PLANE COMPRESSIBLE MHD FLOWS*

By

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1. Introduction. Martin [1] developed a new approach for plane viscous flows of incompressible fluids. Nath and Chandna [2] and Chandna and Garg [3] extended Martin's approach to incompressible magnetohydrodynamic fluids.

In the present paper we follow Martin's work and study the steady plane flows of an inviscid compressible fluid of infinite electrical conductivity when the magnetic field vector lies in the flow plane and makes a non-zero constant angle with the velocity vector. We introduce curvilinear coordinates ϕ, ψ in the physical plane in which the coordinates lines ψ = constant are the streamlines and the lines ϕ = constant are magnetic lines.

The plan of the paper is as follows. In Sec. 2 we start with the basic equations of flow and employ results from differential geometry to recast these equations in (ϕ, ψ) coordinates. The following sections are devoted to applications of the new form of equations and we establish the following results:

i) If the velocity magnitude is constant on each individual streamline, then the streamlines must be concentric circles or parallel straight lines.

ii) If the magnetic field is irrotational, then the velocity magnitude is constant on each individual streamline.

iii) If the flows are irrotational, orthogonal and the fluid obeys the product equation of state, then the flows are homentropic radial or parallel flows.

iv) If the streamlines are straight lines, then vorticity is identically zero.

Finally, solutions of parallel constantly inclined flows and orthogonal circular flows are obtained.

2. Flow equations. The steady, plane adiabatic flow of an inviscid, compressible fluid of infinite electrical conductivity is governed by the following system of equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \tag{1}$$

$$\frac{\partial p}{\partial x} + \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\mu H_2\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right), \tag{2}$$

$$\frac{\partial p}{\partial y} + \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \mu H_1\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right) \tag{3}$$

$$uH_2 - vH_1 = K, \tag{4}$$

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$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0, \tag{5}$$

$$u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = 0, \tag{6}$$

together with an appropriate equation of state $p = p(\rho, s)$. This is a system of seven equations wherein (u, v) are the velocity components, H_1, H_2 the components of the coplanar magnetic field vector, p the pressure function, ρ the density function, s the specific entropy, μ the constant magnetic permeability and K an arbitrary constant which is zero for aligned flows and nonzero in the case of non-aligned flows.

On introducing the functions

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \qquad j = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \tag{7, 8}$$

the system of equations (1)–(6) is replaced by the following system:

$$\begin{aligned}
&\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, && \text{(continuity)}\\
&\left.\begin{aligned}
&\frac{\partial p}{\partial x} + \frac{1}{2}\rho\frac{\partial q^2}{\partial x} - \rho v\omega = -\mu j H_2,\\
&\frac{\partial p}{\partial y} + \frac{1}{2}\rho\frac{\partial q^2}{\partial y} + \rho u\omega = \mu j H_1,
\end{aligned}\right\} && \text{(linear momentum)}\\
&uH_2 - vH_1 = K, && \text{(diffusion)}\\
&\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0, && \text{(solenoidal condition)}\\
&\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega, && \text{(vorticity)}\\
&\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} = j, && \text{(current density)}\\
&u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = 0, && \text{(adiabatic)}
\end{aligned} \tag{9}$$

wherein $q^2 = u^2 + v^2$. Equations of continuity and solenoidal condition imply the existence of a streamfunction $\psi(x, y)$ and a magnetic flux function $\phi(x, y)$ such that

$$\begin{aligned}
\partial\psi/\partial x &= -\rho v, & \partial\psi/\partial y &= \rho u\\
\partial\phi/\partial x &= H_2, & \partial\phi/\partial y &= -H_1.
\end{aligned} \tag{10}$$

We shall now study the flows in which the magnetic field vector $\mathbf{H} = (H_1, H_2)$ and the velocity field vector $\mathbf{V} = (u, v)$ are everywhere non-aligned to each other in the flow region. Using (10) in the diffusion equation, we get

$$\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x} = \frac{\partial(\phi, \psi)}{\partial(x, y)} = \rho K \neq 0 \tag{11}$$

for our flows.

Let

$$x = x(\phi, \psi), \qquad y = y(\phi, \psi), \tag{12}$$

define the curvilinear net with the squared element of arc length along any curve given by

$$ds^2 = E(\phi, \psi)\, d\phi^2 + 2F(\phi, \psi)\, d\phi\, d\psi + G(\phi, \psi)\, d\psi^2 \tag{13}$$

where

$$\begin{aligned} E &= (\partial x/\partial\phi)^2 + (\partial y/\partial\phi)^2, \\ F &= \frac{\partial x}{\partial\phi}\frac{\partial x}{\partial\psi} + \frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\psi}, \\ G &= (\partial x/\partial\psi)^2 + (\partial y/\partial\psi)^2. \end{aligned} \tag{14}$$

