TRAVELLING WAVES IN HYPERELASTIC RODS*

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1. Introduction. In this paper we study the global qualitative behavior of travelling waves in an initially straight hyperelastic rod that deforms in space by bending, twisting, stretching, and shearing. The motion of such a rod is governed by a twelfth-order, quasilinear, hyperbolic system of partial differential equations; travelling waves are solutions of a corresponding system of ordinary differential equations. We show that if the constitutive functions meet a certain isotropy condition, then this system of quasilinear ordinary differential equations is completely integrable. We use this fact to obtain a qualitative description of the travelling waves. The analysis is far more complicated than that for the corresponding static problem (cf. [3]) because the hyperbolicity of the full system of partial differential equations may well destroy the monotonicity of the principal part of the travelling wave equations. These very complications lead to a number of striking results, in particular, to some strange families of solutions. We conclude our paper with discussions of shock waves in hyperelastic rods.

For the sake of contrast, it is worthwhile considering the simplest model of a nonlinear wave equation arising in nonlinear elasticity:

$$\sigma(u_x)_x = u_{tt}\,. \tag{1.1}$$

The study of this second-order conservation law has presented and still presents serious obstacles to analysis. We avoid facing these obstacles by seeking travelling wave solutions of the form $u(x, t) = v(x - ct)$. Then v satisfies the ordinary differential equation

$$\sigma(v')' = c^2 v'', \tag{1.2}$$

which has the integral

$$\sigma(v') - c^2 v' = N(\text{const}). \tag{1.3}$$

If there are no intervals on which the derivative of the function σ is constant, then the continuous solutions v', if any, of (1.3) are constant functions of $x - ct$. It can be shown that discontinuous travelling wave solutions of (the weak form of) (1.1) must also satisfy (1.3). Thus such solutions, if they exist, are characterized by v' being a piecewise constant function of $x - ct$. The physically realizable of the many possible discontinuous solutions are those that satisfy a suitable entropy condition. (We discuss the nature of discontinuous solutions in Sec. 6.) Thus the travelling waves, continuous or not, that can be sustained by (1.1) have a particularly degenerate form. We shall show that our twelfth-order system

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gives rise to travelling waves with a very rich structure. This richness can be attributed to the presence of flexural effects and of couplings of other modes of deformation with these.

We formulate the governing equations in Sec. 2. In Sec. 3 we derive a number of integrals, the existence of which supports the subsequent qualitative analysis. Sec. 4 is devoted to the general properties of solution trajectories in phase space. The material following (4.6) is a rather technical treatment of the novel ways by which singular points can arise in our complicated system; the subsequent material does not depend in a critical way on this analysis, however. In Sec. 5, we obtain a qualitative analysis of travelling waves solutions of our equations. In Figs. 11 and 12 we illustrate some travelling waves of strange form that can be sustained by our equations. In Sec. 6, we study the nature of discontinuous travelling waves that are compatible with the Rankine-Hugoniot conditions and a suitable entropy condition. We show that travelling shocks must have a special character that is largely determined by the "longitudinal" response.

Notation. Latin indices have range 1, 2, 3 and Greek indices have range 1, 2. Unless there is a statement to the contrary, such twice-repeated indices are summed over their range. We conventionally abbreviate an expression such as $f(y_1, y_2, y_3, u_1, u_2, u_3)$ by $f(y_k, u_k)$. If $\mathbb{R}^p \ni \mathbf{x} \to \mathbf{f}(\mathbf{x}) \in \mathbb{R}^q$ is continuously differentiable we denote its derivative by $\partial\mathbf{f}/\partial\mathbf{x} = \mathbf{f}_\mathbf{x}$ where $(\partial\mathbf{f}/\partial\mathbf{x})(\mathbf{x})\cdot\mathbf{h} \equiv \mathbf{f}_\mathbf{x}(\mathbf{x})\cdot\mathbf{h} \equiv (\partial/\partial\epsilon)\,\mathbf{f}(\mathbf{x} + \epsilon\mathbf{h})|_{\epsilon=0}$. A matrix of $\mathbf{f}_\mathbf{x}$ is the matrix of partial derivatives of components of $\mathbf{f}$ with respect to components of $\mathbf{x}$.

2. Formulation of the governing equations.

Kinematics. In the theory we employ, the motion of an infinitely long rod is described by three vector functions

$$\mathbb{R} \times \mathbb{R} \ni (x, t) \to \mathbf{r}(x, t),\ \mathbf{d}_1(x, t),\ \mathbf{d}_2(x, t) \in \mathbb{E}^3 \tag{2.1}$$

with $\mathbf{d}_1$ and $\mathbf{d}_2$ orthonormal. We assume that the natural state of the rod is prismatic. We take x to be the coordinate along the line of centroids of the natural state. We call the material curve of the line of centroids the *axis*. The material sections of the rod are thus identified by x. $\mathbf{r}(\cdot, t)$ is interpreted as the image of the line of centroids in the configuration at time t. $\mathbf{d}_1(x, t)$ and $\mathbf{d}_2(x, t)$ are interpreted as defining the orientation of the material section x in the configuration at time t. We set

$$\mathbf{d}_3(x, t) = \mathbf{d}_1(x, t) \times \mathbf{d}_2(x, t). \tag{2.2}$$

We introduce strains y_1, y_2, y_3 by

$$\partial\mathbf{r}/\partial x = y_k\mathbf{d}_k\ \cdot \tag{2.3}$$

We require that

$$y_3 \equiv (\partial\mathbf{r}/\partial x)\cdot\mathbf{d}_3 > 0. \tag{2.3}$$

This ensures that the rod is not so severely compressed that the local ratio of deformed to natural length of the axis is reduced to zero and that the rod is not so severely sheared that a section x is tangent to the axis at x.

The orthonormal basis $\{\mathbf{d}_k\}$ is related to a fixed orthonormal basis $\{\mathbf{e}_l\}$ by the Euler angles θ, ψ, ϕ:

$$\begin{aligned}\mathbf{d}_1 = {} & (-\sin\psi\ \sin\phi + \cos\psi\ \cos\phi\ \cos\theta)\mathbf{e}_1 \\ & + (\cos\psi\ \sin\phi + \sin\psi\ \cos\phi\cos\theta)\mathbf{e}_2 - \cos\phi\ \sin\theta\ \mathbf{e}_3\,,\end{aligned}$$

$$\mathbf{d}_2 = (-\sin\psi\ \cos\phi - \cos\psi\ \sin\phi\ \cos\theta)\mathbf{e}_1 + (\cos\psi\ \cos\phi - \sin\psi\ \sin\phi\ \cos\theta)\mathbf{e}_2 + \sin\phi\ \sin\theta\mathbf{e}_3$$

$$\mathbf{d}_3 = \cos\psi\ \sin\theta\ \mathbf{e}_1 + \sin\psi\ \sin\theta\ \mathbf{e}_2 + \cos\theta\ \mathbf{e}_3\ . \tag{2.4}$$

(These angles are the same as those of [2, 3, 9].) Since $\{\mathbf{d}_k\}$ is orthonormal, there is a vector $\mathbf{u}$ such that

$$\partial\mathbf{d}_k/\partial x = \mathbf{u} \times \mathbf{d}_k\ . \tag{2.5}$$

The components of $\mathbf{u}$ with respect to the basis $\{\mathbf{d}_k\}$ are

$$u_k = \tfrac{1}{2}\, e_{klm}\, (\partial\mathbf{d}_l/\partial x)\cdot\mathbf{d}_m\ , \tag{2.6}$$

where $\{e_{klm}\}$ are the components of the alternating tensor. The substitution of (2.4) into (2.6) yields

$$\begin{aligned} u_1 &= \theta_x\ \sin\phi - \psi_x\ \sin\theta\ \cos\phi, \\ u_2 &= \theta_x \cos\phi + \psi_x \sin\theta\ \sin\phi, \\ u_3 &= \theta_x + \psi_x \cos\theta. \end{aligned} \tag{2.7}$$

The functions $\{u_k\}$ measure flexure and twist. The full set of strains for our problem are $\{y_k\ ,\ u_k\}$. Our assumption that the natural state is prismatic implies that in this state $\mathbf{d}_3$ coincides with $\mathbf{r}_x$ and $\{\mathbf{d}_k\}$ are constant functions of x. The values of the strains in the natural state are accordingly

$$y_1 = y_2 = 0, \qquad y_3 = 1, \qquad u_k = 0. \tag{2.8}$$

Equations of motion. Let

$$\mathbf{n}(x,\ t) \equiv n_k(x,\ t)\mathbf{d}_k(x,\ t) \tag{2.9}$$

be the resultant force and

$$\mathbf{m}(x,\ t) = m_k(x,\ t)\mathbf{d}_k(x,\ t) \tag{2.10}$$

be the resultant couple acting across the section x at time t. The inertia of the rod is characterized by the functions

$$\mathbb{R} \ni x \to (\rho A)(x),(\rho J_1)(x),(\rho J_2)(x) \in (0,\ \infty). \tag{2.11}$$

$(\rho A)(x)$ represents the natural mass density per unit length and $(\rho J_1)(x)$ and $(\rho J_2)(x)$ represent the principal mass moments of inertia of the cross section in its natural state. In our work we assume that ρA, ρJ_1, ρJ_2 are constants and that the cross-section is "dynamically symmetric", i.e., that

$$J_1 = J_2 \equiv J. \tag{2.12}$$

If there are no net forces or couples distributed along the length of the rod, then the equations of motion of the rod are

$$\mathbf{n}_x = \rho A \mathbf{r}_{tt}, \tag{2.13}$$

$$\mathbf{m}_x + \mathbf{r}_x \times \mathbf{n} = \rho J \frac{\partial}{\partial t}\left(\mathbf{d}_\sigma \times \frac{\partial}{\partial t}\mathbf{d}_\sigma\right) \tag{2.14}$$

(cf. [1]).

Constitutive equations. We assume that the rod consists of a homogeneous, transversely isotropic, hyperelastic material. The distinguished direction of isotropy is along the axis. This material is characterized by a strain-energy function

$$W: \{y_k, u_k: y_3 > 0\} \to [0, \infty), \tag{2.15}$$

which is an isotropic scalar function of the two 2-vectors $y_\alpha \mathbf{d}_\alpha$ and $u_\alpha \mathbf{d}_\alpha$ and of the two scalars y_3 and u_3. For simplicity, we assume that W is twice-continuously differentiable, is strictly convex, and satisfies

$$W \to \infty \quad \text{as} \quad y_\rho \to \pm\infty, \qquad y_3 \to 0, \qquad y_3 \to \infty, \qquad u_k \to \pm\infty. \tag{2.16}$$

We set

$$\hat{n}_k = \partial W/\partial y_k, \qquad \hat{m}_k = \partial W/\partial u_k. \tag{2.17}$$

Then the strict convexity of W implies that the matrix of partial derivatives of $\{n_k, m_k\}$ with respect to $\{y_l, u_l\}$ is positive-definite. Moreover, our assumption of transverse isotropy implies that if f denotes W, $\hat{n}_3$, or $\hat{m}_3$ and if g_ρ denotes $\hat{n}_\rho$ or $\hat{m}_\rho$, then

$$f(y_\sigma, y_3, u_\sigma, u_3) = f(Q_{\sigma\nu}y_\nu, y_3, Q_{\sigma\nu}u_\nu, u_3), \tag{2.18a}$$

$$g_\rho(y_\sigma, y_3, u_\sigma, u_3) = Q_{\tau\rho}g_\tau(Q_{\sigma\nu}y_\nu, y_3, Q_{\sigma\nu}u_\nu, u_3) \tag{2.18b}$$

for all 2×2 orthogonal tensors with components $\{Q_{\tau\rho}\}$. The identity (2.18b) implies that $\hat{n}_2$ and $\hat{m}_2$ vanish when y_2 and u_2 vanish. We strengthen this condition by requiring that

$$\hat{n}_2 = 0 \quad \text{if and only if} \quad y_2 = 0, \tag{2.19a}$$

$$\hat{m}_2 = 0 \quad \text{if and only if} \quad u_2 = 0. \tag{2.19b}$$

Note that (2.18b) implies that the relations obtained from (2.19) by replacing the index 2 with 1 are likewise valid.

