

FUNDAMENTAL SOLUTIONS FOR IDEAL FLUIDS IN UNIFORM MOTION*

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Abstract. In this paper the perturbations produced by mass, momentum and energy sources in the uniform flow of an ideal fluid are determined. The case of instantaneous sources is considered. The matrix solutions thus determined are by definition the fundamental matrices of the systems of equations of fluid mechanics. As an application, the perturbations caused by sources acting permanently for $t > 0$ as well as the perturbations produced by sources moving along the direction of the free flow or perpendicular to this direction are determined.

1. In [1] we have determined the perturbations produced by mass, momentum and energy sources in an ideal (or viscous) fluid at rest. In [2] we have constructed the fundamental matrix for this fluid.

Extending the methods of [1] and [2], in this paper we shall determine the perturbations induced by mass, momentum and energy sources in an ideal fluid in uniform motion. We shall build up the fundamental matrix in this case and we shall apply the results to problems of great practical interest. The case of dissipative fluids will be analysed in [3].

We assume that the basic state of the fluid is characterized by the constant parameters ρ_0 , \mathbf{V}_0 , p_0 and T_0 ($p_0 = \rho_0 R T_0$) and we shall choose the reference system such that the Ox_1 -axis has the direction of the vector \mathbf{V}_0 . At an arbitrary time $t > 0$, the fluid state will be characterized by the variables

$$\rho = \rho_0(1 + \rho^*), \quad \mathbf{V} = V_0(\mathbf{i}_1 + \mathbf{v}), \quad p = p_0 + \rho_0 V_0^2 p^* \quad (1)$$

and $T = T_0(1 + T^*)$, the quantities $\rho_0 \rho^*$, $V_0 \mathbf{v}$, $\rho_0 V_0^2 p^*$ and $T_0 T^*$ representing the perturbations induced by the sources. The functions ρ , \mathbf{V} , p and T must satisfy Eqs. (1)–(4) of [1] in which the dissipative terms are neglected. In order that the equations be treated in a unitary way, we shall assume that the sources are acting in the origin of the system of coordinates. The problem concerning the sources acting in different points does not present additional difficulties [1]. Also, for the sake of uniformity, we shall introduce the dimensionless variables x_i^* and t^* defined by the relations $x_i = L_0 x_i^*$, $V_0 t = L_0 t^*$, where L_0 is an arbitrary characteristic length, and we shall drop the sign *. Linearizing the motion equations in order to determine the perturbation, we obtain the following system:

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$$\begin{aligned}
\partial_t \rho + \partial_1 \rho + \operatorname{div} \mathbf{v} &= f_0 \delta(t, \mathbf{x}), \quad \delta(t, \mathbf{x}) = \delta(t) \delta(\mathbf{x}), \\
\partial_t v_i + \partial_1 v_i + \partial_i p &= f_i \delta(t, \mathbf{x}) \quad (i = 1, 2, 3), \\
\partial_t T + \partial_1 T + (\gamma - 1) \operatorname{div} \mathbf{v} &= (\gamma f_4 - f_0) \delta(t, \mathbf{x}), \\
\gamma M^2 p &= \rho + T,
\end{aligned} \tag{2}$$

where f_j ($j = 0, 1, 2, 3, 4$) are expressed with the aid of the intensities m_0 , Γ^0 and Q_0 of the mass, momentum and energy sources using the following expressions:

$$\begin{aligned}
\rho_0 L_0^3 f_0 &= m_0, & \rho_0 L_0^3 V_0 f_i &= f_i^0 - \delta_{i1} m_0 V_0, \\
c_v \rho_0 L_0^3 T_0 \gamma f_4 &= Q_0 + \frac{1}{2} m_0 V_0^2 - f_1^0 V_0,
\end{aligned}$$

where δ_{ij} is the Kronecker delta. As usual, we have used the notation $\gamma c_v = c_p$, $a_0 M = V_0$, $a_0^2 = \gamma R T_0$, where c_v and c_p are the specific heats of the fluid, while R is the universal constant of gases. Adding $(2)_1$ to $(2)_3$ and taking into account $(2)_4$, we get

$$M^2(\partial_t + \partial_1)p + \operatorname{div} \mathbf{v} = f_4 \delta(t, \mathbf{x}) \tag{3}$$