Eqs. (12) can be solved to determine ϕ, ψ as functions of x, y so that

$$\frac{\partial x}{\partial\phi} = J\frac{\partial\psi}{\partial y}, \qquad \frac{\partial x}{\partial\psi} = -J\frac{\partial\phi}{\partial y}, \qquad \frac{\partial y}{\partial\phi} = -J\frac{\partial\psi}{\partial x}, \qquad \frac{\partial y}{\partial\psi} = J\frac{\partial\phi}{\partial x} \tag{15}$$

where $0 < |J| < \infty$, and by (14),

$$J = \partial(x, y)/\partial(\phi, \psi) = \pm(EG - F^2)^{1/2} = \pm W \quad \text{(say)} \tag{16}$$

is the transformation Jacobian.

Denoting by α the local angle of inclination of the tangent to the coordinate line ψ = constant, directed in the sense of increasing ϕ, we have from differential geometry the following (cf. Martin [1]):

$$\partial x/\partial\phi = \sqrt{E}\cos\alpha, \qquad \partial y/\partial\phi = \sqrt{E}\sin\alpha, \tag{17}$$

$$\partial x/\partial\psi = \frac{F}{\sqrt{E}}\cos\alpha - \frac{J}{\sqrt{E}}\sin\alpha, \qquad \partial y/\partial\psi = \frac{F}{\sqrt{E}}\sin\alpha + \frac{J}{\sqrt{E}}\cos\alpha, \tag{18}$$

$$\partial\alpha/\partial\phi = \frac{J}{E}\Gamma_{11}^{2}, \qquad \partial\alpha/\partial\psi = \frac{J}{E}\Gamma_{12}^{2}, \tag{19}$$

$$\bar{K} = \frac{1}{W}\left[\frac{\partial}{\partial\psi}\left(\frac{W}{E}\Gamma_{11}^{2}\right) - \frac{\partial}{\partial\phi}\left(\frac{W}{E}\Gamma_{12}^{2}\right)\right] = 0, \tag{20}$$

$$\begin{aligned} \frac{\partial}{\partial\phi}\left(\frac{E}{2W^2}\right) &= \frac{1}{W^2}\{F\Gamma_{11}^{2} - E\Gamma_{12}^{2}\}, \\ \frac{\partial}{\partial\psi}\left(\frac{E}{2W^2}\right) &= \frac{1}{W^2}\{F\Gamma_{12}^{2} - E\Gamma_{22}^{2}\}, \end{aligned} \tag{21}$$

$$\frac{\partial}{\partial\phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial\psi}\left(\frac{E}{W}\right) = \frac{1}{W}\{G\Gamma_{11}^{2} - 2F\Gamma_{12}^{2} + E\Gamma_{22}^{2}\}, \tag{22}$$

where

$$\Gamma_{11}^{2} = \frac{1}{2W^2}\left[-F\left(\frac{\partial E}{\partial\phi}\right) + 2E\left(\frac{\partial F}{\partial\phi}\right) - E\left(\frac{\partial E}{\partial\psi}\right)\right],$$

$$\Gamma_{12}^{2} = \frac{1}{2W^2}\left[E\left(\frac{\partial G}{\partial\phi}\right) - F\left(\frac{\partial E}{\partial\psi}\right)\right]$$

and

$$\Gamma_{22}^{2} = \frac{1}{2W^2}\left(E\frac{\partial G}{\partial \psi} - 2F\frac{\partial F}{\partial \psi} + F\frac{\partial G}{\partial \phi}\right) \tag{23}$$

and $\bar{K}$ is the Gaussian curvature. Having recorded the above results, we now take the eight Eqs. (9) and develop these flow equations in a new form in the new variables ϕ, ψ. In the following work, we consider, without any loss of generality, that the fluid flows towards higher parameter values of ϕ so that $J = W > 0$.

Linear momentum equations. On employing (10) in the linear momentum equations, we have

$$\frac{\partial p}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial p}{\partial \psi}\frac{\partial \psi}{\partial x} + \frac{1}{2}\rho\left\{\frac{\partial q^2}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial q^2}{\partial \psi}\frac{\partial \psi}{\partial x}\right\} + \omega\frac{\partial \psi}{\partial x} = -\mu j\frac{\partial \phi}{\partial x},$$

$$\frac{\partial p}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial p}{\partial \psi}\frac{\partial \psi}{\partial y} + \frac{1}{2}\rho\left\{\frac{\partial q^2}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial q^2}{\partial \psi}\frac{\partial \psi}{\partial y}\right\} + \omega\frac{\partial \psi}{\partial y} = -\mu j\frac{\partial \phi}{\partial y}.$$