The constitutive equations are

$$\mathbf{n}(x, t) = \hat{n}_k(y_l(x, t), u_l(x, t))\mathbf{d}_k(x, t), \tag{2.20a}$$

$$\mathbf{m}(x, t) = \hat{m}_k(y_l(x, t), u_l(x, t))\mathbf{d}_k(x, t). \tag{2.20b}$$

The full equations of motion for our transversely isotropic rod are the system of hyperbolic conservation laws consisting of (2.2), (2.3), (2.4), (2.7), (2.13), (2.14), (2.20). The constitutive theory of this model was developed in [2, 3]. As in [3], the assumption (2.17) is made for simplicity only. The monotonicity of $\{y_k, u_k\} \to \{\hat{n}_l(y_k, u_k), \hat{m}_l(y_k, u_k)\}$ and the growth condition (2.16), which play critical roles in [2, 3], are not central in our development here. The isotropy condition (2.18) is critical for our analysis. (Note that in [2, 3] some indices in the definition of isotropy appear in the wrong order; this does not affect the subsequent analysis in these papers.)

Travelling waves. Let f represent any dependent variable appearing in the equations of motion. We seek travelling wave solutions of this system in the form

$$f(x, t) = \bar{f}(x - ct) \tag{2.21}$$

where $c \in \mathbb{R}$. c is called the *wave-speed.* If we substitute representations of the form (2.21) into the equations of motion, set

$$s = x - ct, \tag{2.22}$$

drop the bars over the functions of s, and denote differentiation with respect to s by a prime, then (2.13) and (2.14) reduce to

$$\mathbf{n}' = \rho c^2 A \mathbf{r}'', \tag{2.23}$$

$$\mathbf{m}' + \mathbf{r}' \times \mathbf{n} = \rho c^2 J(\mathbf{d}_\rho \times \mathbf{d}_\rho')'. \tag{2.24}$$

3. Integrals of the governing equations. The integral of (2.23) is

$$\mathbf{n} - \rho c^2 A \mathbf{r}' = N\mathbf{e}_3\,, \tag{3.1}$$

where, without loss of generality, we have taken the constant vector of integration to be a scalar multiple N of the fixed vector $\mathbf{e}_3$. (We could contemplate problems in which $\mathbf{n} - \rho c^2 A\mathbf{r}'$ is merely piecewise constant, but, as we show in Sec. 6, this would prevent $\{\mathbf{r}, \mathbf{d}_\alpha\}$ from being a weak solution of the equations. Thus $N\mathbf{e}_3$ is a true constant. These same remarks apply to the constants α, β, h to be introduced below.) The replacement of $\mathbf{n}(s)$ by $\hat{\mathbf{n}}_k(y_l(s), u_l(s))\mathbf{d}_k(s)$ reduces (3.1) to the three integrals

$$\hat{n}_1(y_l\,, u_l) - \rho c^2 A y_1 = -N \sin\theta\, \cos\phi, \tag{3.2}$$

$$\hat{n}_2(y_l\,, u_l) - \rho c^2 A y_2 = N \sin\theta\, \sin\phi, \tag{3.3}$$

$$\hat{n}_3(y_l\,, u_l) - \rho c^2 A y_3 = N \cos\theta. \tag{3.4}$$

From (3.1) and (2.24) we get

$$\mathbf{m}\cdot\mathbf{e}_3 - \rho c^2 J(\mathbf{d}_\sigma \times \mathbf{d}_\sigma')\cdot\mathbf{e}_3 = \alpha(\text{const}). \tag{3.5}$$

The constitutive relation (2.20b) reduces (3.5) to the integral

$$\begin{aligned}&\{[-\hat{m}_1(y_l\,, u_l)\cos\phi + \hat{m}_2(y_l\,, u_l)\sin\phi] - \rho c^2 J[-u_1\cos\phi + u_2\sin\phi]\}\sin\theta \\ &\quad + [\hat{m}_3(y_l\,, u_l) - 2\rho c^2 J u_3]\cos\theta = \alpha.\end{aligned} \tag{3.6}$$

We dot (2.23) with $\mathbf{r}'$ and (2.24) with $\mathbf{u}$ and then add the resulting expressions to get

$$\mathbf{r}'\cdot[\mathbf{n}' - \mathbf{u}\times\mathbf{n}] + \mathbf{u}\cdot[\mathbf{m}' - \mathbf{u}\times\mathbf{m}] = \rho c^2[A\mathbf{r}'\cdot\mathbf{r}'' + J(\mathbf{d}_\rho \times \mathbf{d}_\rho')'\cdot\mathbf{u}], \tag{3.7a}$$

which reduces to

$$y_k n_k' + u_k m_k' = \rho c^2[A\mathbf{r}'\cdot\mathbf{r}'' + (\mathbf{u} + u_3\mathbf{d}_3)'\cdot\mathbf{u}]. \tag{3.7b}$$

The substitution of (2.20) into (3.7b) and the use of (2.17) convert (3.7b) to

$$y_k\left(\frac{\partial W}{\partial y_k}\right)' + u_k\left(\frac{\partial W}{\partial u_k}\right)' = \rho\frac{c^2}{2}[A y_k y_k + J(u_\sigma u_\sigma + 2u_3^2)]', \tag{3.7c}$$

which yields the integral

$$y_k\left(\frac{\partial W}{\partial y_k}\right) + u_k\left(\frac{\partial W}{\partial u_k}\right) - W - \rho\frac{c^2}{2}[A y_k y_k + J(u_\sigma u_\sigma + 2u_3^2)] = h(\text{const}). \tag{3.8}$$

We have thus obtained five integrals. To render our system completely integrable we require a sixth integral. This we obtain by invoking the isotropy condition (2.18) and the uncoupling condition (2.19).

Now (2.20) and (2.24) imply that

$$\hat{m}_3' + u_1\hat{m}_2 - u_2\hat{m}_1 + y_1\hat{n}_2 - y_2\hat{n}_1 = 2\rho c^2 J u_3' . \tag{3.9}$$

If $(u_1, u_2) \neq (0, 0)$, then we choose

$$(Q_{\rho\sigma}) = [u_\gamma u_\gamma]^{-1/2}\begin{pmatrix} u_1 & u_2 \\ u_2 & -u_1 \end{pmatrix} \tag{3.10}$$

in (2.18). Then (2.18b) and (2.19b) imply that

$$u_1\hat{m}_2 - u_2\hat{m}_1 = -\hat{m}_2\left(\frac{u_1y_1 + u_2y_2}{(u_\gamma u_\gamma)^{1/2}}, \frac{u_2y_1 - u_1y_2}{(u_\gamma u_\gamma)^{1/2}}, y_3, (u_\gamma u_\gamma)^{1/2}, 0, u_3\right) = 0. \tag{3.11}$$

In a similar way (2.18b) and (2.19a) imply that

$$y_1\hat{n}_2 - y_2\hat{n}_1 = 0. \tag{3.12}$$

Thus (3.9) yields the integral

$$\hat{m}_3(y_l, u_l) - 2\rho c^2 J u_3 = \beta(\text{const}). \tag{3.13}$$

For the purpose of simplifying our integrals we set

$$\begin{aligned}
x_1 &= y_1^* = -y_1 \cos\phi + y_2 \sin\phi, \\
x_2 &= y_2^* = y_1 \sin\phi + y_2 \cos\phi, \\
x_3 &= y_3^* = y_3, \\
x_4 &= u_1^* = -u_1 \cos\phi + u_2 \sin\phi = \psi' \sin\theta, \\
x_5 &= u_2^* = u_1 \sin\phi + u_2 \cos\phi = \theta', \\
x_6 &= u_3^* = u_3, \qquad \mathbf{x} = (x_1, \cdots, x_6).
\end{aligned} \tag{3.14}$$

Using (2.18) with

$$(Q_{\rho\sigma}) = \begin{pmatrix} -\cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \tag{3.15}$$

we reduce (3.2)–(3.4), (3.6, (3.8), (3.13) to

$$\hat{n}_1(\mathbf{x}) - \rho c^2 A y_1^* - N \sin\theta = 0, \tag{3.16a}$$

$$\hat{n}_2(\mathbf{x}) - \rho c^2 A y_2^* = 0, \tag{3.16b}$$

$$\hat{n}_3(\mathbf{x}) - \rho c^2 A y_3^* - N \cos\theta = 0, \tag{3.16c}$$

$$\hat{m}_1(\mathbf{x}) - \rho c^2 J u_1^* - (\alpha - \beta\cos\theta)/\sin\theta = 0, \tag{3.16d}$$

$$y_k^*\hat{n}_k(\mathbf{x}) + u_k^*\hat{m}_k(\mathbf{x}) - W(\mathbf{x}) - \rho\frac{c^2}{2}\{A y_k^* y_k^* + J[u_2^* u_2^* + 2(u_3^*)^2]\} = h, \tag{3.16e}$$

$$\hat{m}_3(\mathbf{x}) - 2\rho c^2 J u_3^* = \beta. \tag{3.16f}$$

(The simplifications embodied in the reduction of (3.2)–(3.4), (3.6), (3.8), (3.13) to (3.16) via (3.14), (3.15) are equivalent to a judicious choice of the axes $\mathbf{e}_1$, $\mathbf{e}_2$. We had previously introduced various simplifications by our choice of $\mathbf{e}_3$.) We can substitute (3.16a-d, f) into 3.16e) to reduce (3.16e) to the form

$$y_1^*N \sin\theta + y_3^*N\cos\theta + \rho(c^2/2)Ay_k^*y_k^* + u_1^*\left(\frac{\alpha-\beta\cos\theta}{\sin\theta}\right) + \rho(c^2/2)J(u_1^*)^2$$
$$+ u_2^*\hat{m}_2(\mathbf{x}) - \rho(c^2/2)J(u_2^*)^2 + u_3^*\beta + \rho c^2 J(u_3^*) - W(\mathbf{x}) = h. \tag{3.17}$$

By differentiating (3.16e) or (3.17) with respect to s, using (3.16), and cancelling $u_2^* = \theta'$, we obtain

$$\frac{\partial\hat{m}_2}{\partial y_k}(\mathbf{x})y_k^{*\prime} + \frac{\partial\hat{m}_2}{\partial u_k}(\mathbf{x})u_k^{*\prime} - \rho c^2 J u_2^{*\prime}$$
$$= N(y_3^*\sin\theta - y_1^*\cos\theta) - \gamma(\theta;\alpha,\beta)u_1^* \equiv \delta(\mathbf{x},\theta;\alpha,\beta,N) \tag{3.18}$$

where

$$\gamma(\theta;\alpha,\beta) \equiv (\beta - \alpha\cos\theta)/\sin^2\theta. \tag{3.19}$$

4. General qualitative properties of solutions. For fixed values of the parameters α, β, N, c^2, h, the system (3.16) may determine a family of curves in $(\mathbf{x}, \theta)$-space, which is the half-space of $\mathbb{R}^7$ defined by $x_3 \equiv y_3^* > 0$. By determining the qualitative behavior of these curves we determine the qualitative behavior of travelling waves. To handle the unwieldy system (3.16) we study the projections of these curves on the (θ, θ')-plane. Now for fixed values of

$$(\alpha, \beta, N, c^2) \equiv \lambda, \tag{4.1}$$

the subsystem (3.16a-d,f) may possess multivalued solutions for y_1^*, y_2^*, y_3^*, u_1^*, u_3^* as functions of $\theta, u_2^* = \theta'$, and λ. The substitution of these multivalued solutions into (3.16e), or equivalently into (3.17), reduces these equations to the family of equations we denote by

$$G(\theta, \theta'; \lambda) = h, \tag{4.2}$$

G being a multivalued function. The curves in the (θ,θ')-plane given by (4.2) are the projections onto this plane of the curves defined by (3.16). We shall call these curves the (*phase plane*) *trajectories* of (4.2).

