LARGE FINITE STRAIN MEMBRANE PROBLEMS*

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Abstract. Nonlinear membrane problems involving large finite strains are considered. It is found that explicit asymptotic solutions are possible for a rather large class of problems. Two distinct types of asymptotic solutions, roughly depending on whether the strain energy density function is dominated by I_1 or I_2, are found to exist.

1. Introduction. The theory of the finite deformation of a nonlinearly elastic membrane has been studied by many authors (see for example [1–5]). The application of this theory to particular problems is difficult because of the nonlinearity of the equations involved. Aside from the few problems solved by the use of semi-inverse methods (see [6], for example), exact solutions are few.

In the context of the theory of plane stress, the solutions to the class of axisymmetric problems, originally studied by Rivlin and Thomas [1], may be considered exact in that the governing equations may be reduced to two uncoupled first-order ordinary differential equations. This reduction was accomplished by Yang [7]. The class of solutions obtained by Varley and Cumberbatch [8] is exact, but is based on an assumed strain-energy density function. While the authors went into great detail to justify the form of their assumed density function, their explicit result, however, may also be interpreted as an approximate solution in a certain sense.

In the general case where at least either the undeformed or the deformed surface is not a plane, the number of exact solutions is even fewer. The deformation from a tube to an annulus [9] is perhaps a nontrivial addition to the list of semi-inverse solutions mentioned in [6]. The problem of inflation of cylindrical membranes may also be reduced to quadratures [10, 11]. They all resulted from the fact that one of the two equations of equilibrium may explicitly be integrated [12].

Perturbation or iteration methods are often useful in generating approximate but explicit solutions to problems that are otherwise difficult to solve. For these methods to be practically useful, a problem must have a corresponding limiting problem whose solution is explicitly obtainable. For this reason perturbation methods have been applied to the following classes of problems: (1) small finite strain (see [3] for example), (2) small deformation superposed on known finite deformation (see [4] for example), and (3) infinitely large strain. We shall confine our attentions to the class of problems involving large finite strains. Problems of this nature have been studied for membranes made of either a neo-Hookean material or a Mooney material. The nature of the mathematical reduction as well as the character of the solution for these two classes of problems, however, are completely different.

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The first large finite strain analysis of neo-Hookean membranes appears to be given by Foster [13]. He considered the class of axisymmetric problems, and showed that for large meridional strains (greater than 100%) the governing equations can all be reduced to quadratures. Using this fact, a water-bag problem was solved by Yu and Valanis [14]. When the same kind of assumption is applied to a plane stress problem, the governing equations reduce to two uncoupled linear equations. This was discovered by Wong and Shield [15]. In terms of practical utility, the reduction obtained in [15] is most profound in that it reduces a nonlinear problem to one that is even simpler than a linear problem.

The first asymptotic analysis carried out for membranes made of Mooney material was done by Isaacson [16]. He showed that the shape of an inflated axially symmetric balloon approaches to a spherical surface as the inflating pressure tends to inifinity. A two-term explicit solution was then obtained by Wu [17]. The simplification resulting from this reduction originates from the fact that an infinitely strained Mooney membrane is uniformly stressed. This property has been exploited by Wu in solving a series of axially symmetric problems (see for example [18–20]).

In this paper we show that the two types of asymptotic analysis may be applied to a slightly more general class of materials and a quite more general class of problems. A review of the basic equations is given in Sec. 2, where the two classes of asymptotic problems are defined. The deductions associated with these problems are discussed in Secs. 3 and 4.

When an asymptotic state is used as a basis of iteration, higher-order iterative solutions are usually desirable (see [15] for example). When an asymptotic state is reached via a process of scaling, a boundary-layer type of analysis is usually needed (see [18, 19] for example). These types of calculations are not included in this paper.

No specific problems, other than a few simple illustrative examples mingled in the text, are solved. However, the equations resulting from the deductions give a clear indication as to what can and what cannot be done.

2. Nonlinear membrane theory. Let $x_i = x^i$ be a set of rectangular cartesian coordinates with unit vectors $\mathbf{e}_i = \mathbf{e}^i$. The position vector $\mathbf{x} = \mathbf{Z}$ of a point on a surface S may be expressed in terms of two surface coordinates $\theta^\alpha = \theta_\alpha$. Specifically, we write

$$\mathbf{Z} = \mathbf{Z}(\theta^\alpha) = Z^i(\theta_\alpha)\mathbf{e}_i = Z_i(\theta_\alpha)\mathbf{e}^i. \tag{2.1}$$