Making use of the transformation equations (15), we get

$$\frac{\partial p}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial p}{\partial \psi}\frac{\partial y}{\partial \phi} + \frac{1}{2}\rho\left\{\frac{\partial q^2}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial q^2}{\partial \psi}\frac{\partial y}{\partial \phi}\right\} - \omega\frac{\partial y}{\partial \phi} = -\mu j\frac{\partial y}{\partial \psi},$$

$$-\frac{\partial p}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial p}{\partial \psi}\frac{\partial x}{\partial \phi} + \frac{1}{2}\rho\left\{-\frac{\partial q^2}{\partial \phi}\frac{\partial x}{\partial \psi} + \frac{\partial q^2}{\partial \psi}\frac{\partial x}{\partial \phi}\right\} + \omega\frac{\partial x}{\partial \phi} = \mu j\frac{\partial x}{\partial \psi}.$$

Multiplying these two equations by $\partial x/\partial \phi$, $\partial y/\partial \phi$ respectively and adding gives one equation; again, multiplying by $\partial x/\partial \psi$, $\partial y/\partial \psi$ respectively and adding gives the second equation of the following set of a new equivalent form of linear momentum equations. The two linear momentum equations are:

$$\frac{\partial p}{\partial \phi} + \frac{1}{2}\rho\frac{\partial q^2}{\partial \phi} + \mu j = 0, \tag{24}$$

$$\frac{\partial p}{\partial \psi} + \frac{1}{2}\rho\frac{\partial q^2}{\partial \psi} + \omega = 0. \tag{25}$$

Continuity and diffusion equations. Martin [1] has obtained the necessary and sufficient conditions for the flow of a fluid, along the coordinate lines ψ = constant of a curvilinear coordinate system (12) with ds^2 given by (13), to satisfy the principle of conservation of mass as

$$\rho W q = \sqrt{E}, \qquad u + iv = (\sqrt{E}/\rho J)\exp(i\alpha) \tag{26}$$

where $i = \sqrt{-1}$. Nath and Chandna [2] have proven that the solenoidal condition yields

$$WH = \sqrt{G}, \qquad H_1 + iH_2 = \frac{\sqrt{G}}{J}\exp(i\beta) \tag{27}$$

where $H = (H_1^2 + H_2^2)^{1/2}$ and β is the angle between the tangent to the coordinate line ϕ = constant, directed in the sense of increasing ψ, and the x-axis.

Vorticity equation. Employing (10) and making use of the transformation equations (15), we find

$$\omega = \frac{1}{W}\left\{\frac{\partial v}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial v}{\partial \psi}\frac{\partial y}{\partial \phi} + \frac{\partial u}{\partial \phi}\frac{\partial x}{\partial \psi} - \frac{\partial u}{\partial \psi}\frac{\partial x}{\partial \phi}\right\}.$$

By eliminating derivatives of x, y by use of Eqs. (17), (18) and substituting $u = q\cos\alpha$, $v =$

$q \sin \alpha$, the expression for vorticity takes the form

$$\omega = \frac{1}{\sqrt{EW}} \left\{ F\frac{\partial q}{\partial \phi} - E\frac{\partial q}{\partial \psi} + qW\frac{\partial \alpha}{\partial \phi} \right\} . \tag{28}$$

From Eq. (26), we have

$$\frac{\partial q}{\partial \phi} = \frac{1}{2q\rho^2} \frac{\partial}{\partial \phi}\left(\frac{E}{W^2}\right) - \frac{E}{qW^2\rho^3} \frac{\partial \rho}{\partial \phi} \quad , \frac{\partial q}{\partial \psi} = \frac{1}{2q\rho^2} \frac{\partial}{\partial \psi}\left(\frac{E}{W^2}\right) - \frac{E}{qW^2\rho^3} \frac{\partial \rho}{\partial \psi} .$$

Using Eqs. (21) and (22), we get

$$\begin{aligned} \frac{\partial q}{\partial \phi} &= \frac{1}{q\rho^2W^2} [F\Gamma_{11}{}^2 - E\Gamma_{12}{}^2] - \frac{E}{qW^2\rho^3} \frac{\partial \rho}{\partial \phi}, \\ \frac{\partial q}{\partial \psi} &= \frac{1}{q\rho^2W^2} [F\Gamma_{12}{}^2 - E\Gamma_{22}{}^2] - \frac{E}{qW^2\rho^3} \frac{\partial \rho}{\partial \psi} . \end{aligned} \tag{29}$$

By eliminating q, $\partial q/\partial \phi$, $\partial q/\partial \psi$ and $\partial \alpha/\partial \phi$ from Eq. (28) by use of (26), (29) and (19), the expression for vorticity becomes

$$\omega = \frac{1}{\rho W^2} \left[\{G\Gamma_{11}{}^2 - 2F\Gamma_{12}{}^2 + E\Gamma_{22}{}^2\} - \frac{1}{\rho}\left\{F \frac{\partial \rho}{\partial \phi} - E \frac{\partial \rho}{\partial \psi}\right\}\right] .$$