(4.3) PROPOSITION. The trajectories of (4.2) are symmetric about the θ-axis, are symmetric about the θ'-axis, and have period 2π in θ.

Proof. The use of

$$(Q_{\rho\sigma}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.4}$$

in (2.18) shows that (3.16a-d, e) is invariant under the substitution $u_2^* \to -u_2^*$. The use of the negative of (4.4) in (2.18) shows that (3.16) is invariant under the substitution $(\theta, y_1^*, u_1^*) \to (-\theta, -y_1^*, -u_1^*)$. Direct analysis of (3.16) shows that these equations are invariant under the substitution $\theta \to \theta + 2\pi$. It then follows that (4.2) is invariant under the substitutions $\theta \to -\theta$, $\theta \to \theta + 2\pi$, $\theta' \to -\theta'$.

The curves of (3.16) are the integral curves of the seventh-order system of ordinary differential equations

$$[\mathbf{M}(\mathbf{x}) - c^2\mathbf{K}]\mathbf{x}' = \mathbf{a}(\mathbf{x},\theta;\alpha,\beta,N), \tag{4.5a}$$

$$\theta' = x_5 = u_2^*, \tag{4.5b}$$

where

$$\mathbf{M} = \begin{pmatrix} \dfrac{\partial \hat{n}_k}{\partial y_l} & \dfrac{\partial \hat{n}_k}{\partial u_l} \\ \dfrac{\partial \hat{m}_k}{\partial y_l} & \dfrac{\partial \hat{m}_k}{\partial u_l} \end{pmatrix}, \rho^{-1}\mathbf{K} = \operatorname{diag}(A,A,A,J,J,2J), \mathbf{a}(\mathbf{x},\theta;\alpha,\beta,N) = \begin{pmatrix} u_2^* N \cos\theta \\ 0 \\ -u_2^* N \sin\theta \\ u_2^*\gamma(\theta;\alpha,\beta) \\ \delta(y_1^*,\theta;\alpha,\beta,N) \\ 0 \end{pmatrix}. \tag{4.5c}$$

This system is obtained from (3.16) by differentiating it with respect to s. (Cf. (3.18).) It is equivalent to a subset of our original system of governing equations. We shall study singular points of (4.5) and the singular points of the projections of the integral curves on the (θ,θ')-plane.

Since the matrix $\mathbf{M}(\mathbf{x}) - c^2\mathbf{K}$ can be singular and since $\mathbf{a}$ can be unbounded, the determination of singular points for systems of the form (4.5) is more delicate than that for systems in standard form. Let $(\bar{\mathbf{x}},\bar{\theta}) \in \mathbb{R}^7$ with $\bar{x}_3 > 0$. We say that the algebraic system

$$[\mathbf{M}(\bar{\mathbf{x}}) - c^2\mathbf{K}]\mathbf{x}' = \mathbf{a}(\bar{\mathbf{x}},\bar{\theta};\alpha,\beta,N), \qquad \theta' = \bar{x}_5 \tag{4.6}$$

(cf. (4.5)) has an *infinite solution* if (4.6) fails to have a solution $(\bar{\mathbf{x}}',\bar{\theta}')$ in $\mathbb{R}^7$. (If (4.6) has an infinite solution, then $|\mathbf{a}(\bar{\mathbf{x}}, \bar{\theta}; \alpha, \beta, N)| = \infty$ or $\det[\mathbf{M}(\bar{\mathbf{x}}) - c^2\mathbf{K}] = 0$.) Otherwise (4.6) is said to have a *finite solution*. We say that a solution $(\bar{\mathbf{x}}',\bar{\theta}')$, finite or not, of (4.6) is the solution of (4.6) at $(\bar{\mathbf{x}},\bar{\theta})$. An *extended solution* is a solution that is either finite or infinite. Thus (4.6) always has an extended solution. Of the several kinds of infinite solutions, the most important for our work are:

(i) There is an orthogonal transformation $\mathbf{A}$ from $\mathbb{R}^6$ to itself and there are real numbers $\bar{z}_2, \cdots, \bar{z}_6, \bar{\theta}'$ with the following property: for arbitrary $\epsilon > 0$, there is a neighborhood of $(\bar{\mathbf{x}},\bar{\theta})$ such that if $(\mathbf{x},\theta)$ is in this neighborhood then (4.5) has a unique solution $(\mathbf{x}', \theta') \in \mathbb{R}^7$ with $z_1 > \epsilon^{-1}$, $|\theta' - \bar{\theta}'| + \sum_{k=2}^{6} |z_k - \bar{z}_k| \geq \epsilon$, where $\mathbf{z} = \mathbf{A}^{-1}\mathbf{x}'$. In this case we can represent the solution of (4.6) as $(\bar{\mathbf{x}}',\bar{\theta}') = (\mathbf{A}(\infty,\bar{z}_2, \ldots, \bar{z}_6),\bar{\theta}')$. Thus $(\bar{\mathbf{x}}',\bar{\theta}')$ determines an oriented line through the origin of $\mathbb{R}^7$; this line is in the direction of the vector $(\mathbf{A}(1,0,0,0,0,0),0)$. This case is very special.

(ii) There is an orthogonal transformation $\mathbf{A}$ and there are real numbers $\bar{z}_2, \cdots, \bar{z}_6, \bar{\theta}'$ with the following property: for arbitrary $\epsilon > 0$ there is a neighborhood V of $(\mathbf{x},\theta)$ such that

$$V = V_+ \cup V_0 \cup V_- \text{ with } \{V_+,V_0,V_-\} \text{ disjoint}, \tag{4.7}$$

$$\operatorname{meas} V_+ > 0, \operatorname{meas} V_0 = 0, \operatorname{meas} V_- > 0, \tag{4.8}$$

if $(\mathbf{x}_\pm,\theta_\pm) \in V_\pm$, then (4.6) has unique solutions $(\mathbf{x}_\pm',\theta_\pm')$ at $(\mathbf{x}_\pm,\theta_\pm)$ with $z_1^+ > \epsilon^{-1}$, $z_1^- < -\epsilon^{-1}$, $|\theta_\pm' - \bar{\theta}'| + \sum_{k=2} |z_k^\pm - \bar{z}_k| < \epsilon$ where $\mathbf{z}^\pm = \mathbf{A}^{-1}\mathbf{x}_\pm'$ (4.9)

In this case we represent the solution of (4.6) as $(\bar{\mathbf{x}}',\bar{\theta}') = (\mathbf{A}(\pm\infty, \bar{z}_2, \cdots, \bar{z}_6),\bar{\theta}')$. Thus $(\bar{\mathbf{x}}',\bar{\theta}')$ determines the *unoriented* line passing through the origin and $(\mathbf{A}(1,0,0,0,0,0),0)$. This case is not uncommon. To see how it arises, let $\mathbf{A}(\mathbf{x})$ be an orthogonal matrix such that $\mathbf{A}^{-1}(\mathbf{x})[\mathbf{M}(\mathbf{x}) - c^2\mathbf{K}]\mathbf{A}(\mathbf{x})$ is diagonal. (Note that $\mathbf{M} - c^2\mathbf{K}$ is symmetric.) If $\operatorname{rank}[\mathbf{M}(\bar{\mathbf{x}}) - c^2\mathbf{K}] = 5$ and if $\mathbf{a}(\bar{\mathbf{x}},\bar{\theta};\alpha,\beta,N)$ is finite, say, then (4.5c) decomposes into five scalar equations with unique real solutions $z_2, \cdots, z_6$ and a sixth equation of the form

$$[\mu(x) - \rho c^2 k]z_1 = a(\mathbf{x},\theta';\alpha,\beta,N) \tag{4.10}$$

where $\mu(\bar{\mathbf{x}}) = \rho c^2 k$. If rank$[\mathbf{M}(\mathbf{x}) - c^2\mathbf{K}] = 6$ a.e. in a neighborhood of $\bar{\mathbf{x}}$, then $\mu(\mathbf{x}) - c^2 k$ assumes both signs near $\bar{\mathbf{x}}$. In this case, if $a(\bar{\mathbf{x}},\bar{\theta};\alpha,\beta,N) \neq 0$, then we may denote the solution of (4.10) at $(\mathbf{x},\theta) = (\bar{\mathbf{x}},\bar{\theta})$ as $z_1 = \pm\infty$.

(iii) Eq. (4.6) has an infinite solution for which the infinite values of the components cannot be restricted to a single direction by the introduction of a suitable orthogonal transformation as in cases (i) and (ii) above. This situation can arise when the diagonalization of (4.6) yields at least two scalar equations of the form (4.10) with $a \neq 0$.

We say that system (3.16) has a *singular* point at $(\bar{\mathbf{x}},\bar{\theta}) \in \mathbb{R}^7$ with $\bar{x}_3 > 0$ wherever the set of extended solutions $(\mathbf{x}',\theta')$ of the (4.6) fails to define a unique oriented tangent line at $(\bar{\mathbf{x}},\bar{\theta})$. Otherwise, $(\bar{\mathbf{x}},\bar{\theta})$ is called a *regular point*. Thus $(\bar{\mathbf{x}},\bar{\theta})$ is regular if $\det(\mathbf{M}(\bar{\mathbf{x}}) - c^2\mathbf{k}) \neq 0$ and $0 < |\mathbf{a}(\bar{\mathbf{x}},\bar{\boldsymbol{\theta}};\alpha,\beta,N)| < \infty$ or if (4.6) has an infinite solution of type (i). The most important ways that singular points can arise are:

a) System (4.6) has the unique solution $(\mathbf{x}', \theta') = (0, 0)$. In this case $\det[\mathbf{M}(\bar{\mathbf{x}}) - c^2\mathbf{K}] \neq 0$ and $\mathbf{a}(\bar{\mathbf{x}}, \bar{\theta}; \alpha, \beta, N) = \mathbf{0}$. If $N \neq 0$, this latter requirementis equivalent to $\bar{u}_2^* \equiv \bar{\theta}' = 0$, $\delta(\bar{y}_i^*, \bar{\theta}; \alpha, \beta, N) = 0$.

b) The solutions of (4.6) form a one-dimensional linear manifold in $\mathbb{R}^7$. In this case rank$[\mathbf{M}(\bar{\mathbf{x}}) - c^2\mathbf{k}] = 5$ and $\mathbf{a}(\bar{\mathbf{x}},\bar{\theta};\alpha,\beta,N)$ is orthogonal to the null space of $\mathbf{M}(\bar{\mathbf{x}}) - c^2\mathbf{K}$.

c) The solutions of (4.6) form a linear manifold of dimension ≥ 2 in $\mathbb{R}^7$. In this case rank$[\mathbf{M}(\mathbf{x}) - c^2\mathbf{K}] \leq 4$ and $\mathbf{a}(\bar{\mathbf{x}}, \bar{\theta}; \alpha, \beta, N)$ is orthogonal to the null space of $\mathbf{M}(\bar{\mathbf{x}}) - c^2\mathbf{K}$.

d) System (4.6) has an infinite solution of type (ii).

e) System (4.6) has an infinite solution of type (iii). Note that in cases (b) and (d), the solutions of (4.6) define a unique unoriented tangent line at $(\bar{\mathbf{x}},\bar{\theta})$. We do not pause to give an exhaustive classification of the singular points of (3.16). We merely note that case (a) arises when $\bar{u}_2^* = 0$, $\delta(\bar{\mathbf{x}},\bar{\theta};\alpha,\beta,\gamma) = 0$ and the remaining cases imply that $\det[\mathbf{M}(\bar{\mathbf{x}}) - c^2\mathbf{K}] = 0$ or $|\gamma(\theta;\alpha,\beta)| = \infty$.