Therefore we shall consider Eqs. $(2)_1$, $(2)_2$ and (3) with the unknowns ρ , v_i and p . These equations will have to be integrated all over the space in which the space conditions are

$$\lim_{|\mathbf{x}| \rightarrow \infty} (\rho, v_i, p) = 0 \tag{4}$$

and the temporal conditions

$$\rho = 0, \quad v_i = 0, \quad p = 0, \quad t \leq 0. \tag{5}$$

Eq. (4) expresses the damping condition of the perturbation at infinity and (5) the condition of nonexistence of the perturbation before the source begins to act.

Hereafter we shall use the matrix method in order to determine the solution of the system (2), (3); in this way we have the possibility of putting into evidence the basic structure of the solution and at the same time of writing the general solution of the system (2), (3) in the case in which in the right-hand side of the equalities the arbitrary functions $f(t, \mathbf{x})$ would appear. Introducing the matrices

$$\begin{aligned}
A_1 &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & M^{-2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}, & A_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M^{-2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \\
A_3 &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M^{-2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, & V &= \begin{pmatrix} \rho \\ v_1 \\ v_2 \\ v_3 \\ M^2 p \end{pmatrix}, & f &= \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix},
\end{aligned}$$

we can write the system consisting of Eqs. $(2)_1$, $(2)_2$ and (3) as follows:

$$\partial_t V + A_k \partial_k V = f \delta(t, \mathbf{x}). \tag{6}$$

The fundamental matrix $K(t, \mathbf{x})$ introduced by the formula $V = Kf$ will be determined by equation

$$\partial_t K + A_k \partial_k K = E \delta(t, \mathbf{x}) \tag{7}$$

where E is the unit matrix with 5×5 elements, and by the conditions

$$K(t \leq 0, \mathbf{x}) = 0, K(t, \infty) = 0 \tag{8}$$

with the usual notations.

2. As is already known (Duhamel's principle, see also [4]), the solution of the problem (7), (8) is of the form $K = H(t) K^*(t, \mathbf{x})$ where $H(t)$ is the Heaviside function and K^* the matrix defined by equation

$$\partial_t K^* + A_k \partial_k K^* = 0 \tag{9}$$

and by the conditions

$$K^*(0, \mathbf{x}) = E \delta(\mathbf{x}), \quad K^*(t, \infty) = 0. \tag{10}$$

For

$$\bar{K} = \mathfrak{F}(K^*) = \int \int \int_{-\infty}^{+\infty} K \exp i(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3) dx_1 dx_2 dx_3$$

we obtain

$$\dot{\bar{K}} = \bar{\mathcal{Q}} \bar{K}, \quad \bar{K}(0, \alpha) = E \tag{12}$$

where

$$\bar{\mathcal{Q}} = i \alpha_k A_k = i \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ 0 & \alpha_1 & 0 & 0 & \alpha_1 M^{-2} \\ 0 & 0 & \alpha_1 & 0 & \alpha_2 M^{-2} \\ 0 & 0 & 0 & \alpha_1 & \alpha_3 M^{-2} \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_1 \end{pmatrix}.$$

The solution of Eq. (12) is

$$\bar{K} = E \exp (\bar{\mathcal{Q}} t). \tag{13}$$

In order to determine the function $\exp (\bar{\mathcal{Q}} t)$ we use the method of the minimal polynomial. The roots of the characteristic polynomial of the matrix \mathcal{Q} are

$$\lambda_1 = i \alpha_1, \lambda_2 = i(\alpha_1 + |\alpha| M^{-1}), \lambda_3 = i(\alpha_1 - |\alpha| M^{-1}) \tag{14}$$

where multiplicity of the first root is equal to three. Since the minimal polynomial $m(\lambda)$ has the expression

$$m(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - a_2 \lambda^2 - a_1 \lambda - a_0, \\ a_2 = 3 i \alpha_1, a_1 = 3 \alpha_1^2 - |\alpha|^2 M^{-2}, a_0 = i \alpha_1 (|\alpha|^{-2} M^{-2} - \alpha_1^2),$$