The covariant base vectors $\mathbf{A}_\alpha$ and the components $A_{\alpha\beta}$ of the covariant metric tensor are just

$$\mathbf{A}_\alpha = \mathbf{Z}_{,\alpha} = Z_{,\alpha}{}^i\mathbf{e}_i\,, \tag{2.2}$$

$$A_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{A}_\beta = Z_{,\alpha}{}^i Z_{,\beta}{}^j\,\delta_{ij} \tag{2.3}$$

where δ_{ij} is the Kronecker delta. The unit vector $\mathbf{A}_3 = \mathbf{A}^3$ normal to S is defined by

$$\mathbf{A}_3 = \mathbf{A}^3 = A^{-1/2}\,e_{ijk}\,Z_{,1}{}^j\,Z_{,2}{}^k\,\mathbf{e}^i \tag{2.4}$$

where e_{ijk} are the components of the three-dimensional alternator, and

$$A = \det\,[A_{\alpha\beta}]. \tag{2.5}$$

The components $A^{\alpha\beta}$ of the contravariant metric tensor may be defined by

$$A^{\alpha\beta} = A^{-1}\,e^{\alpha\lambda}\,e^{\beta\mu}\,A_{\lambda\mu} \tag{2.6}$$

where $e^{\alpha\beta}$ are the components of the two-dimensional alternator.

Suppose that the surface S is deformed to a new surface s. Let $\mathbf{x} = \mathbf{z}$ be the position vector of a point on s which, in the undeformed state, had position $\mathbf{Z}$. The deformation may be defined by

$$\mathbf{z} = \mathbf{z}(\theta^\alpha) = z^i(\theta^\alpha)\mathbf{e}_i = z_i(\theta^\alpha)\mathbf{e}^i. \tag{2.7}$$

On the deformed surface s the base vectors $\mathbf{a}_\alpha$, normal vector $\mathbf{a}_3 = \mathbf{a}^3$, and components $a_{\alpha\beta}$ and $a^{\alpha\beta}$ of the metric tensors may be derived from (2.2)–(2.6) by replacing the kernel letters by their lower-case counterparts.

Let dL and dl be, respectively, the arc elements on S and s. Then

$$\Lambda^2 = (dl/dL)^2 = (a_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta)/(A_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta) \tag{2.8}$$

where Λ is the stretch ratio. It follows that

$$\det\,[a_{\alpha\beta} - \Lambda^2 A_{\alpha\beta}] = 0. \tag{2.9}$$

The two invariants J_1 and J_2 involved in the deformation are just

$$J_1 = \Lambda_1{}^2 + \Lambda_2{}^2 = A^{\alpha\beta}a_{\alpha\beta}\,, \qquad J_2 = \Lambda_1{}^2\Lambda_2{}^2 = a/A, \tag{2.10}$$

where Λ_1 and Λ_2 are the principal stretch ratios. We find it convenient to define a few related quantities:

$$I = \Lambda_1 + \Lambda_2\,, \qquad J = \Lambda_1\Lambda_2\,, \qquad I_1 = J_1 + J_2{}^{-1}, \qquad I_2 = J_2 + J_1 J_2{}^{-1}. \tag{2.11}$$

For a homogeneous isotropic elastic material, the strain energy density per unit volume of the undeformed solid is a function of the three (three-dimensional) invariants involved, and the corresponding strain energy density W per unit area of the undeformed surface may be written as a function of I and J.

Let $\mathbf{P}$ be the body force per unit area of the undeformed surface. If we wish to consider surface load as a part of the body force, then it is sometimes more convenient to define a body force $\mathbf{p}$ per unit area of the deformed surface. The governing equations may be derived from the principle of virtual work, viz.,

$$\delta \iint W\sqrt{A}\, d\theta^1\, d\theta^2 = \iint \mathbf{p}\cdot\delta\mathbf{z}\sqrt{a}\, d\theta^1\, d\theta^1 + \int \mathbf{T}\cdot\delta\mathbf{z}\, dl \tag{2.12}$$

where $\delta\mathbf{z}$ is the virtual displacement, and $\mathbf{T}\, dl$ the traction vector on the deformed line element dl. Carrying out the variation, we get

$$\iint \left[\frac{\partial}{\partial\theta^\alpha}\left(\frac{\partial W}{\partial z_{,\alpha}{}^i} A^{1/2}\right) + p_i\, a^{1/2}\right] \delta z^i\, d\theta^1\, d\theta^2$$
$$+ \int \left(\frac{\partial W}{\partial z_{,\alpha}{}^i} A^{1/2}\, e_{\alpha\beta}\, \frac{d\theta^\beta}{dl} - T_i\right) \delta z^i\, dl = 0 \tag{2.13}$$

which yields the displacement equations of equilibrium

$$\frac{\partial}{\partial\theta^\alpha}\left(\frac{\partial W}{\partial z_{,\alpha}{}^i} A^{1/2}\right) + a^{1/2} p_i = 0 \tag{2.14}$$

and the boundary conditions

$$\frac{\partial W}{\partial z_{,\alpha}{}^i} A^{1/2}\, e_{\alpha\beta}\, \frac{d\theta^\beta}{dl} = T_i\,. \tag{2.15}$$