For a more refined analysis of these cases and for the subsequent analysis, we note that (4.6) uncouples when $\bar{u}_2^* = 0$. Let $\mathbf{M}_{|5|}$ and $\mathbf{K}_{|5|}$ denote the matrices obtained by deleting the fifth row and column from $\mathbf{M}$ and $\mathbf{K}$ and let $\mathbf{x}_{|5|}$ and $\mathbf{a}_{|5|}$ denote the 5-vectors obtained by deleting the fifth entry from $\mathbf{x}$ and $\mathbf{a}$. If $\bar{u}_2^* = 0$, then (2.17) and (2.19) ensure that (4.6) reduces to

$$[\mathbf{M}_{|5|}(\bar{\mathbf{x}}) - c^2\mathbf{K}_{|5|}]\cdot\mathbf{x}_{|5|}' = \mathbf{a}_{|5|}(\theta,u_2^*;\alpha,\beta,N), \tag{4.11a}$$

$$[(\partial\hat{m}_2/\partial u_2)(\bar{\mathbf{x}}) - c^2 J](u_2^*)' = \delta(\bar{\mathbf{x}},\bar{\theta};\alpha,\beta,N). \tag{4.11b}$$

We now turn to the study of the singular points of (4.2), which is the equation of the projections of the trajectories of (3.16) or (4.5) on the (θ,θ')-plane. We must proceed with some care because the collection of singular points of the projection need not be comparable to the collection of projections of the singular points. Suppose that (3.16a-d, f) has a multivalued solution for $\{y_k^*, u_1^*, u_3^*\}$ in terms of $(\theta, \theta'; \lambda)$ for $(\theta, \theta', \lambda)$ in some region $\mathfrak{D}$ of $\mathbb{R}^6$. We set

$$v = u_2^* = \theta'. \tag{4.12}$$

This solution defines a multivalued mapping

$$\mathfrak{D} \ni (\theta, v; \lambda) \to \tilde{\mathbf{x}}(\theta, v; \lambda) \in \{\mathbf{x} \in \mathbb{R}^6: x_3 > 0\}. \tag{4.17}$$

For any function $(\mathbf{x}, \theta; \lambda) \to f(\mathbf{x}, \theta; \lambda)$ we set $\tilde{f}(\theta, v; \lambda) = f(\tilde{\mathbf{x}}(\theta, v; \lambda), v; \lambda)$. From (4.5) we get

$$q(\theta, v, \lambda)v' \equiv \left[\frac{\partial \tilde{m}_2}{\partial v}(\theta, v; \lambda) - c^2 J\right]v' = \tilde{\delta}(\theta, v; \lambda) - \frac{\partial \tilde{m}_2}{\partial \mathbf{x}_{[5]}}(\mathbf{x}(\theta, v; \lambda))\tilde{\mathbf{x}}'_{[5]} \equiv p(\theta, v; \lambda), \tag{4.18a}$$

$$\theta' = v, \tag{4.18b}$$

where $\mathbf{x}'_{[5]}$ satisfies

$$[\tilde{\mathbf{M}}_{[5]}(\theta, v; \lambda) - c^2\mathbf{K}_{[5]}]\mathbf{x}'_{[5]} = \mathbf{a}_{[5]}(\theta, v; \alpha, \beta, N). \tag{4.19}$$

A point (θ, v) with $(\theta, v, \lambda) \in \mathfrak{D}$ is a *singular point* of (4.2) for a given branch $\tilde{\mathbf{x}}$ if (4.18) evaluated on that branch fails to define a unique oriented tangent line at (θ, v) in the (θ, v)-plane.

If $q(\theta, v; \lambda) = 0$, then (θ, v) is singular. In this case (θ', v') defines an unoriented tangent line in the plane, which is parallel to the v-axis. If $q(\theta, v; \lambda) \neq 0$ we have several cases. If $0 < |v| + |p(\theta, v; \lambda)| < \infty$ then (θ, v) is regular. If $v = p(\theta, v, \lambda) = 0$, then (θ, v) is singular. (In this case $p(\theta, v; \lambda) = \tilde{\delta}(\theta, 0; \lambda)$.) Since $(\theta', v') = (0, 0)$, relation (4.18) does not even define an unoriented tangent line at $(\theta, 0)$. If $|\gamma(\theta; \alpha, \beta)| = \infty$, then neither $p(\theta, v; \lambda)$ nor $q(\theta, v; \lambda)$ need be well-defined without further constitutive hypotheses. If these values are well-defined (as extended limits of the values of p and q at neighboring points), then we can ascertain whether (θ, v) is singular. If $|p(\theta, v; \lambda)| = \infty$, then (4.18a) implies that $|v'| = \infty$. Whether (θ, v) is singular or regular, Eq. (4.18) defines at least an unoriented tangent line parallel to the v-axis.

These considerations also show that a curve of (4.2) has a tangent (not necessarily oriented) parallel to the v-axis where

a) $v = 0, p(\theta, 0; \lambda) \neq 0$,
b) $v = 0, p(\theta, 0; \lambda) = 0, q(\theta, 0, \lambda) = 0$,
c) $|p(\theta, v; \lambda)| = \infty$.

Case (c) is the most noteworthy: It can occur only where $|\gamma(\theta; \alpha, \beta)| = \infty$. Since γ is independent of v, the condition $|\gamma(\theta; \alpha, \beta)| = \infty$ determines a family of separatrices (possibly unoriented) that are parallel to the v-axis.

5. Global behavior of solutions of special problems. We have noted that the global qualitative behavior of travelling waves can be determined from the trajectories of (4.2). The analysis of this multivalued equation devolves upon the nature of the solutions of the fifth-order algebraic system (3.16a-d, f). For $c \neq 0$ the operator in this system is strictly monotone and coercive in $\mathbf{x}_{[5]}$; the equations accordingly have a unique solution for $\mathbf{x}_{[5]}$ in terms of θ, v, λ (cf. [2,3]). For $c \neq 0$, the luxury of a global implicit function theorem is not generally available to us. Thus an exhaustive study of (4.2), as performed in [3] for $c = 0$, seems to be a task that is at best tedious and at worst fruitless. The study of a few simple cases, however, exhibits a budget of novel effects, which seem typical of this class of problems and which illustrate the role of constitutive functions.

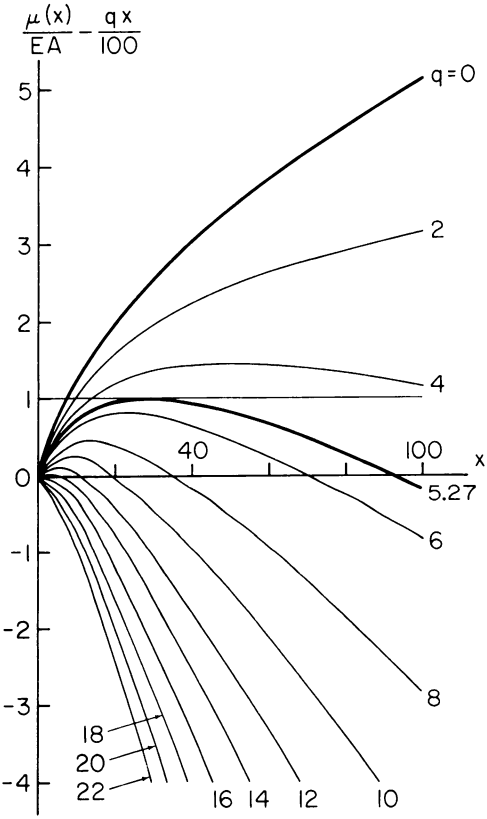
We say that a function f: $[0, \infty) \to \mathbb{R}$ is *sublinear* if there is a number K such that $f(x) \leq K(1 + x)$ for $x \in [0, \infty)$. We say that a function f: $\mathbb{R} \to \mathbb{R}$ is *sublinear* if its restriction to $[0, \infty)$ is sublinear and if $(-\infty, 0) \ni x \to -f(-x)$ is sublinear. An increasing function that is not sublinear is called *superlinear*.

Let μ represent any one of the five functions

$$y_\sigma \to \hat{n}_\sigma(y_l, u_l), \qquad \sigma = 1, 2; \qquad u_k \to \hat{m}_k(y_l, u_l), \qquad k = 1, 2, 3.$$

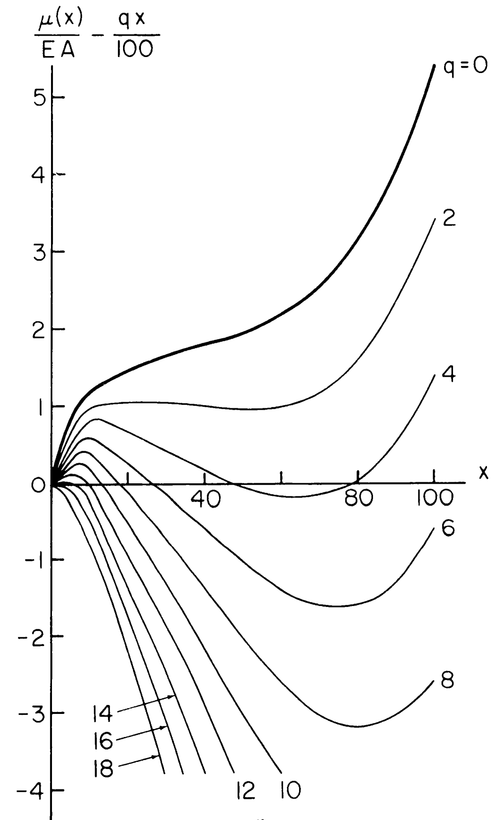
Conditions (2.18) and (2.19) imply that the first four of these functions are odd; we assume the same for the fifth. Thus μ is odd. The strict convexity of W implies that μ is strictly increasing. In Fig. 1 we depict typical forms of $x \to \mu(x) - ax$, $a \geq 0$ for μ sublinear and superlinear. (a clearly stands for $\rho c^2 A$, $\rho c^2 J$, or $2\rho c^2 J$.) Note that in the sublinear case $\mu(x) - ax \to -\infty$ as $x \to \infty$ for $a > 0$ and that for sufficiently large a the function $x \to \mu(x) - ax$ is decreasing. In the superlinear case, $\mu(x) - ax \to \infty$ as $x \to \infty$. In Fig. 2 we depict the form of $y_3 \to \hat{n}_3(y_k, u_k) - ay_3$ when $y_3 \to \hat{n}_3(y_k, u_k)$ is sublinear. We account for (2.16) by requiring that $\hat{n}_3(y_k, u_k) \to -\infty$ as $y_3 \to 0$. From Figure 2, we immediately read off

(5.1) Proposition. Let $y_3 \to \hat{n}_3(y_k, u_k)$ be uniformly sublinear. (This means that the constant K in the definition can be taken to be independent of $\{y_\sigma, u_l\}$.) To each N there corresponds a number $a_0(N) > 0$ such that (3.16c) has no real solution y_3^* if $\rho c^2 A > a_0$. There is a number $a_1 > 0$ such that if $\rho c^2 A > a_1$, then (3.16c) has no real solution y_3^* when $|N|$ is sufficiently small. (There can be no travelling waves in either of these cases.) Finally



(a)

Fig. 1a. Forms of $(\mu(x)/EA) - (qx/100)$ when μ is sublinear. (E is a constitutive constant.) The actual curves shown correspond to the constitutive equation $3(EA)^{-1}\mu(x) = (4 + 3x)^{1/2} - 2$. This figure has an odd continuation to the left.