it follows that the solution of system (12) is

$$\bar{K} = E g_0(t) + \bar{\mathcal{Q}} g_1(t) + \bar{\mathcal{Q}}^2 g_2(t) \tag{15}$$

where g_2, g_0 and g_1 are determined by the equations

$$g_2''' - a_2 g_2'' - a_1 g_2' - a_0 g_2 = 0, \quad g_2(0) = g_2'(0) = 0, \quad g_2''(0) = 1, \\ g_0' = a_0 g_2, \quad g_0(0) = 1; \quad g_1' = a_1 g_2 + g_0, \quad g_1(0) = 0,$$

We obtain

$$\begin{aligned}
 g_0 &= \left[1 + \frac{\alpha_1^2 M^2}{|\alpha|^2} \left(\cos \frac{|\alpha|}{M} t - 1 \right) - \frac{i\alpha_1 M}{|\alpha|} \sin \frac{|\alpha|}{M} t \right] \exp(i\alpha_1 t), \\
 g_1 &= \left[\frac{M}{|\alpha|} \sin \frac{|\alpha|}{M} t + \frac{2i\alpha_1 M^2}{|\alpha|^2} \left(\cos \frac{|\alpha|}{M} t - 1 \right) \right] \exp(i\alpha_1 t), \\
 g_2 &= \frac{M^2}{|\alpha|^2} \left(1 - \cos \frac{|\alpha|}{M} t \right) \exp(i\alpha_1 t).
 \end{aligned} \tag{15'}$$

Taking into account formulae (A.1)–(A.3) in the Appendix, we obtain the following fundamental matrix:

$$\begin{aligned}
 K^* &= E\delta(t - x_1)\delta(x_2)\delta(x_3) + \frac{M^2}{4\pi t} (E\partial_1 + \mathcal{Q}) \delta\{M^{-1}t - [(x_1 - t)^2 + r_0^2]^{1/2}\} \\
 &+ \frac{M^2}{4\pi} (E\partial_{11}^2 + 2\mathcal{Q}\partial_1 + \mathcal{Q}^2) \frac{H\{M^{-1}t - [(x_1 - t)^2 + r_0^2]^{1/2}\}}{[(x_1 - t)^2 + r_0^2]^{1/2}},
 \end{aligned} \tag{16}$$

where the matrix-operator \mathcal{Q} is obtained from $\bar{\mathcal{Q}}$ by substituting $-\partial_j \leftarrow i\alpha_j$. We have, for instance,

$$M^2\mathcal{Q}^2 = \begin{pmatrix} M^2\partial_{11}^2 & 2M^2\partial_{11}^2 & 2M^2\partial_{12}^2 & 2M^2\partial_{13}^2 & \partial_{ii}^2 \\ 0 & (1 + M^2)\partial_{11}^2 & \partial_{12}^2 & \partial_{13}^2 & 2\partial_{11}^2 \\ 0 & \partial_{12}^2 & M^2\partial_{11}^2 + \partial_{22}^2 & \partial_{23}^2 & 2\partial_{12}^2 \\ 0 & \partial_{13}^2 & \partial_{23}^2 & M^2\partial_{11}^2 + \partial_{33}^2 & 2\partial_{13}^2 \\ 0 & 2M^2\partial_{11}^2 & 2M^2\partial_{12}^2 & 2M^2\partial_{13}^2 & M^2\partial_{11}^2 + \partial_{ii}^2 \end{pmatrix},$$

$$\partial_{ii}^2 \equiv \partial_{11}^2 + \partial_{22}^2 + \partial_{33}^2.$$

3. As can be proved [2], the general solution of the equation

$$\partial_t V + A_k \partial_k V = f(t, \mathbf{x}) \tag{17}$$

with the conditions

$$V(0, \mathbf{x}) = 0, V(t, \infty) = 0,$$

is

$$V(t, \mathbf{x}) = \int_0^t K^{**} f \, d\tau$$

where

$$K^{**} f = \int \int \int_{-\infty}^{+\infty} K^*(t - \tau, \mathbf{x} - \xi) f(\tau, \xi) d\xi, \quad d\xi \equiv d\xi_1 d\xi_2 d\xi_3.$$