Considering W as a function of I and J, we have

$$\frac{\partial W}{\partial z_{,\alpha}{}^{i}} A^{1/2} = a^{1/2} \tau^{\alpha\delta} z_{,\delta}{}^{i} \tag{2.16}$$

where

$$\tau^{\alpha\delta} = \frac{1}{IJ} \frac{\partial W}{\partial I} A^{\alpha\delta} + \left(\frac{1}{I} \frac{\partial W}{\partial I} + \frac{\partial W}{\partial J} \right) a^{\alpha\delta}. \tag{2.17}$$

The condition (2.15) may be written as

$$\mathbf{T}\, dl = \frac{\partial W}{\partial z_{,\alpha}{}^{i}} \left(\frac{A}{a} \right)^{1/2} \left(a^{1/2} e_{\alpha\beta} \frac{d\theta^{\beta}}{dl} \right) \mathbf{e}_i \, dl \tag{2.18}$$

which, in view of (2.16) and (2.17), implies that $\mathbf{T}$ is tangent to the deformed surface and

$$\mathbf{T}\, dl = \tau^{\alpha\delta} \left(a^{1/2} e_{\alpha\beta} \frac{d\theta^{\beta}}{dl} \right) (z_{,\delta}{}^{i}\, \mathbf{e}_i)\, dl. \tag{2.19}$$

It follows that $\tau^{\alpha\delta}$ defined by (2.17) are just the components of the contravariant Cauchy stress tensor. Using (2.16), we may obtain from (2.14) the stress equations of equilibrium:

$$\tau^{\alpha\beta}|_{\alpha} + a^{\gamma\beta} p_i z_{,\gamma}{}^{i} = 0, \tag{2.20}$$

$$\tau^{\alpha\beta} b_{\alpha\beta} + p_i\, a_3{}^{i} = 0, \tag{2.21}$$

where a vertical bar denotes the covariant differentiation on the deformed surface s, and $b_{\alpha\beta}$ are the components of the second fundamental tensor of s defined by

$$b_{\alpha\beta} = z_{,\alpha\beta}{}^{i}\, a_3{}^{j}\, \delta_{ij}\,. \tag{2.22}$$

We reduce these equations to suit two special cases.

Plane stress. Suppose that both S and s are two-dimensional regions in the $x_3 = 0$ plane. If we let the rectangular cartesian coordinates Z_α of a point on S be the surface coordinates, then $Z_\alpha = \theta_\alpha$ and $z_\alpha = z_\alpha(Z_\beta)$. It follows that $z_{\alpha,\beta}$ are the components of the deformation tensor, and the components of the (first) Piola stress tensor $P_{\alpha\beta}$ are just

$$P_{\alpha\beta} = \frac{\partial W}{\partial z_{\alpha,\beta}} = \frac{1}{I} \frac{\partial W}{\partial I} z_{\alpha,\beta} + \left(\frac{1}{I} \frac{\partial W}{\partial I} + \frac{\partial W}{\partial J} \right) e_{\alpha\mu} e_{\beta\nu} z_{\mu,\nu}\,. \tag{2.23}$$

Setting $A = 1$ and $\mathbf{p} = 0$ in (2.14), we obtain

$$P_{\alpha\beta,\beta} = 0 \tag{2.24}$$

or

$$\left(\frac{1}{I} \frac{\partial W}{\partial I} z_{\alpha,\beta} \right)_{,\beta} + e_{\alpha\mu} e_{\beta\nu} \left(\frac{1}{I} \frac{\partial W}{\partial I} + \frac{\partial W}{\partial J} \right)_{,\beta} z_{\mu,\nu} = 0 \tag{2.25}$$

Eqs. (2.24) may be satisfied by expressing $P_{\alpha\beta}$ in terms of two stress functions $\Phi_\alpha(Z_\alpha)$, i.e.,

$$P_{\alpha\beta} = e_{\alpha\mu} e_{\beta\nu} \Phi_{\mu,\nu}\,. \tag{2.26}$$

Consider now the boundary ∂S of S defined by $Z_\alpha = B_\alpha(L)$ where L measures the arc length along ∂S such that S is on the left of the tangent vector $\mathbf{S}$ defined by

$$\mathbf{S} = S_\alpha \mathbf{e}_\alpha = (dB_\alpha/dL)\mathbf{e}_\alpha\,. \tag{2.27}$$

The outward normal is

$$\mathbf{N} = N_\alpha \mathbf{e}_\alpha = e_{\alpha\beta} S_\beta \mathbf{e}_\alpha . \tag{2.28}$$

The force vector (2.19) may then be written as

$$\mathbf{T}\, dl = P_{\alpha\beta} N_\beta \, dL\, \mathbf{e}_\alpha . \tag{2.29}$$

Displacement and traction boundary conditions are, respectively,

$$z_\alpha(B_\beta(L)) = D_\alpha(L), \tag{2.30}$$

$$P_{\alpha\beta}(B_\beta(L))N_\beta(L) = T_\alpha(L), \tag{2.31}$$

where D_α and T_α are given functions. In terms of the stress functions defined by (2.26), (2.31) takes the form

$$\frac{d}{dL}\, \Phi_\alpha(B_\alpha(L)) = e_{\alpha\beta}\, T_\beta(L). \tag{2.32}$$