(b)

Fig. 1b. Forms of $(\mu(x)/EA) - (qx/100)$ when μ is superlinear. This figure has an odd continuation to the left.

to each fixed N there is a number $a_2(N)$ with $0 < a_2(N) < a_0(N)$ such that there are values of θ for which (3.16c) fails to have a real solution when $a_2(N) < \rho c^2 A \leq a_0(N)$. (This means that such travelling waves as may exist under these circumstances are characterized by having θ confined to a fixed range.)

In the rest of this section we restrict our attention to the case in which $\hat{n}_1$ depends only on y_1, $\hat{n}_2$ depends only on y_2, etc. This assumption eliminates a number of interesting effects due to nonlinear couplings (cf. [2, 3]), but of course preserves the effects due to the presence of c^2, or more precisely, to the hyperbolicity of the original system. (The classical example of uncoupled equations is the kinetic analogy of Kirchhoff [6], cf. [9].)

In Figs. 3, 4, 5 we exhibit the nature of multivalued solutions of (3.16a, c, d). In these figures the curves for different values of c should be compared with the curve for $c = 0$, labelled 0. Note that (3.16b, f) admit only constant solutions. Eq. (3.16b) has the solution $y_2^* = 0$; it may have others. In Fig. 6 we show curves giving

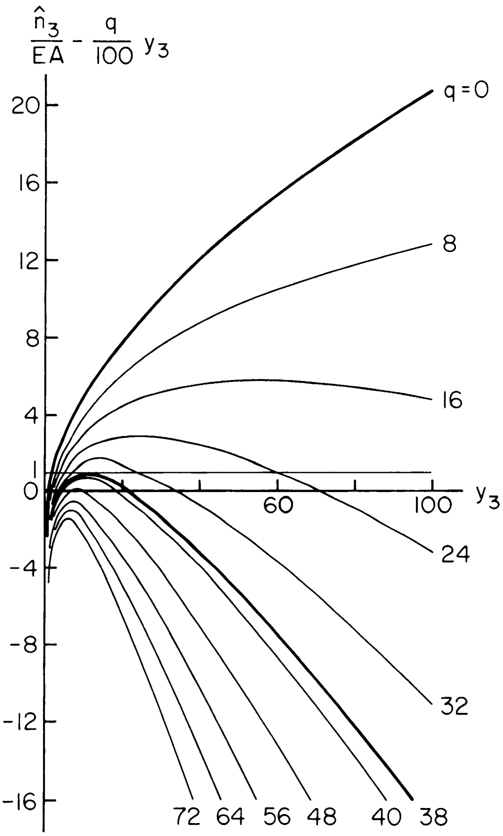


FIG. 2. The form of $(\hat{n}_3/EA) - (qy_3/100)$ when $y_3 \to \hat{n}_3(\mathbf{x})$ is sublinear. The actual curves shown correspond to the constitutive equation $3(EA)^{-1}\hat{n}_3(y_3) = 4\{[4 + 3(y_3 - 1]^{1/2} - 2\}$ for $y_3 \geq 1$, $(EA)^{-1}n_3(y_3) = 1 - (y_3)^1$ for $y_3 \in (0, 1]$.

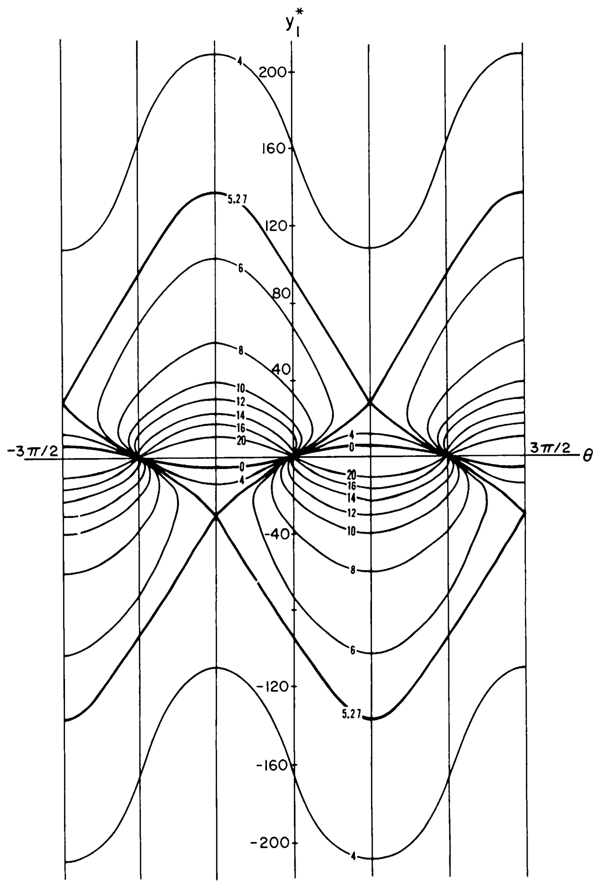


FIG. 3. The solutions of (3.16a) when $N > 0$ for y_1^* as multivalued functions of θ for different values of q. The actual curves plotted are based upon $\hat{n}_1$ being given by μ of Fig. 1a and upon $N = EA$. (The numbering of curves corresponds to that of Fig. 1a.) The corresponding figure for superlinear $\hat{n}_1$ looks like this figure turned upside-down.

$$E_1 = y_1^* N \sin\theta + \rho(c^2/2)A(y_1^*)^2 - \int_0^{y_1^*} \hat{n}_1(y)dy \tag{5.2}$$

as a function of θ when y_1^* is as in Fig. 3. E_1 is the contribution of the y_1^* terms to the left side of (3.17).

To obtain a specific example that will illustrate the general qualitative properties of travelling waves, we choose

$$\hat{n}_1(y_1) = (EA/3)[(4 + 3y_1)^{1/2} - 2] \text{ for } y_1 \geq 0, \tag{5.3a}$$

$$\hat{n}_3(y_3) = (EA/2)(y_3 - 1/y_3), \tag{5.3b}$$

$$\hat{m}_1(u_1) = EJu_1, \qquad \hat{m}_3(u_3) = 2EJu_3, \tag{5.3c,d}$$

where E is a positive constant. $\hat{n}_2$ and $\hat{m}_2$ have the same form as $\hat{n}_1$ and $\hat{m}_1$ (by the isotropy assumption). The nonlinearity in (5.3b) is necessary to handle (2.16). That in (5.3a) is the source of the special nonlinear effects we are investigating. The system (5.3) is designed so that its linearization about $y_\sigma = 0$, $y_3 = 1$, $u_k = 0$ reduces to the classical linear rod theory with E the elastic modulus and with the shear modulus taken to be $(E/3)$. Fig. 1a is obtained by setting μ equal to $\hat{n}_1$ of (5.3a). The curve in Fig. 1a numbered q corresponds to c^2 satisfying $\rho c^2/E = q/100$. Fig. 3 is then obtained for $N = EA$. (Note that curves of Figs. 4 and 5 do not correspond to (5.3b, c).)

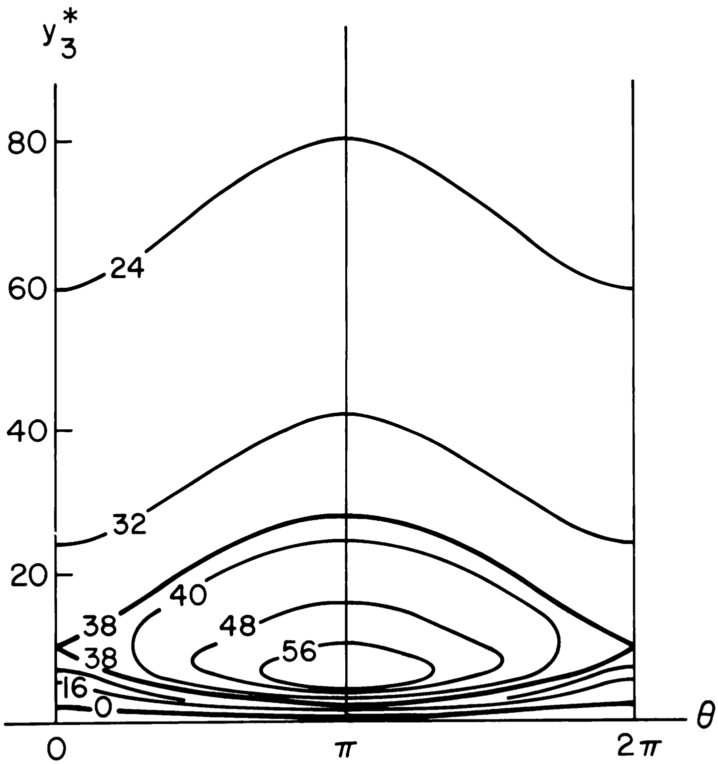


FIG. 4. The solutions of (3.16c) when $N > 0$ for y_3^* as multivalued functions of θ for different values of q. The actual curves plotted are based upon $\hat{n}_3$ having the sublinear form shown in Fig. 2 and upon $N = EA$.

Under these conditions, we find that

$$E_3 \equiv y_3^* N \cos\theta + \rho(c^2/2)A(y_3^*)^2 - \int_1^{y_3^*} \hat{n}_3(y)dy \tag{5.4}$$

makes a negligible contribution to the nonconstant terms of (3.17) (or (4.2)) when c^2 has the values corresponding to the curves of Fig. 6, namely $q \equiv 100\rho c^2/E = 5.27, 8$. Now we write (3.17) as

$$(u_2^*)^2 \equiv (\theta')^2 = F(\theta) - H \tag{5.5}$$

where

$$-H = \frac{2h}{G_q} + \frac{A}{J}(y_2^*) - \frac{2\beta^2}{(JG_q)^2}(\text{const}), \tag{5.6}$$

$$F(\theta) = \frac{-2}{JG_q}(E_1 + E_3) - \frac{1}{(JG_q)^2}\left[\frac{\alpha - \beta\cos\theta}{\sin\theta}\right]^2, \tag{5.7}$$

$$G_q = E[1 - (.01)q]. \tag{5.8}$$

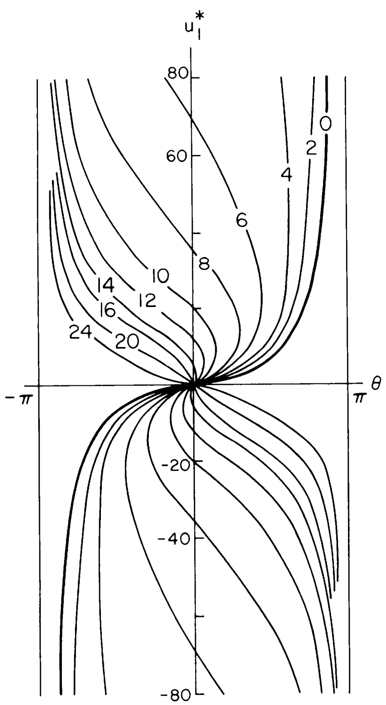


FIG. 5a. The solutions of (3.16d) when $\alpha = \beta > 0$ for u_1^* as multivalued functions of θ for different values of q. The actual curves plotted correspond to the constitutive equation $3(EJ)^{-1}\hat{m}_1(u_1^*) = 4[(4 + 3u_1^*)^{1/2} - 2]$ and to $\alpha = \beta = EJ$.