Taking into account the expression of the fundamental matrix (16), it follows that the general solution of Eq. (17) is

$$\begin{aligned}
 V(t, \mathbf{x}) &= \int_0^t f(t - \tau, x_1 - \tau, x_2, x_3) d\tau \\
 &+ \frac{M^2}{4\pi} (E\partial_1 + \mathcal{Q}) \int_0^t \frac{d\tau}{\tau} \int \int \int_{-\infty}^{+\infty} f(t - \tau, \mathbf{x} - \xi) \delta\{M^{-1}\tau - [(\xi_1 - \tau)^2 + \xi_0^2]^{1/2}\} d\xi \\
 &+ \frac{M^2}{4\pi} (E\partial_{11}^2 + 2\mathcal{Q}\partial_1 + \mathcal{Q}^2) \int_0^t d\tau \int \int \int_{-\infty}^{+\infty} f(t - \tau, \mathbf{x} - \xi) \frac{H\{M^{-1}\tau - [(\xi_1 - \tau)^2 + \xi_0^2]^{1/2}\}}{[(\xi_1 - \tau)^2 + \xi_0^2]^{1/2}} d\xi
 \end{aligned} \tag{18}$$

where $\xi_0^2 = \xi_2^2 + \xi_3^2$.

As a first application we consider the case in which the uniform motion of the fluid is perturbed by sources which in the origin act permanently for $t > 0$, therefore the case in which

$$f(t, \mathbf{x}) = f^0 H(t) \delta(\mathbf{x}) \tag{19}$$

where $H(t)$ is, as throughout this paper, Heaviside's function. We obtain

$$\begin{aligned}
 V(t, \mathbf{x}) &= f^0 [H(x_1) - H(x_1 - t)] \delta(x_2) \delta(x_3) + \frac{M^2}{4\pi} (E\partial_1 + \mathcal{Q}) f^0 \int_0^t \frac{d\tau}{\tau} \delta(M^{-1} - R_\tau) \\
 &+ \frac{M^2}{4\pi} (E\partial_{11}^2 + 2\mathcal{Q}\partial_1 + \mathcal{Q}^2) f^0 \int_0^t \frac{H(M^{-1}\tau - R_\tau)}{R_\tau} d\tau, \quad R_\tau = [(x_1 - \tau)^2 + r_0^2]^{1/2}.
 \end{aligned} \tag{20}$$

For the calculation of the integrals occurring here the zeros of the function $h(\tau) = M^{-1}\tau - R_\tau$ will have to be determined. We have $h(0) < 0$, $\lim_{\infty} h(\tau) = (1 - M^2)\infty$ and

$$\beta^2 \tau_{\pm} = -M^2 x_1 \pm MR, \quad R \equiv (r^2 - M^2 r_0^2)^{1/2}, \quad \beta^2 \equiv 1 - M^2.$$

For $M < 1$, $h(\tau)$ has a single zero, τ_+ . For $M > 1$, $h(\tau)$ has two zeros if $h_0 = h(\tau_0) > 0$, where τ_0 is the root of the derivative $h'(\tau)$, and no zero if $h_0 < 0$. We obtain $M h_0 = x_1 - r_0 \beta_1$, $\beta_1 \equiv (M^2 - 1)^{1/2}$, $\tau_+ < \tau_-$ and $\tau_{\pm} h'(\tau_{\pm}) = \pm R$. At points τ_+ the function h is increasing ($h' > 0$) and at points τ_- it is decreasing ($h' < 0$). Using the formula [5]

$$\delta(h(\tau)) = \sum_i \frac{\delta(\tau - \tau_i)}{|h'(\tau_i)|} \tag{21}$$

where τ_i are the zeros of the function h , we deduce

$$\begin{aligned}
 \int_0^t \frac{d\tau}{\tau} \delta(M^{-1}\tau - R_\tau) &= R^{-1} H(t - \tau_+), && \text{if } M < 1 \\
 &= R^{-1} [H(t - \tau_+) + H(t - \tau_-)], && \text{if } M > 1 \text{ and } h_0 > 0 \\
 &= 0, && \text{if } M > 1 \text{ and } h_0 < 0.
 \end{aligned}$$