Making use of the identities (23), we get

$$\begin{aligned} \omega &= \frac{1}{\rho W} \left[\frac{\partial}{\partial \phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{W}\right)\right] - \frac{1}{W\rho^2}\left[\frac{F}{W}\frac{\partial \rho}{\partial \phi} - \frac{E}{W}\frac{\partial \rho}{\partial \psi}\right] \\ &= \frac{1}{W}\left[\frac{\partial}{\partial \phi}\left(\frac{F}{\rho W}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{\rho W}\right)\right] . \end{aligned} \tag{30}$$

Current density. Following Nath and Chandna [2], the new form for the current density equation is given by

$$j = \frac{1}{W}\left[\frac{\partial}{\partial \phi}\left(\frac{G}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{F}{W}\right)\right] . \tag{31}$$

Adiabatic condition. Using (10) in the adiabatic condition, we have

$$\frac{1}{\rho}\left(\frac{\partial s}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial s}{\partial \psi}\frac{\partial \psi}{\partial x}\right)\frac{\partial \psi}{\partial y} - \frac{1}{\rho}\left(\frac{\partial s}{\partial \phi}\frac{\partial \phi}{\partial y} + \frac{\partial s}{\partial \psi}\frac{\partial \psi}{\partial y}\right)\frac{\partial \psi}{\partial x} = \frac{1}{\rho}\frac{\partial(\phi, \psi)}{\partial(x, y)}\frac{\partial s}{\partial \phi} = 0$$

which implies, for non-aligned flows, that

$$\frac{\partial s}{\partial \phi} = 0 \qquad \text{or} \qquad s = s(\psi). \tag{32}$$

Summing up the results of this section and using (26) in (24) and (25), we have

THEOREM 1. If the streamlines $\psi(x, y)$ = constant and the magnetic lines $\phi(x, y)$ = constant generate a curvilinear net in the physical plane of compressible MHD non-aligned fluid, then the flow in independent variables ϕ, ψ is governed by the system

$$\frac{\partial p}{\partial \phi} + \frac{1}{2}\rho\frac{\partial}{\partial \phi}\left(\frac{E}{\rho^2W^2}\right) + \mu j = 0, \qquad \text{(linear momentum)}$$

$$\frac{\partial p}{\partial \psi} + \frac{1}{2}\rho\frac{\partial}{\partial \psi}\left(\frac{E}{\rho^2W^2}\right) + \omega = 0,$$

$$\omega = \frac{1}{W}\left[\frac{\partial}{\partial\phi}\left(\frac{F}{\rho W}\right) - \frac{\partial}{\partial\psi}\left(\frac{E}{\rho W}\right)\right], \qquad \text{(vorticity)}$$

$$j = \frac{1}{W}\left[\frac{\partial}{\partial\phi}\left(\frac{G}{W}\right) - \frac{\partial}{\partial\psi}\left(\frac{F}{W}\right)\right], \qquad \text{(current density)}$$

$$\frac{\partial}{\partial\psi}\left[\frac{W}{E}\Gamma_{11}^2\right] - \frac{\partial}{\partial\phi}\left[\frac{W}{E}\Gamma_{12}^2\right] = 0, \qquad \text{(Gauss)}$$

$$\partial s/\partial\phi = 0, \qquad \text{(adiabatic)}$$

$$EG - F^2 = 1/\rho^2K^2, \qquad \text{(diffusion)}$$

$$\rho = \rho(p, s) \qquad \text{(state)} \qquad (33)$$

of eight equations for eight unknowns E, F, G, ω, j, p, ρ and s as functions of ϕ, ψ.

Given a solution of this system, the flow in the physical plane and hodograph plane is given by:

$$z = x + iy = \int \frac{\exp(i\alpha)}{\sqrt{E}}\{E\,d\phi + (F + iJ)\,d\psi\},$$

$$u + iv = \frac{\sqrt{E}}{\rho W}\exp(i\alpha), \qquad H_1 + iH_2 = \frac{\sqrt{G}}{W}\exp(i\beta),$$

where

$$\alpha = \int \frac{W}{E}\{\Gamma_{11}^2\,d\phi + \Gamma_{12}^2\,d\psi\}, \qquad \beta = \int \frac{W}{G}\{\Gamma_{12}^2\,d\phi + \Gamma_{11}^2\,d\psi\}. \qquad (34)$$

3. Constant velocity magnitude on each streamline. In this section, we investigate the possible flow patterns when $q = q(\psi)$. Using the diffusion equation in the continuity equation and employing the assumption that $q = q(\psi)$, we find that

$$\sqrt{E} = A(\psi) \qquad (35)$$

where $A(\psi)$ is an arbitrary function of ψ.