Axisymmetric problem. We introduce cylindrical coordinates and define the surfaces S and s by the expressions:

$$S\colon Z_1 = R(L)\cos\theta, \qquad Z_2 = R(L)\sin\theta, \qquad Z_3 = Z(L), \tag{2.33}$$

$$s\colon z_1 = r(L)\cos\theta, \qquad z_2 = r(L)\sin\theta, \qquad z_3 = z(L), \tag{2.34}$$

where L measures the arc length along the undeformed meridian curve. It follows that

$$\Lambda_1 = dl/dL = [(dr/dL)^2 + (dz/dL)^2]^{1/2}, \qquad \Lambda_2 = r/R \tag{2.35}$$

where l measures the arc length along the deformed meridian curve. The two principal stresses τ_1, τ_2 are

$$\tau_1 = (1/\Lambda_2)(\partial W/\partial \Lambda_1), \qquad \tau_2 = (1/\Lambda_1)(\partial W/\partial \Lambda_2), \tag{2.36}$$

and the two equations of equilibrium become

$$\frac{d}{dL}(r\tau_1) = \tau_2 \frac{dr}{dL}, \tag{2.37}$$

$$\frac{d}{dL}\left(r\tau_1 \frac{dz}{dl} + \tfrac{1}{2} p r^2\right) = 0, \tag{2.38}$$

where p is an inflating pressure, the only surface load assumed.

With the aim of solving boundary-value problems in mind, we must augment the above equations with a specific strain energy density function W. The few commonly known density functions[1] are:

Neo-Hookean [3]	$W = C_1(I_1 - 3),$	(2.39)
Mooney [3]	$W = C_1(I_1 - 3) + C_2(I_2 - 3),$	(2.40)
Rivlin [3]	$W = C_1(G_1 - 3) + f(I_2 - 3),$	(2.41)
Harmonic [21]	$W = 2\mu[H(I) - J],$	(2.42)[2]

[1] Compare (2.10) and (2.11) for definitions of invariants. All unidentified symbols in (2.39) – (2.46) are taken from the original references.

[2] The W used in [8] is a harmonic material, and satisfies the relation $(H')^{k-1} - (H')^{-1} = \text{const.}\,(H' - I)$.

Standard [5,21] $$W = 2\mu\left[\frac{\lambda + 2\mu}{4\mu} I^2 - J - \frac{\lambda + \mu}{\mu} I + \frac{\lambda + \mu}{\mu}\right], \tag{2.43}$$

Blatz-Ko [22] $$W = \frac{\mu}{2}\left[3J^{2/3} + \frac{I^2}{J^2} - \frac{2}{J} - 5\right] \text{ (plane stress)}, \tag{2.44}$$

Hart-Smith [23] $$\frac{\partial W}{\partial I_1} = G \exp k_1(I_1 - 3)^2 \, (0 \leq k_1 < 0.0004), \tag{2.45}$$

$$\frac{\partial W}{\partial I_2} = Gk_2 I_2^{-1} \, (0 \leq k_2 < 1.5),$$

Soap-film [24] $$W = TJ \qquad (T = \text{constant}). \tag{2.46}$$

A study of these density functions reveals that they fall into one of the two categories for which the type of asymptotic analysis mentioned in the introduction is possible.

Type-I asymptotic solution. When large strains are involved and when W is dominated by a function W_1 defined by

$$W_1 = C[I^2 - (2 + c)J], \tag{2.47}$$

where C and c are appropriate constants deduced from W, simplifications similar to those obtained in [13] and [15] become apparent. This includes the W functions defined by (2.39), (2.43) and possibly (2.41), (2.42). It also includes (2.45), as long as the strains are not so large that $\partial W/\partial I_1$ is practically unity. This is possible because k_1 is very small.

Type-II asymptotic solution. Consider first the Rivlin material defined by (2.41). We assume that for large strain the function f satisfies the condition

$$f(x) = C_2/(n + 1)x^{n+1} \tag{2.48}$$

where $n > -\frac{1}{2}$ is assumed. The cases $n \leq -\frac{1}{2}$ are covered in Type-I, and $n = 0$ corresponds to the Mooney material. This class of problems is discussed in Sec. 4. We shall see that some of the general results obtained may also be applied to materials defined by (2.44) and (2.46).

3. Type-I asymptotic solution. When the dominant portion of W takes the form

$$W_1 = C[I^2 - (2 + c)J], \tag{3.1}$$

where C and c are appropriate constants depending on the specific form of a given energy density function W, the equations of equilibrium permit certain simplifications. We consider a number of specific cases.