We also have

$$\begin{aligned}
 \int_0^t \frac{H(M^{-1}\tau - R_\tau)}{R_\tau} d\tau &= H(t - \tau_+) \ln \left| \frac{x_1 + R}{(x_1 - t + R_t)(M + 1)} \right|, \text{ for } (M < 1) \cup \\
 &\quad (M > 1, h_0 > 0, t < \tau_-) \\
 &= H(t - \tau_-) \ln \left| \frac{x_1 + R}{x_1 - R} \right|, \text{ for } (M > 1, h_0 > 0) \\
 &= 0, \text{ for } (M > 1, h_0 < 0).
 \end{aligned}$$

In this way the solution (20) is completely determined.

4. As a second application let us determine the perturbation induced by sources moving uniformly in the flow of the free fluid. Let us assume for the beginning that the sources are moving along the direction of the free flow. This means that in (18) we must write

$$f(t, \mathbf{x}) = f^0 \delta(x_1 - k_0 t) \delta(x_2) \delta(x_3) \tag{22}$$

where k_0 is the ratio between the velocity of the sources and the velocity of the free flow; k_0 is a positive number if both velocities are oriented in the same direction and a negative number if the velocities have opposite directions. From (18) we get

$$V(t, \mathbf{x}) = f^0 \delta(x_2) \delta(x_3) \int_0^t \delta(x_1 - k_0 t + k\tau) d\tau + \frac{M^2}{4\pi} (E\partial_1 + \mathcal{Q}) f^0 \int_0^t \frac{d\tau}{\tau} \delta(M^{-1}\tau - R_\tau) + \frac{M^2}{4\pi} (E\partial_{11}^2 + 2\mathcal{Q}\partial_1 + \mathcal{Q}^2) f^0 \int_0^t \frac{H(M^{-1}\tau - R_\tau)}{R_\tau} d\tau \tag{23}$$

where

$$\int_0^t \delta(x_1 - k_0 t + k\tau) d\tau = \frac{1}{k} \int_0^t \delta\left(\tau + \frac{x_1 - k_0 t}{k}\right) d\tau = \frac{1}{K} H\left(\frac{x_1}{k} - \frac{t}{k}\right) - \left(k_0 \frac{t}{k} - \frac{x_1}{k}\right),$$

$$R_\tau = [(x_1 - k_0 t + k\tau)^2 + r_0^2]^{1/2}, \quad k = k_0 - 1.$$

The study of the zeros of the function $h(\tau) = M^{-1}\tau - R_\tau$ is carried out as shown above. We have $h(0) < 0, h(\infty) = (1 - k^2 M^2)\infty$;

$$\tau_\pm = \frac{k^2 M^2 (x_1 - k_0 t) \pm MR}{1 - M^2 k^2}, \quad R = [(x_1 - k_0 t)^2 + (1 - M^2 k^2) r_0^2]^{1/2}.$$

If $k^2 M^2 < 1$ then there is a single positive root, namely τ_+ . If $1 < k^2 M^2$ then there are two roots τ_+ and τ_- ($\tau_+ < \tau_-$) if $h_0 > 0$ and no root at all if $h_0 < 0$, where

$$Mh_0 = -\frac{r_0}{|k|} (M^2 k^2 - 1)^{1/2} - \frac{x_1 - k_0 t}{k}.$$

Since we also have the relation $\tau_\pm h'(\tau_\pm) = \pm R$, we obtain

$$\int_0^t \frac{d\tau}{\tau} \delta(M^{-1}\tau - R_\tau) = R^{-1} H(t - \tau_+), \quad \text{if } k^2 M^2 < 1$$

$$= R^{-1} [H(t - \tau_+) + H(t - \tau_-)], \quad \text{if } (1 < k^2 M^2, h_0 > 0)$$

