epsilon^{n-1} \sigma_n (k) \tag{13}$$

$$k(\epsilon) = k_c + \epsilon^2 \kappa + O(\epsilon^3) \tag{14}$$

where $\phi_0 = z/b$.

In (14), one could include an $O(\epsilon)$ term on the right-hand side, but it turns out to be zero anyway.

One obtains upon substitution of (10)–(14) into (2)–(9):

$O(\epsilon)$:

$$z > 0: \qquad \phi_{1xx} + \phi_{1zz} = 0, \tag{15}$$

$$z < 0: \qquad \Omega_{1xx} + \Omega_{1zz} = 0, \tag{16}$$

$$z = 0: \qquad \Omega_{1z} - \sigma_1 \eta_{1\tau} = 0, \tag{17}$$

$$\phi_1 = - \eta_1 \phi_{0z}, \tag{18}$$

$$\beta k_c^2 \eta_{1xx} - \eta_1 + \frac{\alpha k_c^3}{4\pi} \phi_{0z} \phi_{1z} - \sigma_1 \Omega_{1\tau} = 0, \tag{19}$$

$$z = b: \qquad \phi_1 = 0, \tag{20}$$

$$z \Rightarrow - \infty: \qquad \Omega_1 \Rightarrow 0, \tag{21}$$

$$\tau = 0: \qquad \eta_1 = \cos x; \tag{22}$$

$O(\epsilon^2)$:

$$z > 0: \qquad \phi_{2xx} + \phi_{2zz} = 0, \tag{23}$$

$$z < 0: \qquad \Omega_{2xx} + \Omega_{2zz} = 0, \tag{24}$$

$$z = 0: \qquad \Omega_{2z} - \sigma_1 \eta_{2\tau} = \eta_{1x} \Omega_{1x} - \eta_1 \Omega_{1zz} + \sigma_2 \eta_{1\tau}, \tag{25}$$

$$\phi_2 = - \eta_1 \phi_{1z} - \eta_2 \phi_{0z}, \tag{26}$$

$$\begin{aligned} \beta k_c^2 \eta_{2xx} - \eta_2 + \frac{\alpha k_c^3}{4\pi} \phi_{0z} \phi_{2z} - \sigma_1 \Omega_{2\tau} = {} & \frac{1}{2} (\Omega_{1x}^2 + \Omega_{1z}^2) \\ & - \frac{\alpha k_c^3}{8\pi} \{\phi_{1z}{}^2 + 2\phi_{0z} (\phi_{2z} + \eta_1 \phi_{1zz} + \\ & - \eta_{1x} \phi_{1x})\} + \sigma_2 \Omega_{1\tau} + \sigma_1 \eta_1 \Omega_{1\tau z}, \end{aligned} \tag{27}$$

$$z = b: \qquad \phi_2 = 0, \tag{28}$$

$$z \Rightarrow - \infty: \qquad \Omega_2 \Rightarrow 0, \tag{29}$$

$$\tau = 0: \qquad \eta_2 = 0, \tag{30}$$

$O(\epsilon^3)$:

$$z > 0: \qquad \phi_{3xx} + \phi_{3zz} = 0 \tag{31}$$

$$z < 0: \qquad \Omega_{3xx} + \Omega_{3zz} = 0 \tag{32}$$

$$\begin{aligned} z = 0: \qquad \Omega_{3z} - \sigma_1 \eta_{3\tau} = {} & -(\Omega_{1zz} \eta_2 + \Omega_{2zz} \eta_1) - \tfrac{1}{2} \Omega_{1zz} \eta_1^2 + \Omega_{2x} \eta_{1x} \\ & + \Omega_{1xz} \eta_1 \eta_{1x} + \Omega_{1x} \eta_{2x} + \sigma_2 \eta_{2\tau} - \sigma_3 \eta_{1\tau}, \end{aligned} \tag{33}$$

$$\phi_3 = -\eta_1 \phi_{2z} - \eta_2 \phi_{1z} - \eta_3 \phi_{0z} - \tfrac{1}{2} \eta_1^2 \phi_{1zz}, \tag{34}$$

$$\beta k_c^2 \eta_{3xx} - \eta_3 + \frac{\alpha k c^3}{4\pi} \phi_{0z} \phi_{3z} - \sigma_1 \Omega_{3\tau}$$

$$= (\Omega_{1x}\Omega_{2x} + \Omega_{1x}\Omega_{1xz}\eta_1 + \Omega_{1z}\Omega_{2z} + \Omega_{1z}\Omega_{1zz}\eta_1) + \sigma_2(\Omega_{2\tau} + \Omega_{1\tau z}\eta_1) + \sigma_3\Omega_{1\tau}$$

$$+ \sigma_1(\Omega_{2\tau z}\eta_1 + \Omega_{1\tau z}\eta_2 + \tfrac{1}{2}\Omega_{1\tau zz}\eta_1^2) + \frac{3}{2}\beta k_c^2\eta_{1xx}\eta_{1x}^2 - 2\beta k_c\kappa\eta_{1xx}$$

$$- \frac{3\alpha k_c^2}{4\pi}\kappa\phi_{0z}\phi_{1z} - \frac{\alpha k_c{}^3}{8\pi}\Big\{2\phi_{1z}(\phi_{2z} + \eta_1\phi_{1zz} - \eta_{1x}\phi_{1x}) - \phi_{1z}^3$$