A. *Plane stress.* The displacement equations (2.25) reduce to

$$z_{\alpha,\beta\beta} = 0 \tag{3.2}$$

and the relations (2.23) become

$$P_{\alpha\beta} = 2C\left[\delta_{\alpha\mu}\delta_{\beta\nu} - \frac{c}{2}\, e_{\alpha\mu}e_{\beta\nu}\right] z_{\mu,\nu} \tag{3.3}$$

or

$$z_{\alpha,\beta} = \frac{1}{2C\left(1 - \frac{c^2}{4}\right)}\left[\delta_{\alpha\mu}\delta_{\beta\nu} + \frac{c}{2}\, e_{\alpha\mu}e_{\beta\nu}\right] P_{\mu\nu}\,. \tag{3.4}$$

It follows from (3.4), (3.2) and (2.26) that the stress functions Φ_α are also harmonic, i.e.,

$$\Phi_{\alpha,\beta\beta} = 0 \cdot \tag{3.5}$$

Eqs. (3.2) and (3.5) were first derived by Wong and Shield [15] for neo-Hookean material which corresponds to the case $c = 0$. We prefer to express the general solution in terms of two analytic functions.

To this end we introduce

$$Z = Z_1 + iZ_2, \qquad z = z_1 + iz_2, \tag{3.6}$$

$$C(2 + c)\phi = \Phi = \Phi_1 + i\Phi_2, \tag{3.7}$$

$$2(\ )_{,Z} = \partial(\ )/\partial Z_1 - i\partial(\ )/\partial Z_2\,, \tag{3.8}$$

$$2(\ )_{,\bar{Z}} = \partial(\ )/\partial Z_1 + i\partial(\ )/\partial Z_2\,, \tag{3.9}$$

where, and throughout this paper, $(\bar{\ })$ indicates the complex conjugate of ( ). We also assume that the undeformed plane S is a multiply-connected region with K holes. The resultant force on the kth hole is denoted by $X_k + iY_k$.

Eq. (3.2) implies that

$$z = \Omega(Z) + \overline{\Psi(Z)} + \sum_{k=1}^{K} \alpha_k[\ln(Z - Z_k{}^0)(\bar{Z} - \bar{Z}_k{}^0)] \tag{3.10}$$

where Ω and Ψ are arbitrary holomorphic functions, α_k are arbitrary constants, and $Z_k{}^0$ is a point inside the kth hole. The single-valuedness of z has been used in fixing the form of the last term. The stress-strain relations (3.3), together with (2.26), (3.7), (3.8) and (3.9), imply

$$2C(2 + c)\phi_{,Z} = (P_{11} + P_{22}) + i(P_{21} - P_{12}) = 2C(2 - c)z_{,Z}\,, \tag{3.11}$$

$$2C(2 + c)\phi_{,Z} = (P_{22} - P_{11}) - i(P_{21} - P_{12}) = -2C(2 + c)z_{,\bar{Z}}\,. \tag{3.12}$$

It follows from (3.10) and the above that

$$\phi = \frac{2-c}{2+c}\,\Omega(Z) - \overline{\Psi(Z)} + \sum_{k=1}^{K} \alpha_k\left[\frac{2-c}{2+c}\ln(Z - Z_k{}^0) - \ln(\bar{Z} - \bar{Z}_k{}^0)\right] \cdot \tag{3.13}$$

Eq. (2.32) may be written as

$$d\phi = -\frac{i}{C(2+c)}\,[T_1(L) + iT_2(L)]\,dL \cdot \tag{3.14}$$

Applying (2.14) to the kth hole, we obtain

$$\alpha_k = \frac{1}{8\pi C}\,(X_k + iY_k) \cdot \tag{3.15}$$

In case $\alpha_k = 0$, the resultant moment M_k on the kth hole is

$$M_k = C(2 + c)\,\mathrm{Re}\oint\left[\Psi(Z) - \frac{2-c}{2+c}\,\overline{\Omega(Z)}\right] dZ \cdot \tag{3.16}$$

B. *Zero surface load.* We consider the case where the deformation is purely a result of stretching, i.e., $\mathbf{p} = 0$. For this case the equations of equilibrium (2.14) become

$$\frac{\partial}{\partial\theta^\alpha}\left(\frac{\partial W}{\partial z_{,\alpha}{}^i}\,A^{1/2}\right) = 0 \cdot \tag{3.17}$$

Substituting (3.1) for W, we obtain from (2.16) and (2.17)

$$\frac{\partial W}{\partial z_{,\alpha}{}^{i}} A^{1/2} = 2C\left(A^{1/2}A^{\alpha\delta} - \frac{c}{2}a^{1/2}a^{\alpha\delta}\right) z_{,\delta}{}^{i} \tag{3.18}$$

which, for neo-Hookean material, reduces to

$$\frac{\partial W}{\partial z_{,\alpha}{}^{i}} A^{1/2} = 2CA^{1/2}A^{\alpha\delta} z_{,\delta}{}^{i} \qquad (c = 0) \cdot \tag{3.19}$$