$$= 0, \quad \text{if } (1 < k^2 M^2, h_0 < 0)$$

$$\int_0^t \frac{H(M^{-1}\tau - R_\tau)}{R_\tau} d\tau = \frac{H(t - \tau_+)}{k} \ln \left| \frac{(1 - kM)\{x_1 - t + [(x_1 - t)^2 + r_0^2]^{1/2}\}}{x_1 - k_0 t + R} \right|$$

for $(k^2 M^2 < 1) \cup (1 < k^2 M^2, h_0 > 0, t < \tau_-)$

$$= \frac{H(t - \tau_-)}{k} \ln \left| \frac{x_1 - k_0 t - R}{x_1 - k_0 t + R} \right|, \quad \text{for } (1 < k^2 M^2, h_0 < 0).$$

$$= 0, \quad \text{for } (1 < k^2 M^2, h_0 < 0).$$

Let us now determine the solution in the case in which the sources are moving perpendicular to the direction of the free flow. This means that we write

$$f(t, \mathbf{x}) = f^0 \delta(x_1) \delta(x_2 - k_0 t) \delta(x_3) \quad (23)$$

in (18). We get

$$\begin{aligned} V(t, \mathbf{x}) &= f^0 \delta(x_3) \int_0^t \delta(x_1 - \tau) \delta(x_2 - k_0 t + k_0 \tau) d\tau \\ &+ \frac{M^2}{4\pi} (E \partial_1 + \mathcal{Q}) f^0 \int_0^t \frac{d\tau}{\tau} \delta(M^{-1} \tau - R_\tau) \\ &+ \frac{M^2}{4\pi} (E \partial_{11}^2 + 2\mathcal{Q} \partial_1 + \mathcal{Q}^2) f^0 \int_0^t \frac{H(M^{-1} \tau - R_\tau)}{R_\tau} d\tau \end{aligned} \quad (25)$$

where, this time,

$$\begin{aligned} \int_0^t \delta(x_1 - \tau) \delta(x_2 - k_0 t + k_0 \tau) d\tau &= \delta(k_0 x_1 + x_2 - k_0 t) \quad \text{if } (0 < x_1 < t, 0 < t - k_0^{-1} x_2 < t) \\ &= 0 \quad \text{otherwise} \end{aligned}$$

$$R_\tau = [(x_1 - \tau)^2 + (x_2 - k_0 t + k_0 \tau)^2 + x_3^2]^{1/2}.$$

In this case, too, the zeros of the function $h(\tau) = M^{-1} \tau - R_\tau$ are determined. We obtain

$$\tau_\pm = \frac{-M^2[x_1 - k_0(x_2 - k_0 t)] \pm MR}{1 - k^2 M^2}$$

where

$$\begin{aligned} R^2 &= M^2[x_1 - k_0(x_2 - k_0 t)]^2 + (1 - k^2 M^2) R_0^2, \\ R_0^2 &= x_1^2 + (x_2 - k_0 t)^2 + x_3^2, \quad k^2 = 1 + k_0^2. \end{aligned}$$

For R and k the positive determination is taken into account. If $k^2 M^2 < 1$ then the function h has a single zero τ_+ . If $1 < k^2 M^2$ and $h_0 > 0$, where

$$k^2 M h_0 = x_1 - k_0(x_2 - k_0 t) - (k^2 M^2 - 1)^{1/2} \{k^2 R_0^2 - [x_1 - k_0(x_2 - k_0 t)]^2\}^{1/2},$$

then the function $h(\tau)$ has two zeros, τ_+ and τ_- ($\tau_+ < \tau_-$). Finally, if $1 < k^2 M^2$ and $h_0 < 0$, the function $h(\tau)$ has no zero at all. Taking into account $\tau_\pm h'(\tau_\pm) = \pm R$, it follows that

$$\begin{aligned} \int_0^t \frac{d\tau}{\tau} \delta(M^{-1} \tau - R_\tau) &= R^{-1} H(t - \tau_+), \quad \text{if } k^2 M^2 < 1 \\ &= R^{-1} [H(t - \tau_+) + H(t - \tau_-)], \quad \text{if } (1 < k^2 M^2, h_0 > 0) \\ &= 0, \quad \text{if } (1 < k^2 M^2, h_0 < 0) \end{aligned}$$