$$+ 3\phi_{0z}\left(\eta_{2x}\phi_{1x} + \eta_{1x}\phi_{2x} + \eta_{1x}\eta_1\phi_{1xz} - \phi_{3z} - \eta_1\phi_{2zz} - \frac{1}{2}\eta_1^2\phi_{1zzz}\right)\Big\}, \quad (35)$$

$$z = b: \qquad \phi_3 = 0, \quad (36)$$

$$z \Rightarrow -\infty: \qquad \Omega_3 \Rightarrow 0, \quad (37)$$

$$\tau = 0: \qquad \eta_3 = 0. \quad (38)$$

From (15)–(22), one obtains

$$\eta_1 = e^\tau \cos x, \quad (39)$$

$$\phi_1 = \frac{e^\tau}{b}\cos x \cdot \frac{\sinh(z-b)}{\sinh b}, \quad (40)$$

$$\Omega_1 = \sigma e^\tau e^z \cos x, \quad (41)$$

$$\sigma_1{}^2 = \frac{\alpha k_c{}^3}{4\pi b^2}\coth b - \beta k_c^2 - 1 = 0. \quad (42)$$

The destabilizing effect of the electrostatic forces on the wave motion at the free surface is obvious.

For long waves, i.e. $k \ll 1$ (one then requires $\alpha/(4\pi b'^3) \approx 1$; see below), the linear cut-off wavenumber is given by

$$k_c \approx [(\alpha/4\pi b'^3) - 1]^{-1/2}. \quad (43)$$

One may expect to construct uniformly valid solutions only for wavenumbers larger than k_c. But it turns out upon a consideration of the nonlinear problem in the following that this is possible only for wavenumbers greater than k_c by a definite amount.

For wavenumbers near k_c (i.e. for $\sigma_1 = 0$), one obtains from (23)–(30),

$$\eta_2 = (e^\tau - 1)\cos x, \quad (44)$$

$$\Omega_2 = \sigma_2 e^\tau e^z \cos x, \quad (45)$$

$$\phi_2 = \frac{e^\tau}{b}\cos x \cdot \frac{\sinh(z-b)}{\sinh b} + \text{higher harmonics}. \quad (46)$$

Using (39)–(41), (44)–(46), in (31)–(38), the removal of the secular terms in (35), in particular, requires

$$\sigma_2{}^2 - \frac{3}{8}\beta k_c^2 + 2\beta k_c\kappa - \kappa\cdot\frac{3\alpha}{4\pi b'^2}\coth b + O(k_c^3) = 0, \quad (47)$$

from which

$$\sigma_2 \approx \left[\frac{3}{8}\beta k_c^2 + \frac{3\alpha}{4\pi b'^3 k_c}\kappa\right]^{1/2}, \quad (48)$$

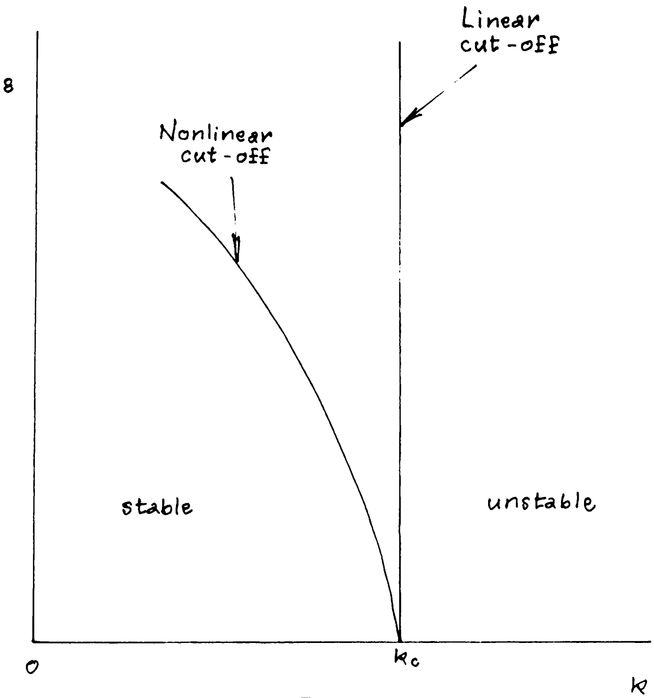

FIG. 2.

so that one has

$$\kappa > -\frac{\pi}{2}\frac{\beta}{\alpha}b^3: \quad \text{instability},$$

$$\kappa = -\frac{\pi}{2}\frac{\beta}{\alpha}b^3: \quad \text{neutral stability},$$

$$\kappa \leq -\frac{\pi}{2}\frac{\beta}{\alpha}b^3: \quad \text{stability},$$

and corresponding to neutral stability, one has

$$k = kc - \left(\frac{\pi}{2}\frac{\beta}{\alpha}b^3\right)\epsilon^2 + O(\epsilon^3), \tag{49}$$

which is graphically represented in Fig. 2.

It thus appears that:

(1) the waves at the free surface grow even at $k = k_c$, despite the cut-off predicted by the linear theory;

(2) the electrostatic forces continue to have a destabilizing effect on the wave motion at the free surface in the nonlinear case.

REFERENCES

[1] D. H. Michael, *Nonlinear effects in electrohydrodynamic surface wave propagation*, Quart. Appl. Math. **35**, 345 (1977)

[2] Lord Rayleigh, *The theory of sound*, The Macmillan Company, 1894