The displacement equations of equilibrium may then be obtained by substituting either (3.18) or (3.19) into (3.17). It is clear that for neo-Hookean materials the equations are linear but with variable coefficients. For $c \neq 0$, the equations are also linear if the deformed surface is a plane. The following sub-cases are considered

B1. *Plane-to-surface deformation of neo-Hookean membranes.* For this case $c = 0$, and we may also let $\theta^\alpha = Z^\alpha$ and $Z^3 = 0$. The equations of equilibrium are just

$$\left(\frac{\partial^2}{\partial Z_1{}^2} + \frac{\partial^2}{\partial Z_2{}^2}\right) z_i = 0 \cdot \tag{3.20}$$

This result represents a substantial generalization of [15] in that the deformed surface is three-dimensional. Traction boundary conditions are in general more difficult to handle. However, for a traction-free boundary the conditions may be easily derived from (2.15). They are just

$$dz_i/dN = 0 \tag{3.21}$$

where N indicates the direction normal to the undeformed boundary.

B2. *Surface-to-plane deformation.* We let $z_3 = 0$, and the two displacement equations of equilibrium are

$$(A^{1/2}A^{\alpha\delta}z_{,\delta}{}^{\beta})_{,\alpha} = 0, \tag{3.22}$$

where the relation

$$e^{\delta\nu}z_{,\delta}{}^{\beta}z_{,\nu}{}^{\gamma} = e^{\beta\gamma}a^{1/2} \tag{3.23}$$

has been used to eliminate the term involving c and, as a result, (3.22) holds for all c. The two equations (3.22) are linear and uncoupled. Once again, the traction-free boundary conditions may be derived from (2.15). They are

$$\left(A^{1/2}A^{\alpha\delta} - \frac{c}{2}a^{1/2}a^{\alpha\delta}\right) z_{,\delta}{}^{\beta} e_{\alpha\gamma}\left(\frac{d\theta^\gamma}{dL}\right) = 0 \tag{3.24}$$

where $d\theta^\gamma/dL$ may be derived from the spatial boundary curve defined on S. It is clear that this boundary condition is linear only for the case $c = 0$ (neo-Hookean material).

If the undeformed surface is isometric with the plane, i.e., $A_{\alpha\beta} = \delta_{\alpha\beta}$ for a special choice of θ^α, then (3.22) becomes

$$\left(\frac{\partial^2}{\partial\theta_1{}^2} + \frac{\partial^2}{\partial\theta_2{}^2}\right) z_\beta = 0 \cdot \tag{3.25}$$

However, the interpretation of (3.25) must be consistent with the coordinates θ_α. For example, if S is a cylindrical surface of radius R defined by

$$Z_1 = R\cos\theta_2\,, \qquad Z_2 = R\sin\theta_2\,, \qquad Z_3 = \theta_1\,, \tag{3.26}$$

then z_β are harmonic in the "cartesian" $\theta_1\theta_2$-plane, and periodic in θ_2 with period 2π. It should be mentioned that one of the equations involved in axisymmetric deformations of an initially cylindrical membrane can be explicitly integrated [12]. The simplification implied by (3.25), however, is not restricted to axisymmetric problems.

B3. *Surface-to-surface deformation of neo-Hookean membranes.* The displacement equations of equilibrium are

$$(A^{1/2}A^{\alpha\delta}z^i_{,\delta})_{,\alpha} = 0 \qquad (c = 0) \tag{3.27}$$

and the traction-free conditions are

$$A^{1/2}A^{\alpha\delta}z^i_{,\delta}e_{\alpha\gamma}\left(\frac{d\theta^\gamma}{dL}\right) = 0. \tag{3.28}$$

The difference between this case and B2 is that c must be zero, even if only displacement conditions are involved.

C. *Neo-Hookean membrane and body force* **P**. Replacing $a^{1/2}p_i$ by $A^{1/2}P_i$ and using (3.19), we obtain from (2.14)

$$(A^{1/2}A^{\alpha\delta}z_{,\delta}{}^i)_{,\alpha} = -\frac{1}{2C}A^{1/2}P_i \tag{3.29}$$

which, for S isometric with the plane, reduces to

$$\left(\frac{\partial^2}{\partial\theta_1{}^2} + \frac{\partial^2}{\partial\theta_2{}^2}\right) z_i = -\frac{1}{2C}P_i\,. \tag{3.30}$$

D. *Inflation of axisymmetric membranes.* When the surface load is a constant inflating pressure p and when W is replaced by (3.1), the equations of equilibrium (2.14) do not seem to reduce to linear equations. The class of axisymmetric problems, however, can be solved explicitly. Thus, we turn to Eq. (2.33)–(2.38).