$$\int_0^t \frac{H(M^{-1}\tau - R_\tau)}{R_\tau} d\tau = \begin{cases} k^{-1}H(t - \tau_+) \ln \left| \frac{(1 - kM)(t - x_1 + k_0x_2 + kR_t)}{kR - [x_1 - k_0(x_2 - k_0t)]} \right|, \\ \quad \text{if } (k^2M^2 < 1) \cup (1 < k^2M^2, h_0 > 0, \quad t < \tau_-) \\ k^{-1}H(t - \tau) \ln \left| \frac{kR + [x_1 - k_0(x_2 - k_0t)]}{kR - [x_1 - k_0(x_2 - k_0t)]} \right|, \\ \quad \text{if } (1 < k^2M^2, \quad h_0 > 0) \\ 0, \quad \text{if } (1 < k^2M^2, \quad h_0 < 0). \end{cases}$$

The solutions determined here have not only a theoretical importance but also a practical one. For example, the solution determined in Sec. 3 in the particular case $m_0 = Q_0 = 0$ gives the perturbation induced by a fixed body which can be assimilated to a point, in the presence of a uniform wind, and the solutions determined in Sec. 4 (for $m_0 = Q_0 = 0$) give the perturbations produced at the leading edge of an aeroplane which moves in the direction of the wind (in the same direction or in the opposite direction to the wind) or perpendicular to the direction of the wind respectively.

Appendix. In Sec. 2 the following formulae are used:

$$\tilde{f}^{-1}(\exp(i\alpha_1 t)) = \delta(t - x_1) \delta(x_2) \delta(x_3), \tag{A.1}$$

$$\tilde{f}^{-1}\left(\frac{\exp(i\alpha_1 t)}{|\alpha|} \sin \frac{|\alpha|}{M} t\right) = \frac{M}{4\pi t} \delta\{M^{-1}t - [(x_1 - t)^2 + r_0^2]^{1/2}\}, \tag{A.2}$$

$$\tilde{f}^{-1}\left(\frac{\exp(i\alpha_1 t)}{|\alpha|^2} \left(1 - \cos \frac{|\alpha|}{M} t\right)\right) = \frac{H\{M^{-1}t - [(x_1 - t)^2 + r_0^2]^{1/2}\}}{4\pi [(x_1 - t)^2 + r_0^2]^{1/2}}. \tag{A.3}$$

Formula (A.1) is obvious. In order to deduce formula (A.2) we use cylindrical coordinates in the Fourier space: $\alpha_1 = \alpha_1, \alpha_2 = \rho \cos \theta, \alpha_3 = \rho \sin \theta$ and cylindrical coordinates in the physical space: $x_1 = x_1, x_2 = r_0 \cos \phi, x_3 = r_0 \sin \phi$, as well as the formulae

$$\int_0^\infty \frac{J_0(\rho r_0) \rho \sin t M^{-1}(\rho^2 + \alpha_1^2)^{1/2}}{(\rho^2 + \alpha_1^2)^{1/2}} d\rho = M \frac{H(t - Mr_0)}{(t^2 - M^2 r_0^2)^{1/2}} \cos \frac{\alpha_1}{M} (t^2 - M^2 r_0^2)^{1/2},$$

$$\int_{-\infty}^{+\infty} \exp(-i\alpha_1(x_1 - t)) \cos \frac{\alpha_1}{M} (t^2 - M^2 r_0^2)^{1/2} d\alpha_1 = \pi \left\{ \delta \left[x_1 - t + \frac{1}{M} (t^2 - M^2 r_0^2)^{1/2} \right] \right.$$

$$\left. + \delta \left[x_1 - t - \frac{1}{M} (t^2 - M^2 r_0^2)^{1/2} \right] \right\}, \quad \delta(x - c) + \delta(x + c) = 2c\delta(x^2 - c^2)$$

obtained from [6] and [5]. We get

$$\tilde{f}^{-1}\left(\frac{\exp(i\alpha_1 t)}{|\alpha|} \sin \frac{|\alpha|}{M} t\right)$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \exp(-i\alpha_1(x_1 - t)) d\alpha_1 \int_0^\infty \frac{J_0(\rho r_0) \rho \sin t M^{-1}(\rho^2 + \alpha_1^2)^{1/2}}{(\rho^2 + \alpha_1^2)^{1/2}} d\rho$$