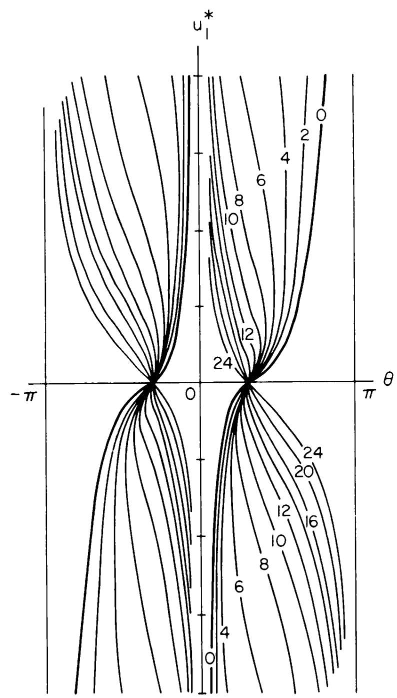
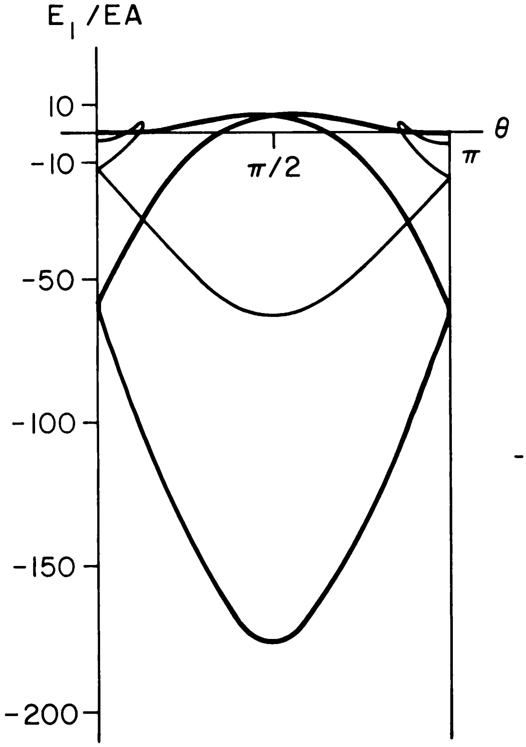


FIG. 5b. The solutions of (3.16d) when $0 < \alpha < \beta$ for u_1^* as multivalued functions of θ for different values of q. The actual curves plotted are based upon $\hat{m}_1$ having the form given in Fig. 5a and upon $\alpha = EJ$, $\beta = 2EJ$.

In Fig. 7 we plot $F(\theta)$ for $q = 5.27$ when $\alpha = \beta$ is taken fairly large and in Fig. 8 we sketch the phase plane trajectories corresponding to the different values of H shown in Fig. 7. The vertical scales of these trajectories vary from figure to figure. In Fig. 9 we plot $F(\theta)$ for $q = 8$ when $\alpha = \beta$ is taken to be a smaller number than that for Fig. 7 and in Fig. 10 we sketch corresponding trajectories. The arrows indicate the motion of (θ, θ') in the phase plane with increasing s. Of course all parts of trajectories in the upper half plane must move to the right and parts of trajectories lying in the lower half plane must move left. Thus there are points off the θ-axis that two trajectories approach from opposite directions. Although the trajectory has a tangent at such points it does not have an oriented tangent. These points are accordingly singular points of the very sort that we discussed from an analytic point of view in Sec. 4. One such trajectory is one of the kidney-shaped trajectories of Fig. 10b. Since θ' is positive near A and negative near B on this trajectory, a point (θ, θ') starting anywhere on the open arc CDA reaches A in a bounded s-interval. Thus the kidney-shaped trajectory $ABCD$ cannot correspond to a travelling wave with a twice continuously differentiable θ. We could contemplate a trajectory in which there is a jump from A to C. Such solutions with θ' discontinuous are discussed in the next section.

The trajectories $ABCD$ and $AFCE$ of Fig. 8b represent periodic travelling waves with θ



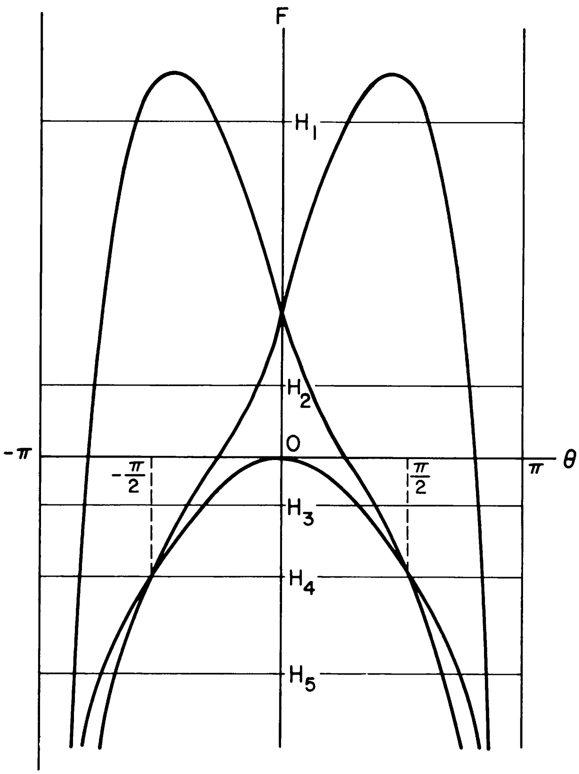


FIG. 6. Plot of E_1 vs. θ for curves $q = 5.27$ and $q = 8$ of Fig. 3 when $N = EA$. The curve $q = 5.27$ is drawn in heavy lines.

FIG. 7. Plot of F vs θ for $q = 5.27$ when $\alpha = \beta$ is taken sufficiently large.

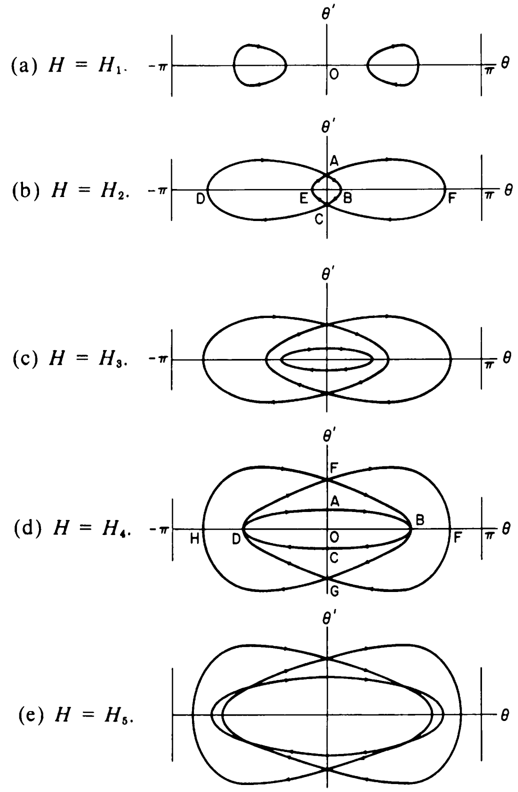


FIG. 8. Phase plane trajectories for different values of H corresponding to Fig. 7. The vertical scales are distorted.

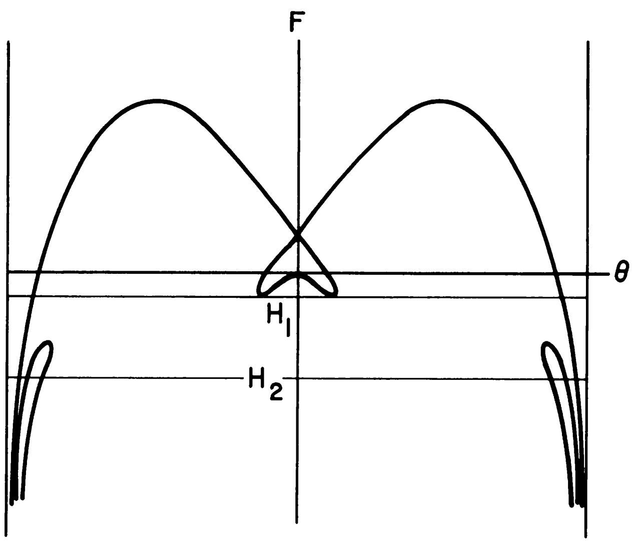


FIG. 9. Plot of F vs. θ for $q = 8$ when $\alpha = \beta$ is taken to be much smaller than its value for Fig. 7.

$\in C^2$. The trajectories $ABCE$ and $AFCD$ represent periodic travelling waves with θ being merely piecewice twice continuously differentiable. Such a θ generates a weak solution of our governing equations. Thus there is a family of solutions described by the point (θ, θ') moving in a clockwise sense about the trajectories in Fig. 8b with these solutions having the option of switching at the singular points A and C from one smooth arc to another. In this manner we obtain an uncountable family of oscillatory piecewise C^2 solutions rather than periodic solutions. Since A and C correspond to $\theta = 0$, we see from Fig. 3 that y_1^* has a jump every time such a switch is made. (In Sec. 6 we show that such discontinuous solutions are not physically realizable.)

A seemingly related family of solutions is that corresponding to Fig. 8d (which occurs only for a very special set of parameters). Since the smooth trajectories $ABCD$ and $EBGH$ are tangent at B (which is a regular point) and since the smooth trajectories $ABCD$ and $FGDE$ are tangent at D, there is an uncountable family of oscillatory travelling waves with

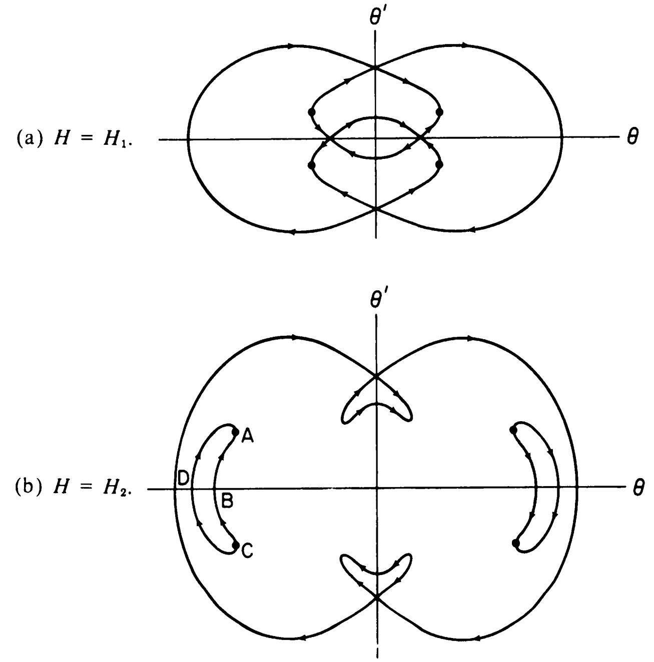


FIG. 10. Phase plane trajectories corresponding to Fig. 9. The vertical scales are distorted.

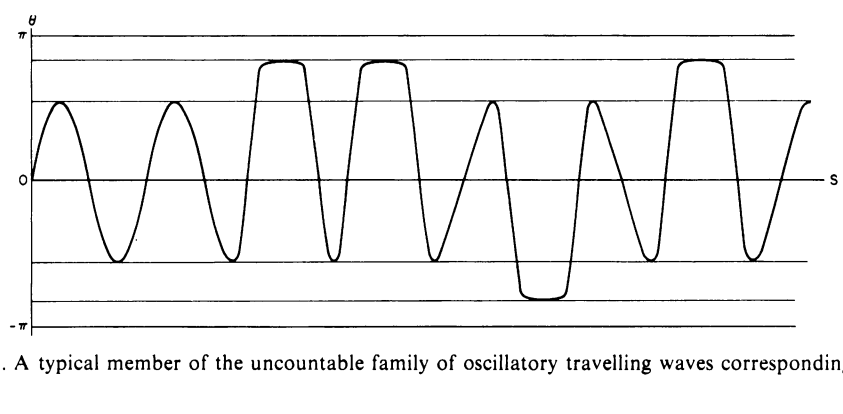


FIG. 11. A typical member of the uncountable family of oscillatory travelling waves corresponding to Fig. 8d.