Substituting (3.1) for W, we obtain from (2.36)

$$\tau_1 = 2C\left(\frac{\Lambda_1}{\Lambda_2} - \frac{c}{2}\right), \qquad \tau_2 = 2C\left(\frac{\Lambda_2}{\Lambda_1} - \frac{c}{2}\right)\,. \tag{3.31}$$

The two equations of equilibrium (2.37), (2.38) immediately reduce to

$$\left(R\,\frac{dl}{dL}\right)^2 - r^2 = K_1\,, \tag{3.32}$$

$$\left(R - \frac{c}{2}\,\frac{r}{dl/dL}\right)\frac{dz}{dL} + \frac{p}{4C}r^2 = K_2\,, \tag{3.33}$$

where K_1 and K_2 are arbitrary constants. Eliminating dl/dL from the above, we get

$$R\,\frac{dz}{dL} = \left(K_2 - \frac{p}{4C}r^2\right)\left[1 - \frac{c}{2}\left(1 + \frac{K_1}{r^2}\right)^{-1/2}\right]^{-1}\,. \tag{3.34}$$

Eq. (3.32) may be written as

$$R\,\frac{dr}{dL} = \pm\left[K_1 + r^2 - \left(R\,\frac{dz}{dL}\right)^2\right]^{1/2}\,. \tag{3.35}$$

In view of (3.34), (3.35) may be integrated to yield r as a function of L for given $R(L)$.

If both r and R vanish at the axis of symmetry then $K_1 = 0$. The quantity $2\pi K_2$ is the total force acting on a cross-section perpendicular to the axis of symmetry. Thus, $K_2 = 0$ if the inflating pressure p is the only external load. Setting $K_1 = 0$ in (3.32), we conclude that $\Lambda_1 = \Lambda_2$. It follows that

$$\tau_1 = \tau_2 = 2C\left(1 - \frac{c}{2}\right) \cdot \tag{3.36}$$

If $K_2 = 0$, Eq. (3.33), which was derived from (2.38), is equivalent to

$$r = -\frac{2\tau_1}{p}\frac{dz}{dl} = -\frac{4C\left(1 - \frac{c}{2}\right)}{p}\frac{dz}{dl} \cdot \tag{3.37}$$

Nothing that l is the arc length along the deformed meridian curve, we conclude from (3.37) that the deformed meridian curve is the circle

$$r^2 + (z - z_c)^2 = \rho^2 \equiv \left[\frac{4C}{p}\left(1 - \frac{c}{2}\right)\right]^2 \tag{3.38}$$

where z_c is a constant. This result holds for a closed S, i.e. a balloon, as well as a supported membrane. It indicates that the deformed surface is always spherical. Moreover, the radius ρ of the sphere depends only on the inflating pressure, and is independent of the initial shape of the membrane, as long as it is axisymmetric. Of course the S- to -s mapping function is dictated by the initial shape S. Setting $K_1 = K_2 = 0$ in (3.34) and (3.35), andintegrating, we obtain

$$\ln \frac{r}{\rho + (\rho^2 - r^2)^{1/2}} = \pm \int \frac{dL}{R(L)} \tag{3.39}$$

where ρ is the radius of the sphere defined by (3.38), and the change of the sign occurs at the equator of the sphere (cf. (3.35)).

This portion represents a slight generalization of Forster's results [13] in that $c \neq 0$. Moreover, the relation (3.36) appears to have escaped his attention. In any case, the immediate consequence of (3.36) is (3.38) where ρ is found to be inversely proportional to the inflating pressure, a characteristic typical of a neo-Hookean material.

4. Type-II asymptotic solution. We consider a Rivlin material defined by (2.41), and assume that the function f satisfies the property

$$f(x) = \frac{C_2}{n+1} x^{n+1} \quad x \to \infty \quad (n > -\tfrac{1}{2}). \tag{4.1}$$

For $n \leq -\frac{1}{2}$, the asymptotic solution is of the Type-I nature. It follows from (4.1) and (2.41) that

$$\frac{1}{I}\frac{\partial W}{\partial I} = 2C_1 + 2C_2 J^{2n-3}, \tag{4.2}$$

$$\frac{\partial W}{\partial J} = -2C_1 + 2C_2 J^{2n+1} \tag{4.3}$$

as Λ_1, $\Lambda_2 \to \infty$. Certain general statements can be made for this class of problems. Substituting (4.2) and (4.3) into (2.17), and keeping only the dominate portion, we obtain

$$\tau^{\alpha\delta} = 2C_2 J^{2n+1} a^{\alpha\delta}. \tag{4.4}$$

Consider now the case of zero surface load. Substituting (4.4) into (2.20) and noting that the covariant derivative of the metric tensor vanishes, we get $J_{,\alpha} = 0$. It follows that

$$J = 1/\Lambda \tag{4.5}$$

where Λ is a (small) constant representing the transverse stretch ratio. An immediate consequence of (4.4) and (4.5) is that the deformed surface is uniformly stressed with the constant principal Cauchy stresses

$$\tau_1 = \tau_2 = 2C_2J^{2n+1} = 2C_2\Lambda^{-(2n+1)}. \tag{4.6}$$

The third equation of equilibrium (2.21), in conjunction with (4.4), becomes

$$b_\alpha{}^\alpha = 0 \tag{4.7}$$

which implies that the mean curvature of the deformed surface is zero. It follows that an infinitely strained surface without surface load is a minimal surface, and the problem becomes a purely geometric one.