$$\begin{aligned}
 &= \frac{H(t - Mr_0)}{2\pi} \delta \left[(x_1 - t)^2 + r_0^2 - \frac{t^2}{M^2} \right] = \frac{MH(t - Mr_0)}{4\pi t} \delta \{ [(x_1 - t)^2 + r_0^2]^{1/2} - M^{-1} t \} \\
 &= \frac{M}{4\pi t} \delta \{ M^{-1} t - [(x_1 - t)^2 + r_0^2]^{1/2} \}.
 \end{aligned}$$

Using the same change of variables we have

$$\begin{aligned}
 \tilde{f}^{-1} \left(\frac{\exp(i\alpha_1 t)}{|\alpha|^2} \right) &= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \exp(i\alpha_1(t - x_1)) d\alpha_1 \int_0^\infty \frac{J_0(\rho r_0) \rho d\rho}{\alpha_1^2 + \rho^2} \\
 &= \frac{1}{2\pi^2} \int_0^\infty K_0(r_0 \alpha_1) \cos \alpha_1(t - x_1) d\alpha_1 = \frac{1}{4\pi} [(t - x_1)^2 + r_0^2]^{-1/2}, \\
 \tilde{f}^{-1} \left(\frac{\exp(i\alpha_1 t)}{|\alpha|^2} \cos \frac{|\alpha|}{M} t \right) &= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} I \exp(i\alpha_1(t - x_1)) d\alpha_1
 \end{aligned}$$

where

$$I(t) = \int_0^\infty \frac{J_0(\rho r_0)}{\rho^2 + \alpha_1^2} \cos \frac{|\alpha|}{M} t d\rho.$$

Since the integral $I(t)$ is not given in tables, we shall calculate it here. Obviously we have

$$I(0) = K_0(|\alpha_1| r_0), \quad I'(t) = -\frac{H(t - Mr_0)}{(t^2 - M^2 r_0^2)^{1/2}} \cos \frac{\alpha_1}{M} (t^2 - M^2 r_0^2)^{1/2},$$

$$I(t) = I(0) + \int_0^t I'(\tau) d\tau = K_0(|\alpha_1| r_0) - H(t - Mr_0) \int_{Mr_0}^t \frac{\cos \alpha_1 M^{-1} (\tau^2 - M^2 r_0^2)^{1/2}}{(\tau^2 - M^2 r_0^2)^{1/2}} d\tau,$$

and consequently

$$\begin{aligned}
 \tilde{f}^{-1} \left(\frac{e^{i\alpha_1 t}}{|\alpha|^2} \left(1 - \cos \frac{|\alpha|}{M} t \right) \right) &= \frac{H(t - Mr_0)}{4\pi} \int_{Mr_0}^t \frac{d\tau}{(\tau^2 - M^2 r_0^2)^{1/2}} \\
 &\cdot \left\{ \delta \left[x_1 - t + \frac{1}{M} (\tau^2 - M^2 r_0^2)^{1/2} \right] + \delta \left[x_1 - t - \frac{1}{M} (\tau^2 - M^2 r_0^2)^{1/2} \right] \right\} \\
 &= \frac{H(t - Mr_0)}{2\pi M} \int_{Mr_0}^t \delta \left[(x_1 - t)^2 + r_0^2 - \frac{\tau^2}{M^2} \right] d\tau \\
 &= \frac{H(t - Mr_0)}{4\pi} \int_{Mr_0}^t \frac{1}{\tau} \delta \{ M^{-1} \tau - [(x_1 - t)^2 + r_0^2]^{1/2} \} d\tau.
 \end{aligned}$$

The final integral is non-vanishing only if $M[(x_1 - t)^2 + r_0^2]^{1/2} < t$, the inequality $r_0 < [(x_1 - r)^2 + r_0^2]^{1/2}$ being always fulfilled. In these conditions, the factor $H(t - Mr_0)$ becomes superfluous and consequently formula (A.3) is obtained.

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