Letting θ = constant $\neq 0$ to be the angle between magnetic lines and streamlines everywhere, we get

$$W = (EG)^{1/2}\sin\theta \qquad (36)$$

and, therefore, from the continuity equation

$$\rho\sqrt{G} = 1/q\sin\theta. \qquad (37)$$

Differentiating (37) with respect to ϕ, we obtain

$$\frac{\partial}{\partial\phi}[\ln\sqrt{G}] = -\frac{1}{\rho}\frac{\partial\rho}{\partial\phi}\,. \qquad (38)$$

Employing the diffusion equation, current density equation, (35) and (36) in the first linear momentum equation, we obtain

$$\frac{\partial p}{\partial\phi} + \frac{\mu}{E\sin^2\theta}\frac{\partial}{\partial\phi}[\ln\sqrt{G}] = 0. \qquad (39)$$

Using $c^2 = (\partial p/\partial \rho)$, the adiabatic condition and (38) in Eq. (39), it follows that

$$(\rho c^2 E - \mu \operatorname{cosec}^2 \theta) \frac{\partial \rho}{\partial \phi} = 0. \tag{40}$$

This equation implies that either $\partial \rho/\partial \phi = 0$ or $\rho c^2 E = \mu \operatorname{cosec}^2 \theta$. However, if the gas is a polytropic gas with the state equation $p = A(s)\rho^\gamma$, then the second possibility takes the form $\rho = (\mu \operatorname{cosec}^2 \theta/\gamma A(s) E)^{1/\gamma}$ and, therefore, $\partial \rho/\partial \phi = 0$. Using $\rho = \rho(\psi)$ in (37), we get

$$\sqrt{G} = B'(\psi) \tag{41}$$

where $B'(\psi)$ is an arbitrary function of ψ related to the arbitrary function $A(\psi)$ through Gauss' equation. Using (35) and (41) in Gauss' equation we get

$$\sqrt{E} = A(\psi) = LB(\psi) + M, \qquad \sqrt{G} = B'(\psi)$$

where L, M are arbitrary constants. Using these restrictions on the $\sqrt{E}$ and $\sqrt{G}$ in (34), we find that

$$\alpha = \alpha_0 - \frac{L}{\sin \theta} \phi - \cot \theta \ln\{LB(\psi) + M\},$$

and, therefore, z is given by

$$\begin{aligned} z &= z_0 + i \frac{\sin \theta}{L} \{LB(\psi) + M\} \exp(i\alpha), \qquad \text{if} \qquad L \neq 0 \\ &= z_0 + \exp(i\alpha_0)\{B(\psi) \exp(i\theta) + M\phi\}, \qquad \text{if} \qquad L = 0 \end{aligned} \tag{42}$$

where z_0 is an arbitrary complex constant and α_0 an arbitrary real constant.

From this result, we conclude that

THEOREM 2. If, for a flow, the velocity magnitude is constant on each individual streamline and the gas is a polytropic gas, then the streamlines ψ = constant are concentric circles for the case $L \neq 0$ and are parallel straight lines for the case $L = 0$.

Furthermore, from (40), we see that for a gas obeying the general equation of state, the streamline pattern will be as stated in this theorem provided $\rho c^2 E \sin^2 \theta \neq \mu$ anywhere in flow region.

4. Flows with irrotational magnetic field. In this section, we study the flow geometry when the magnetic field is irrotational and, therefore, the current density is zero.

Letting θ = constant $\neq 0$ be the angle between the streamlines and magnetic lines everywhere, we have

$$F = (EG)^{1/2} \cos \theta, \tag{43}$$

$$W = \frac{1}{\rho K} = (EG)^{1/2} \sin \theta. \tag{44}$$

Using these results in the current density equation and the assumption that $j = |\text{curl } \mathbf{H}| = 0$, we obtain

$$\rho G = A(\psi) \tag{45}$$

where $A(\psi)$ is an arbitrary function of ψ. Furthermore, Eqs. (44) and (45) give

$$\rho E = \frac{1}{\rho G K^2 \sin^2\theta} = \frac{1}{K^2 \sin^2\theta A(\psi)} \,. \tag{46}$$

Since $j = 0$, the first linear momentum equation, the diffusion equation and the adiabatic condition yield

$$c^2 \frac{\partial \rho}{\partial \phi} + \frac{1}{2}K^2\left\{\frac{\partial}{\partial \phi}(\rho E) - E\,\frac{\partial \rho}{\partial \phi}\right\} = 0 \tag{47}$$

where $c^2 = \partial p/\partial \rho$. Using (46) in (47), we find

$$\left(c^2 - \frac{1}{2}K^2 E\right)\frac{\partial \rho}{\partial \phi} = 0\,. \tag{48}$$