If the surface load is a constant inflating pressure p, then (4.6) still holds and (2.21) becomes

$$b_\alpha{}^\alpha = -p/2C_2J^{2n+1}. \tag{4.8}$$

The inflated shape of the membrane is thus a surface of constant mean curvature. In case the undeformed surface S is closed, then the deformed surface s must be spherical [25]. Using (4.8) and the fact that J is constant, we can determine the radius ρ of the spherical surface s. It is

$$\rho = \frac{4C_2}{p}\left[\frac{p^2S}{64\pi C_2{}^2}\right]^{(2n+1)/(4n+1)} \tag{4.9}$$

where S is the surface area of the undeformed surface S. This result is a generalization of [16] and [17] in that the undeformed surface is completely arbitrary. For $n = 0$ (Mooney), (4.9) reduces to the result obtained in [17] for axisymmetric balloons. The S-to-s mapping associated with the simple result (4.9) is, unfortunately, a complicated nonlinear geometric problem.

For axisymmetric problems, the condition (4.5), which satsifies (2.37), together with (2.38), enables us explicitly to integrate all the equations involved. This is, in fact, the very origin that makes the explicit solutions presented in [17]–[20] possible. Other than the axisymmetric case, Type-II asymptotic analysis does not seem to lead to any substantial simplification. We only include a more detailed discussion on surface-to-plane deformation.

Surface-to-plane deformation. The surface load for this case is, of course, zero. We also set $z_3 = 0$. Substituting (4.2), (4.3) into (2.14) and using (3.23), we obtain

$$\left(\frac{1}{I}\frac{\partial W}{\partial I}A^{1/2}A^{\alpha\delta}z_{\beta,\delta}\right)_{,\alpha} + 2C_2e^{\alpha\mu}e^{\beta\lambda}z_{\lambda,\mu}(J^{2n+1} + J^{2n-3})_{,\alpha} = 0 \tag{4.10}$$

which may be solved to yield

$$(J^{2n+1} + J^{2n-3})_{,\alpha} = -\frac{1}{2C_2J}\left(\frac{1}{I}\frac{\partial W}{\partial I}A^{1/2}A^{\gamma\delta}z_{\mu,\delta}\right)_{,\gamma} z_{\mu,\alpha}\,. \tag{4.11}$$

It follows from (4.11) that

$$e^{\alpha\beta}z_{\mu,\alpha}\left(\frac{1}{I}\frac{\partial W}{\partial I}A^{1/2}A^{\gamma\delta}z_{\mu,\delta}\right)_{,\gamma\beta} = 0. \tag{4.12}$$

If we now let $J \to \infty$, then the dominant portion of (4.11) becomes

$$(J^{2n+1})_{,\alpha} = 0 \tag{4.13}$$

and the solution is

$$J = \frac{\partial z_1}{\partial \theta_1}\frac{\partial z_2}{\partial \theta_2} - \frac{\partial z_1}{\partial \theta_2}\frac{\partial z_2}{\partial \theta_1} = \frac{1}{\Lambda} \tag{4.14}$$

which is just the general conclusion (4.5). Eq. (4.12) now becomes

$$e^{\alpha\beta} z_{\mu,\alpha}(A^{1/2}A^{\gamma\delta}z_{\mu,\delta})_{,\gamma\beta} = 0. \tag{4.15}$$

The asymptotic solution is thus determined by (4.14) and (4.15). We note that (4.15) is an identity for axisymmetric problems.

In case the undeformed surface S is isometric with the plane, then (4.15) becomes

$$e^{\alpha\beta} z_{\mu,\alpha} z_{\mu,\gamma\gamma\beta} = 0, \tag{4.16}$$

and the asymptotic solution is determined by (4.14) and (4.16).

For plane stress problems, the two equations (4.14) and (4.16) remain unchanged, but the surface coordinates θ_α may be replaced by Z_α. These two equations are exactly the same as those obtained for the class of plane-strain-superposed-on-uniform-finite-extension problems discussed by Adkins [26] (see also p. 116 of [3]). It follows that there exists a kind of correspondence principle between the two classes of problems. In particular, the reciprocal theorem due to Adkins [26] also applies. In practice, though, the existence of such a correspondence principle is not really that useful in that there are just not that many nontrivial explicit solutions.

For materials defined by (2.44) and (2.46), relations similar to (4.4) may be obtained. These are

$$\tau^{\alpha\delta} = Ta^{\alpha\delta} \qquad \text{(Soap-film)}, \tag{4.17}$$

$$\tau^{\alpha\delta} = \mu J^{-1/2} a^{\alpha\delta} \qquad \text{(Blatz-Ko)}. \tag{4.18}$$

It follows that the conditions (4.5) and (4.7) also hold for these two cases.

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