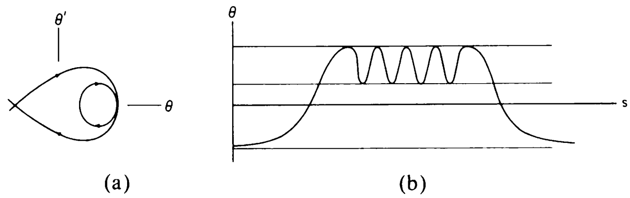


FIG. 12. A typical member (b) of the countably infinite family of travelling pulses resulting from a phase plane trajectory of the form (a).

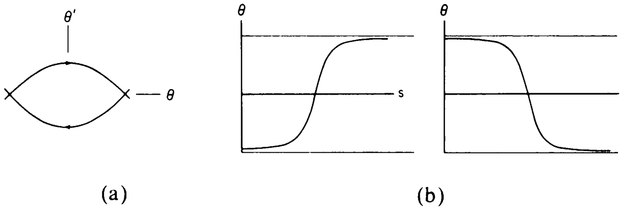


FIG. 13. Travelling bores (b) that result from a phase portrait of the form (a).

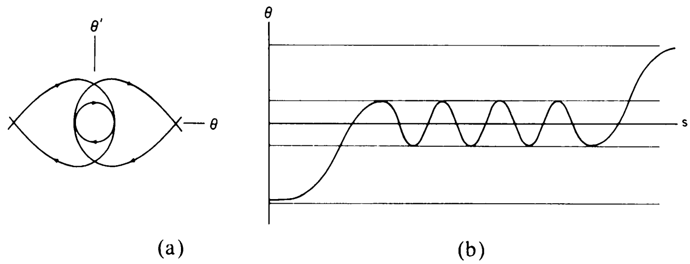


FIG. 14. A typical member (b) of the countably infinite family of travelling bores that would correspond to a phase portrait of the form (a).

$\theta \in C^2$. In Fig. 11 we sketch a typical member of this family.

We can, of course, also allow switching at E and G but this would destroy the C^2 character of solutions. We note that y_1^* (as well as the other strains) is continuous on trajectories that switch only at B and D. (D has abscissa $-\pi/2$ and B has abscissa $\pi/2$. Switching in Fig. 8d at D and B corresponds to switching of the trajectories $q = 5.27$ in Fig. 3 where they cross at $-\pi/2$ and $\pi/2$.)

A related phenomenon, which does not arise in our example for the parameter ranges used (but cf. [3]), is a trajectory of the sort shown in Fig. 12a. This trajectory gives rise to a countably infinite family of pulses of the sort shown in Fig. 12b. The location of the saddle points is determined by the equation $\delta(y_1^*, \theta; \alpha, \beta, N) = 0$. There can also be trajectories joining pairs of saddle points such as in Fig. 13*a* that lead to travelling bores. We have not, however, discovered circumstances under which there is a phase plane trajectory of the form shown in Fig. 14a, which would lead to a countably infinite family of bores, but we feel that such trajectories can exist for very special values of the parameters.

We finally note that the chief qualitative effect of changing the parameter c is to change the signs of various terms of (3.17) or (4.2) and to reduce the range of θs for which (3.16c) has solutions. Once the phase-plane behavior of θ is determined we can readily obtain the motion of $\mathbf{d}_3$ on the unit sphere. Cf. [3] for details of this process.

6. Discontinuous waves. We now study the question of determining which of the many kinds of discontinuous travelling waves suggested by our development of Sec. 5 are physically realizable. Suppose that $\{\mathbf{r}, \mathbf{d}_\sigma\}$ is a piecewise twice continuously differentiable weak solution of the partial differential equations (2.13), (2.14), (2.17). Suppose that $x = g(t)$, with $g \in C^1(\mathbb{R})$, is the equation of a curve across which some components of $\{\mathbf{r}, \mathbf{d}_\sigma\}$ and their first and second derivatives suffer jump discontinuities. Then $\{\mathbf{r}, \mathbf{d}_\sigma\}$ must satisfy the *Rankine-Hugoniot jump conditions* at $(g(t), t)$:

$$[\![\mathbf{n}]\!](g(t), t) + \rho A g'(t)[\![\mathbf{r}_t]\!](g(t), t) = \mathbf{0}, \tag{6.1}$$

$$[\![\mathbf{m}]\!](g(t), t) + \rho J g'(t)\left[\!\!\left[\mathbf{d}_\sigma \times \frac{\partial}{\partial t}\mathbf{d}_\sigma\right]\!\!\right](g(t), t) = \mathbf{0}, \tag{6.2}$$

where

$$[\![f]\!](g(t), t) \equiv \lim_{\epsilon \to 0}\,[f(g(t) + \epsilon, t) - f(g(t) - \epsilon, t)] \tag{6.3}$$

and where $\mathbf{n}$ and $\mathbf{m}$ are given by (2.20). If we seek travelling wave solutions of the form (2.12) and consequently choose $g(t) = ct + \xi$, then (6.1) and (6.2) reduce to

$$[\![\mathbf{n} - \rho c^2 A \mathbf{r}']\!](\xi) = \mathbf{0}, \tag{6.4}$$

$$[\![\mathbf{m} - \rho c^2 J \mathbf{d}_\sigma \times \mathbf{d}_\sigma']\!](\xi) = \mathbf{0}. \tag{6.5}$$

(We are adhering to the notational conventions adopted in the discussion surrounding (2.21)–(2.24).) Relation (6.4) implies that $\mathbf{n} - \rho c^2 A\mathbf{r}'$ is continuous so that $N\mathbf{e}_3$, introduced in (3.1), is a true constant. The componential version of (6.4) and relations (3.2)–(3.4) then imply that θ and ϕ must be (effectively) continuous. Relations (6.5) and (3.5) imply that α is a true constant. Since θ and ϕ are continuous, $\mathbf{d}_3$ is continuous. Relations (6.5) and (3.13) imply that β is a true constant. Since weak solutions conserve energy, the h appearing (3.8), (3.16e) and elsewhere is a true constant. If we take components of (6.4), (6.5) with respect to the basis $\{\mathbf{d}_k\}$ where $\mathbf{e}_1$, $\mathbf{e}_2$ are chosen so as to yield (3.16), then (3.16a-d, f) imply that all the conditions of (6.4) and (6.5) are automatically met save for

$$[\![\hat{m}_2(\mathbf{x}) - c^2 J u_2^*]\!](\xi) = 0. \tag{6.6}$$

It is well known that initial-value problems for quasilinear hyperbolic systems such as (2.13), (2.14), (2.17) can have many weak solutions. The unique weak solution satisfying further entropy conditions is adjudged physically reasonable. (Although such entropy conditions are motivated by physical considerations, often connected with notions of dissipation and stability, the nature of the relationship between uniqueness and physical considerations is not completely understood.)

The entropy condition is usually stated in terms of the characteristic speeds and the shock speed. For the system (2.13), (2.14), (2.17) a *characteristic direction* $\mathbf{w}(\mathbf{x})$ at $\mathbf{x}$ is an eigenvector of $\mathbf{M}(\mathbf{x})$ relative to $\mathbf{K}$ (see (4.5c)) and the corresponding *characteristic speed* $\chi(\mathbf{x})$ at $\mathbf{x}$ is the corresponding eigenvalue:

$$[\mathbf{M}(\mathbf{x}) - \chi(\mathbf{x})\mathbf{K}]\mathbf{w}(\mathbf{x}) = \mathbf{0}. \tag{6.7}$$

Lax [7] proposed a stability criterion for shocks that characterizes a shock as stable if as t increases the characteristic curves impinge on the curve $x = g(t)$ of discontinuity from both sides. This criterion is appropriate for *genuinely nonlinear systems.* Our system (2.13), (2.14), (2.18) would be genuinely nonlinear if

$$\mathbf{w}(\mathbf{x})\cdot\nabla\chi(\mathbf{x}) \neq 0 \tag{6.8}$$

for all $\mathbf{x}$ with $x_3 > 0$ and for all solution pairs χ, $\mathbf{w}$ of (6.7). Since $y_\sigma \to \hat{n}_\sigma(y_l, u_l)$, $u_k \to \hat{m}_k(y_l, u_l)$ are odd functions our system cannot be genuinely nonlinear. The function $[0,\infty) \ni y_3 \to \hat{n}_3(y_l, u_l)$, however, could be concave. In this case, if the corresponding equation were uncoupled from the rest, then (6.8) would apply to this equation and Lax's conditions would supply conditions for the stability of discontinuities in y_3. We discuss these below.

Liu [8] generalized Lax's condition to systems that are not necessarily genuinely nonlinear and used his generalization to prove a uniqueness theorem for the Riemann problem. (In this connection, see [5].) To state Liu's condition, which we shall apply to our problem, we consider the system

$$\begin{aligned}\hat{n}_k(\mathbf{x}) - \hat{n}_k(\mathbf{x}_0) &= \rho\sigma A(y_k^* - y_{0k}^*),\\ \hat{m}_k(\mathbf{x}) - \hat{m}_k(\mathbf{x}_0) &= (1+\delta_{k3})\rho\sigma J(u_k^* - u_{0k}^*)\ (k \text{ not summed}),\end{aligned} \tag{6.9}$$

where δ_{ij} represents the Kronecker delta. Suppose that $\mathbf{x}_0$ is given. Values of σ such that (6.9) has a solution $\mathbf{x}$ are denoted $\sigma(\mathbf{x}_0, \mathbf{x})$. Note that $\sigma(\mathbf{x}_0, \mathbf{x}) = \sigma(\mathbf{x}, \mathbf{x}_0)$ and that $\sigma(\mathbf{x}, \mathbf{x}_0)$, if it exists, satisfies

$$\sigma(\mathbf{x}, \mathbf{x}_0) = \frac{[\hat{n}_k(\mathbf{x}) - \hat{n}_k(\mathbf{x}_0)](y_k^* - y_{0k}^*) + [\hat{m}_k(\mathbf{x}) - \hat{m}_k(\mathbf{x}_0)](u_k^* - u_{0k}^*)}{\rho A(y_k^* - y_{0k}^*)(y_k^* - y_{0k}^*) + \rho J[(u_\alpha^* - u_{0\alpha}^*)(u_\alpha^* - u_{0\alpha}^*) + 2(u_3^* - u_{03}^*)^2]}$$

The strict convexity of W (cf. (2.17)), which is nothing more than a hyperbolicity condition for our problem, implies that $\sigma(\mathbf{x}_0, \mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{x}_0$. We set

$$\mathcal{S}(\mathbf{x}_0) = \{\mathbf{x}: \exists\sigma(\mathbf{x}_0, \mathbf{x}) \text{ such that (6.9) is satisfied}\}. \tag{6.11}$$

Let

$$\mathbf{x}_\pm(g(t), t) = \lim_{\epsilon\to 0} \mathbf{x}(g(t)\pm\epsilon, t). \tag{6.12}$$

A *discontinuity* $\{\mathbf{x}_\pm(g(t), t)\}$ is called *admissible* if it satisfies Liu's *entropy condition:* there is a component of $\mathcal{S}(\mathbf{x}_-)$ connecting $\mathbf{x}_-$ to $\mathbf{x}_+$ and

$$\sigma(\mathbf{x}_-, \mathbf{x}_+) \le \sigma(\mathbf{x}_-, \mathbf{x}) \tag{6.13}$$

for every $\mathbf{x}$ in this component of $\mathcal{S}(\mathbf{x}_-)$. This condition is equivalent to: there is a component of $\mathcal{S}(\mathbf{x}_+)$ connecting $\mathbf{x}_+$ to $\mathbf{x}_-$ and

$$\sigma(\mathbf{x}_-, \mathbf{x}_+) \ge \sigma(\mathbf{x}, \mathbf{x}_+) \tag{6.13b}$$

for every $\mathbf{x}$ in this component of $\mathcal{S}(\mathbf{x}_-)$. ($\mathcal{S}(\mathbf{x}_\pm)$ is called the *shock set* through $\mathbf{x}_\pm$ and $\sigma(\mathbf{x}_-, \mathbf{x}_+)^{1/2}$ is the *shock speed* associated with the discontinuity $\{\mathbf{x}_\pm\}$.)