This equation implies that either $c^2 - \frac{1}{2}K^2E = 0$ or $\partial\rho/\partial\phi = 0$. However, if the gas is polytropic, then by using Eq. (46) in $c^2 - \frac{1}{2}K^2E = 0$, we get

$$\rho c^2 = \gamma A(s)\rho^\gamma = 1/2G\rho \sin\theta \tag{49}$$

which, together with (45), implies that $\rho = \rho(\psi)$ or $\partial\rho/\partial\phi = 0$. For non-polytropic gases, (48) implies that $\partial\rho/\partial\phi = 0$ when $c^2 \neq \frac{1}{2}K^2E$ or $M^2 = q^2/c^2 \neq 2$, that is, the density will be constant on each individual streamline for a flow of nonpolytropic gas provided that the Mach number for the flow is not equal to $\sqrt{2}$. For such flows, using $\partial\rho/\partial\phi = 0$ in (45), we get

$$\sqrt{G} = B'(\psi) \tag{50}$$

and the form of $\sqrt{E}$ is obtained, by using the form of $\sqrt{G}$ in Gauss' equation, as

$$\sqrt{E} = LB(\psi) + M \tag{51}$$

where L, M are arbitrary constants and $B(\psi)$ is an arbitrary function of ψ. Since the limitations on the forms of $\sqrt{E}$, $\sqrt{G}$ given by (50), (51) are identical with those of the previous section, we conclude that

THEOREM 3. If the magnetic field is irrotational, then the streamlines are concentric circles or a family of parallel straight lines for any flow of a polytropic gas and for those flows of non-polytropic gases in which the Mach number is not equal to $\sqrt{2}$ anywhere.

5. Irrotational orthogonal flows. We investigate the flow patterns when $\omega = 0$, $F = 0$ and the gas obeys an equation of state of the form $\rho = P_1(p)S_1(s)$. Using the assumptions $\omega = 0$ and $F = 0$ in the vorticity equation, we have

$$\partial E/\partial\psi = 0, \qquad \text{or} \qquad E = E(\phi). \tag{52}$$

Linear momentum equations, using (52), take the form

$$\frac{\partial p}{\partial \phi} + \frac{1}{2}\rho K^2 \frac{\partial E}{\partial \phi} + \mu\rho\,\frac{\partial}{\partial \phi}\left[\frac{1}{\rho E}\right] = 0\,, \tag{53}$$

$$\partial p/\partial\psi = 0, \qquad \text{or} \qquad p = p(\phi). \tag{54}$$

By using $p = p(\phi)$ and $s = s(\psi)$ in the state equation, the form of the density function is given by

$$\rho = P(\phi)S(\psi) \tag{55}$$

where $P(\phi) = P_1(p)$, $S(\psi) = S_1(s)$.

Taking $G = 1/K^2\rho^2 E$ from the diffusion equation and using (52), (55) in (53) and Gauss' equation, we obtain

$$\left\{p'(\phi) + \frac{1}{2}K^2 S(\psi)P(\phi)E'(\phi)\right\} - \mu\left\{\frac{P'(\phi)E(\phi) + P(\phi)E'(\phi)}{P(\phi)E^2(\phi)}\right\} = 0, \tag{56}$$

$$\frac{1}{E^2(\phi)P^2(\phi)}\left\{E(\phi)P'(\phi) + \frac{1}{2}E'(\phi)P(\phi)\right\} = S(\psi)f(\psi), \tag{57}$$

where primes denote differentiation and $f(\psi)$ is an arbitrary function of ψ. Since ϕ, ψ are independent variables, it follows that each side of (57) is equal to a constant, say A. Taking the left-hand side of (57) equal to A and using in (56), we obtain

$$\frac{1}{2}K^2 S(\psi) = \frac{1}{P(\phi)E'(\phi)}\left[\frac{\mu}{E^2(\phi)}\left\{AE^2(\phi)P(\phi) + \frac{1}{2}E'(\phi)\right\} - p'(\phi)\right]. \tag{58}$$

This equation implies that $S(\psi)$ = constant and, therefore, we obtain that irrotational orthogonal flows of a gas obeying the product equation of state are homentropic flows with equation of state of the form $\rho = \rho(p)$. This physical restriction and Eq. (54) give

$$\rho = \rho(\phi). \tag{59}$$

Using (59) and (52) in $W = 1/\rho K$, we get

$$G = G(\phi). \tag{60}$$

Letting

$$\sqrt{E} = g'(\phi) \tag{61}$$

in Gauss' equation, we get

$$\sqrt{G} = Lg(\phi) + M \tag{62}$$

where L, M are arbitrary constants.

These limitations on the forms of $\sqrt{E}$ and $\sqrt{G}$ are identical to those in Sec. 3, with the roles of ϕ and ψ interchanged, and we therefore conclude that

THEOREM 4. If orthogonal flows are irrotational for a compressible fluid obeying the product equation of state, then these flows are homentropic radial or parallel flows.

6. Straight streamlines. In this section we inquire what plane orthogonal flow patterns are possible when the streamlines are straight lines for compressible flows obeying an equation of state of the product form. In order to approach this problem, we assume that the streamlines are non-parallel straight lines enveloping a curve C. Taking the tangent lines to C and their orthogonal trajectories (the involutes of C) as a system of orthogonal curvilinear coordinates, the squared element of arc length is given by

$$ds^2 = d\xi^2 + (\xi - \sigma)^2\kappa^2\, d\sigma^2 \tag{63}$$

where σ denotes the arc length, κ the curvature of C and ξ the parameter constant along each individual involute. If ν denotes the angle of elevation of the tangent line to C, we have $d\nu/d\sigma = \kappa$ and (63) becomes

$$ds^2 = d\xi^2 + (\xi - \sigma)^2 \, d\nu^2 \tag{64}$$

where $\sigma = \sigma(\nu)$. In this coordinate system, the coordinate curves ξ = constant and ν = constant are respectively the involutes of C and the tangent lines of C.

We now investigate the flows for which

$$\phi = \phi(\xi), \qquad \psi = \psi(\nu). \tag{65}$$

Using (65) in (13), we get

$$ds^2 = E\phi'^2 \, d\xi^2 + 2F\phi'\psi' \, d\xi \, d\nu + G\psi'^2 \, d\nu^2. \tag{66}$$

Comparing (66) and (64), we get

$$E = 1/\phi'^2, \qquad F = 0, \qquad G = \left[\frac{\xi - \sigma(\nu)}{\psi'(\nu)}\right]^2. \tag{67}$$

Substituting for E, F, G in vorticity equation and using the diffusion equation, we obtain that vorticity vanishes. Therefore, we have irrotational orthogonal flows of the previous section and we conclude with the same results.

7. Parallel flows with constantly-inclined magnetic lines. Let the streamlines be the straight lines $y = \eta$, $-\infty < \eta < \infty$, and the magnetic lines be the lines $y = x \tan \theta + \xi$, $-\infty < \xi < \infty$, so that the two families of lines are everywhere constantly inclined to each other at an angle $\theta \neq 0$. Therefore

$$x = \eta \cot \theta + \xi, \qquad y = \eta$$

define the curvilinear net with the squared element of arc length along any curve given by

$$ds^2 = d\xi^2 + 2 \cot \theta \, d\xi \, d\eta + \operatorname{cosec}^2 \theta \, d\eta^2. \tag{68}$$

For the flow under consideration, we require

$$\phi = \phi(\xi), \qquad \psi = \psi(\eta). \tag{69}$$

Using (69) in (13) yields

$$ds^2 = E\phi'^2 \, d\xi^2 + 2F\phi'\psi' \, d\xi \, d\eta + G\psi'^2 \, d\eta^2. \tag{70}$$

Comparing (70) and (68) implies

$$E = 1/\phi'^2, \qquad F = \cot \theta/\phi'\psi', \qquad G = \operatorname{cosec}^2 \theta/\psi'^2 \tag{71}$$

and, therefore, $W = 1/\phi'\psi'$.

Substituting (71) in the linear momentum equations and using the vorticity and current density equations, we have

$$\begin{aligned} &\frac{\partial p}{\partial \phi} - K\psi' \frac{\phi''}{\phi'^3} + \mu \operatorname{cosec}^2 \theta \, \phi'' = 0, \\ &\frac{\partial p}{\partial \psi} - K \cot \theta \frac{\phi''}{\phi'^2} = 0 \end{aligned} \tag{72}$$

where $\phi'' \neq 0$ since $p(\phi, \psi) \neq$ constant. By using the integrability condition $\partial^2 p/\partial \phi \, \partial \psi = \partial^2 p/\partial \psi \, \partial \phi$, eqs. (72) give

$$\cot\theta\left\{\frac{\phi'''}{\phi''}-\frac{\phi''}{\phi'}\right\}=\frac{\psi''}{\psi'}=L \tag{73}$$

where L is an arbitrary constant. Integrating the differential equations (73), we obtain

$$\begin{aligned}\phi'(\xi) &= -L/(N\cot\theta\exp(L\tan\theta\xi)+QL),\\ \psi'(\eta) &= M\exp(L\eta)\end{aligned} \tag{74}$$

where M, N and Q are arbitrary constants. Using (74) in (71), the functions E, F, and G may be determined explicitly. The solution to our flow problem can then be obtained, from (33) and (34), as