We are now ready to study the nature of travelling discontinuities satisfying (6.6) and (6.13). A discontinuity $\{\mathbf{x}_\pm\}$ satisfies (6.6) if and only if there is a number τ such that

$$\hat{m}_2(\mathbf{x}_\pm) - \rho c^2 J\theta'_\pm = \tau. \tag{6.14}$$

It follows from (6.14) that there cannot be jumps in θ' on the curves of Figs. 8 and 9 because $\theta' \to \hat{m}_2(\mathbf{x}) - \rho c^2 J\theta'$ is strictly increasing for the value of c^2 used and for the linear function $\hat{m}_2$ given by (5.3c). For constitutive functions more complicated than (5.3) and for other values of c^2 the possibility of such jumps cannot be so easily dismissed. To handle such problems we invoke Liu's entropy condition. We assume that a component of $\mathcal{S}(\mathbf{x}_-)$ connects $\mathbf{x}_-$ to $\mathbf{x}_+$. (Otherwise the discontinuity would not be admissible.) Then (6.9) and (6.14) imply that

$$\rho J\sigma(\mathbf{x}_-, \mathbf{x}_+)(\theta_+' - \theta_-') = \hat{m}_2(\mathbf{x}_+) - \hat{m}_2(\mathbf{x}_-) = \rho c^2 J(\theta_+' - \theta_-') + \tau. \tag{6.15}$$

But the speed $\sigma(\mathbf{x}_-, \mathbf{x}_+)^{1/2}$ of a travelling shock must be the wave speed c. Hence (6.15) implies that

$$\tau = 0. \tag{6.16}$$

Thus (6.14), which is the only Rankine-Hugoniot condition that is not satisfied identically, implies that jumps can only occur between values of $\mathbf{x}$ for which $\hat{m}_2(\mathbf{x}) - \rho c^2 J\theta' = 0$. The values of θ' for which this equation is satisfied can be read off from a figure like Fig. 1. Let us study shocks in which $\theta_+' \neq \theta_-'$. Since we are assuming that $\mathcal{S}(\mathbf{x}_-)$ connects $\mathbf{x}_-$ to $\mathbf{x}_+$, Liu's entropy condition (6.13a) implies that

$$\rho c^2 J = \frac{\hat{m}_2(\mathbf{x}_+) - \hat{m}_2(\mathbf{x}_-)}{\theta_+' - \theta_-'} \leq \frac{\hat{m}_2(\mathbf{x}) - \hat{m}_2(\mathbf{x}_-)}{\theta' - \theta_-'} \tag{6.17}$$

for all $\mathbf{x}$ on a component of $\mathcal{S}(\mathbf{x}_-)$ to $\mathbf{x}_+$ joining $\mathbf{x}_-$ to $\mathbf{x}_+$ (and a fortiori for all θ', which is the fifth component of $\mathbf{x}$, between θ_-' and θ_+').

Suppose that $\hat{m}_2$ depends only on θ'. By examining the geometric consequences of (6.14), (6.16), (6.17) in light of the oddness of $\theta' \to \hat{m}_2(\theta')$ we readily arrive at

(6.18) PROPOSITION. Let $\hat{m}_2$ depend solely on θ'. Let θ, λ, h be fixed and let $\theta_\pm'$ satisfy (4.2). Then a travelling wave satisfying the Rankine-Hugoniot and Liu conditions can suffer a discontinuity $\{\theta_\pm'\}$ at θ, λ, h with θ_+' and θ_-' having opposite signs only if $\hat{m}_2(\theta') = \rho c^2 J\theta'$ for all θ' between θ_-' and θ_+'.

This result prevents many jumps in θ' for phase-plane trajectories like those of Figs. 8 and 10 that arise for uncoupled nonlinear constitutive equations. (Note that linear constitutive relations are easily handled by the argument following (6.14).) In particular, there are no travelling waves that can correspond to a kidney-shaped trajectory like that of Fig. 10b.

Now we consider jumps in θ' when $\hat{m}_2$ does not satisfy the restriction of Proposition (6.18). We limit our attention to discontinuities of the form

$$\{x_\pm\} = (x_1, x_2, x_3, x_4, \pm\theta_0', x_6), \qquad \theta_0' \neq 0. \tag{6.19}$$

We set

$$\mathbf{z}_\pm = (z_1, z_2, z_3, z_4, \pm\theta', z_6). \tag{6.20}$$

Now the isotropy condition (2.18) implies that $\mathbf{z}_- \in \mathcal{S}(\mathbf{x}_-)$ if and only if $\mathbf{z}_+ \in \mathcal{S}(\mathbf{x}_+)$ and that

$$\sigma(\mathbf{x}_-, \mathbf{z}_-) = \sigma(\mathbf{x}_+, \mathbf{z}_+). \tag{6.21}$$

The entropy condition (6.13) and the equality $\sigma(\mathbf{x}_-, \mathbf{x}_+) = c^2$ then implies that

$$\sigma(\mathbf{x}_-, \cdot) = \sigma(\mathbf{x}_+, \cdot) = \sigma(\mathbf{x}_-, \mathbf{x}_+) = c^2 \tag{6.22}$$

where the domain of $\sigma(\mathbf{x}_-, \cdot)$ is $\mathcal{S}(\mathbf{x}_-)$ and that of $\sigma(\mathbf{x}_+, \cdot)$ is $\mathcal{S}(\mathbf{x}_+)$. This condition means that the problem is degenerate in a sense that generalizes the exceptional case $\hat{m}_2(\theta') = c^2 J\theta'$ of Proposition (6.18). (Cf. (6.10).) To see the nature of this degeneracy, we suppose that

$\{\mathcal{S}(\mathbf{x}_\pm)\}$ contain continuously differentiable curves defined parametrically by functions $\{\zeta \to \hat{\mathbf{z}}_\pm(\zeta)\}$. Then (6.22) and (6.9) imply that

$$\mathbf{M}(\hat{\mathbf{z}}_\pm(\zeta)) \frac{d\hat{\mathbf{z}}_\pm}{d\zeta}(\zeta) = c^2\mathbf{K} \frac{d\hat{\mathbf{z}}_\pm}{d\zeta}(\zeta). \tag{6.23}$$

This means that $(d\hat{\mathbf{z}}_\pm/d\zeta)(\zeta)$ is a characteristic direction at $\hat{\mathbf{z}}_\pm(\zeta)$ and that c^2 is the characteristic speed there. Hence $\hat{\mathbf{z}}_\pm$ define characteristic curves (cf. (6.7)). From (6.9) we then find that $\hat{n}_k(\mathbf{x}) - \rho c^2 A y_k^*$ and $\hat{m}_k(\mathbf{x}) - (1 + \delta_{kn})\rho c^2 J u_k^*$ are constants for $\mathbf{x}$ on these characteristic curves. Note that all of these functions except $\hat{m}_2(\mathbf{x}) - \rho c^2 J\theta'$ are also constant on trajectories. In particular, (6.14) and (6.16) imply that $\hat{m}_2(\mathbf{x}) - \rho c^2 J\theta' = 0$ for $\mathbf{x} \in \mathcal{S}(\mathbf{x}_\pm)$. Hence we have the following companion of Proposition (6.18).

(6.23) PROPOSITION. Let θ, λ, h be fixed and let $\pm\theta_0$ satisfy (4.2). Let the corresponding $\mathbf{x}_\pm$ be given by (6.19). Let $\{\mathcal{S}(\mathbf{x}_\pm)\}$ contain C^1 curves $\{\hat{\mathbf{z}}_\pm\}$ joining $\mathbf{x}_-$ to $\mathbf{x}_+$. Then a travelling wave satisfying the Rankine-Hugoniot and Liu conditions can suffer a discontinuity $\{\mathbf{x}_\pm\}$ only if $\hat{n}_k(\mathbf{x}) - \rho c^2 A y_k^* = a_k$ (const), $\hat{m}_k(\mathbf{x}) - (1 + \delta_{k_3})\rho c^2 J u_k^* = b_k$ (const) with $b_2 = 0$ for $\mathbf{x}$ on the curves $\{\hat{\mathbf{z}}_\pm\}$ (which are characteristic curves).

The development of more general results along these lines seems to lead to statements couched in terms of genericity. We do not pursue such generalizations here.

We now examine the question of whether an admissible discontinuity arises for a trajectory like DAF in Fig. 8b. We see that y_1^* suffers a discontinuity at A corresponding to the positive and negative values of y_1^* on the curve $q = 5.27$ of Fig. 3. Just as in Proposition (6.18), the oddness of $y_1^* \to \hat{n}_1(y_1^*)$ prevents such a discontinuity from being admissible for a nonlinear $\hat{n}_1$. More general results along the lines of Proposition (6.23) can be obtained when $\hat{n}_1$ depends on other arguments besides y_1^*.

We note that this argument would fail were only y_3 to suffer a discontinuity at a point like A of Fig. 8b because $y_3^* \to \hat{n}_3(\mathbf{x})$ is certainly not odd. Indeed, (2.3) and (2.16) suggest that the requirement that this function be concave, at least for small y_3^*, is eminently reasonable. Thus Liu's entropy condition would not necessarily exclude travelling compressional shocks (coupled with other modes of deformation). This suggests that there is a distinguished role for compressional shocks.

The discontinuities at B and D of Fig. 8d are merely contact discontinuities; they are compatible with both the Rankine-Hugoniot and Liu conditions. From the extensive work on hyperbolic conservation laws, there is no evidence to suggest that a travelling wave of the form of Fig. 11 would fail to be the physically realizable solution to a Cauchy problem with the shape of Fig. 11 generating the initial data.

7. Conclusion. We have shown that our twelfth-order, quasilinear hyperbolic system has a rich variety of travelling waves, both classical and weak. Our development depended critically upon the isotropy condition (2.18) and upon (2.19). Our approach differed in a number of significant particulars from that used for the static problem of [3] because the operators of the travelling wave problem do not enjoy the monotonicity of those of the static problem. One minor but typical manifestation of the distinction between the static problem and the travelling wave problem is that in the former problem Eqs. (3.16b, f) (with $c^2 = 0$) have unique solutions.

We found that when the wave speed is characteristic at points of the rod, there can be countably or uncountably infinite families of travelling waves for the same values of parameters. There are also travelling waves when the wave speed is nowhere characteristic. This behavior may be contrasted with that for the classical second-order linear wave equation for which any shape defines a travelling wave when the wave speed is characteristic and there are no travelling waves when the wave speed is not characteristic. It may also be contrasted with that for (1.1) for which all travelling waves have piecewise linear shape.

The study of longitudinal waves in rods is a useful way to determine the constitutive properties of the rods in the nonlinear range (cf. Bell [4]). Bell has informed us, however, that flexural motion causes distortions that render standard experiments useless. Our results, showing the complicated effects that accompany flexural motion, suggest why this is so. They also suggest that experimental procedures accounting for combined flexure, torsion, extension, and shear could conceivably give a more refined picture of constitutive response.

At the level of second-order systems such as (1.1), there is little distinction between equations from gas dynamics and those from nonlinear elasticity. The governing equations that we have analyzed are far more complicated than (1.1); they cannot be confused with equations of gas dynamics. Associated with this very complexity is a rich mathematical structure leading to a complete family of integrals. This mathematical structure, the exactness of the underlying geometry, the generality of the material response, and the availability of rational thermo-visco-elastic generalizations of our equations justify us in submitting that these equations are a worthy and rewarding object of serious analysis.

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