$$\begin{aligned}
p(\xi,\eta) &= \frac{KMN}{L}\cot\theta\exp\{L(\eta+\xi\tan\theta)\}-\frac{1}{2}L^2\mu\operatorname{cosec}^2\theta\, f^2(\xi),\\
\omega(\xi) &= -KN\cot\theta\exp\{\xi L\tan\theta\},\\
j(\xi) &= L^2N\operatorname{cosec}^2\theta\exp\{\xi L\tan\theta\}f^2(\xi),\\
\rho(\xi,\eta) &= \frac{-LM}{K}\exp\{L\eta\}f(\xi),\qquad u(\xi)=\frac{-K}{L}\frac{1}{f(\xi)},\qquad v=0,\\
H_1(\xi) &= -L\cot\theta\, f(\xi),\qquad H_2(\xi)=-Lf(\xi),
\end{aligned} \tag{75}$$

where, for brevity, we have written

$$f(\xi)\equiv[N\cot\theta\exp\{\xi L\tan\theta\}+QL]^{-1}.$$

8. **Circular flows.** In this section we study flows in which the streamline pattern is circular and the magnetic lines are orthogonal to them. Introducing polar coordinates (r, δ), the streamlines are defined by r = constant and the magnetic lines are given by δ = constant. The square of the element of arc length is

$$ds^2 = dr^2 + r^2\, d\delta^2. \tag{76}$$

For our problem we have $\phi = \phi(\delta)$ and $\psi = \psi(r)$ and, therefore, we also have

$$ds^2 = E\phi'^2\, d\delta^2 + 2F\phi'\psi'\, d\delta\, dr + G\psi'^2\, dr^2. \tag{77}$$

Comparison of (76) and (77) gives

$$E = r^2/\phi'^2,\qquad F=0,\qquad G=1/\psi'^2 \tag{78}$$

and, therefore, $W = r/\phi'\psi'$.

Using (78) in the vorticity and current density equations of (33), we find

$$\omega = -2K/\phi',\qquad j=\phi''/r^2. \tag{79}$$

The linear momentum equations, on substituting (78) and (79), become

$$\frac{\partial p}{\partial\delta}+\frac{Kr\phi'\psi'}{2}\frac{d}{d\delta}\left(\frac{1}{\phi'^2}\right)+\frac{\mu\phi'\phi''}{r^2}=0,\qquad \frac{\partial p}{\partial r}-\frac{K\psi'}{\phi'}=0. \tag{80}$$

Applying the integrability condition on $p(r, \delta)$, Eqs. (80) yield

$$\frac{\phi''}{\phi'^2}\{K\psi' + K(r\psi'' + \psi')\} + \frac{2\mu}{r^3}\phi'\phi'' = 0. \tag{81}$$

Eq. (81) implies that

$$\phi'' = 0 \qquad \text{and/or} \qquad 2K\psi' + Kr\psi'' = \frac{-2\mu}{r^3}\phi'^3. \tag{82}$$

However, the second equation in (82) may be separated to give

$$\frac{Kr^3}{2\mu}\{r\psi'' + 2\psi'\} = -\phi'^3 = -a_1^3 \qquad \text{(constant)}, \tag{83}$$

which again implies that $\phi'' = 0$. Hence the integrability condition (81) on $p(r, \delta)$ requires that $\phi'' \equiv 0$. Returning to Eqs. (80), we then find

$$p(r) = \frac{K}{a_1}\psi(r) + a_2, \tag{84}$$

where a_2 is an arbitrary constant. Eqs. (79) give

$$\omega = -2K/a_1, \qquad j = 0. \tag{85}$$

Calculating E, G, and W from (78) and using $\phi' = a_1$, the diffusion equation of (33) gives

$$\rho(r) = a_1\psi'(r)/Kr. \tag{86}$$

Eqs. (84) and (86) are valid for general equations of state and, due to the arbitrariness of the streamfunction $\psi(r)$, represent solutions for a large class of circular flows. In the case of homentropic (constant-entropy) flow of a polytropic gas, with equation of state $p/\rho^\gamma = A$ = constant, an explicit formula for ψ can be obtained as

$$\psi(r) = \frac{1}{K}\left\{\frac{\gamma K^2}{2a_1(Aa_1)^{1/\gamma}(\gamma - 1)} r^2 + a_3\right\}^{\gamma/(\gamma-1)} - \frac{a_1 a_2}{K}$$

where a_3 is another arbitrary constant.

Returning to the general case, the velocity and magnetic field can be obtained from Eqs. (34) as

$$u + iv = \frac{K}{a_1}(-y + ix), \qquad H_1 + iH_2 = \frac{a_1}{x^2 + y^2}(x + iy) \tag{87}$$

where we have used the facts that $\beta = \delta$ and $\alpha = (\pi/2) + \delta$ for these flows. It is interesting to note that the velocity and magnetic fields are independent of the choice for the streamfunction $\psi(r)$.

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