QUARTERLY OF APPLIED MATHEMATICS

VOLUME XXXVI January 1979 No. 4

NEARLY ISOCHORIC ELASTIC DEFORMATIONS: VOLUME CHANGES IN PLANE STRAIN*

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Abstract. Plane deformations of nearly incompressible elastic solids are examined with a view to calculating the volume changes accompanying an arbitrary (plane) deformation. The dilatation is shown to be determined from a knowledge of the deformation appropriate to the corresponding incompressible material under the same boundary conditions. This work parallels that given in a previous paper for three-dimensional deformations and relies on the decomposition of the deformation gradient into its dilatational and isochoric parts.

It is emphasized that the manner in which the response function of an incompressible material is transmitted into that of a nearly incompressible material is of critical importance in the calculation of the dilatation. This is illustrated for a specific boundary-value problem and for isotropic materials. In particular, it is shown that when the incompressible neo-Hookean solid is considered in the compressible context mutually contradictory results may be obtained. These results are assessed in the light of experimental evidence available for rubberlike solids.

1. Introduction. Let $\mathbf{X}$ be the position vector of a material point in the undeformed configuration (assumed free of stress) and $\mathbf{x}(\mathbf{X})$ its position in the deformed configuration. On a rectangular Cartesian basis $\mathbf{X}$ and $\mathbf{x}$ respectively have components X_i and x_i ($i = 1, 2, 3$). Attention is confined here to plane deformations for which $x_3 = X_3$ and such that x_1 and x_2 are independent of X_3.

The (plane) deformation gradient $\partial x_i/\partial X_j$ ($i, j = 1$ or 2) is denoted by α_{ij} and symbolically by $\boldsymbol{\alpha}$. The dilatation, denoted by $\epsilon \equiv J - 1$, is obtained from

$$J = \det \boldsymbol{\alpha}. \tag{1.1}$$

We consider an elastic material possessing a stored-energy function W per unit undeformed volume. In the present context W is interpreted per unit undeformed area of the (1,2)-plane and is regarded as a function of $\boldsymbol{\alpha}$ (subject to its indifference to rigid-body rotations of the material after deformation).

The (in-plane) components of nominal stress are therefore given by

$$s_{ji} = \partial W/\partial \alpha_{ij}, \qquad \mathbf{s} = \partial W/\partial \boldsymbol{\alpha} \tag{1.2}$$

* Received December 29, 1977.

and the equilibrium equations can be put as

$$\operatorname{div} \mathbf{s} = \mathbf{0} \tag{1.3}$$

when there are no body forces, div denoting the divergence operator relative to $\mathbf{X}$.

The stress normal to the (1,2)-plane required to maintain the plane-strain conditions will not be needed explicitly, and no further reference is made to it.

We note that the Cauchy stress $\boldsymbol{\sigma}$ is given by

$$J\boldsymbol{\sigma} = \boldsymbol{\alpha}\mathbf{s} = \boldsymbol{\alpha}(\partial W/\partial\boldsymbol{\alpha}), \tag{1.4}$$

the product $\boldsymbol{\alpha}\mathbf{s}$ being interpreted in the usual matrix sense.

For an incompressible material the subscript zero is attached to all relevant quantities. The deformation is given by $\mathbf{x}_0(\mathbf{X})$, and the deformation gradient denoted $\boldsymbol{\alpha}_0$, where

$$\det \boldsymbol{\alpha}_0 = 1. \tag{1.5}$$

Let $W_0(\boldsymbol{\alpha}_0)$ denote the strain-energy function per unit area of the (1,2)-plane. Eq. (1.5) states that the area does not change with deformation.

The nominal stress $\mathbf{s}_0$ is now given by

$$\mathbf{s}_0 = (\partial W_0/\partial\boldsymbol{\alpha}_0) - p_0\boldsymbol{\beta}_0^T, \tag{1.6}$$

where $\boldsymbol{\beta}_0^T = \boldsymbol{\alpha}_0^{-1}$, superscript T denotes the transpose, and p_0 is an arbitrary (plane) hydrostatic stress.

The corresponding Cauchy stress $\boldsymbol{\sigma}_0$ is given by

$$\boldsymbol{\sigma}_0 = \boldsymbol{\alpha}_0\mathbf{s}_0 = \boldsymbol{\alpha}_0(\partial W_0/\partial\boldsymbol{\alpha}_0) - p_0\boldsymbol{\delta}, \tag{1.7}$$

where $\boldsymbol{\delta}$ is the (two-dimensional) identity. For future reference we note that

$$\tfrac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_0) = \tfrac{1}{2}\boldsymbol{\alpha}_0\cdot(\partial W_0/\partial\boldsymbol{\alpha}_0) - p_0, \tag{1.8}$$

where the dot denotes the scalar product and tr denotes the trace. Thus $\operatorname{tr}(\boldsymbol{\sigma}_0) = \boldsymbol{\delta}\cdot\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_0\cdot\boldsymbol{\delta}$.

2. Decomposition of the deformation. The notation used above parallels that adopted in [1] for three-dimensional deformations. In [1] we decomposed the deformation gradient into its isochoric and dilatational parts along the lines suggested by Flory [2]. Here we employ an analogous decomposition appropriate for plane deformations.

We introduce the *isochoric part* $\boldsymbol{\alpha}^*$ of the deformation gradient $\boldsymbol{\alpha}$ defined by

$$\boldsymbol{\alpha}^* = J^{-1/2}\boldsymbol{\alpha}, \tag{2.1}$$

where J is given by (1.1). We refer to J as the *dilatational part* of the deformation. From (1.1) and (2.1) it follows that

$$\det \boldsymbol{\alpha}^* = 1. \tag{2.2}$$

Let $\boldsymbol{\beta}^T$ denote $\boldsymbol{\alpha}^{-1}$. Then

$$\boldsymbol{\beta}^* = J^{1/2}\boldsymbol{\beta} \tag{2.3}$$

and $\boldsymbol{\beta}^{*T}$ is the inverse of $\boldsymbol{\alpha}^*$.

In what follows we shall require the formulae

$$\partial J/\partial\boldsymbol{\alpha} = J\boldsymbol{\beta}^T \tag{2.4}$$

and, in index notation,

$$\partial\alpha^*_{kl}/\partial\alpha_{ij} = J^{-1/2}(\delta_{ik}\delta_{jl} - \tfrac{1}{2}\alpha^*_{kl}\beta^*_{ij}). \tag{2.5}$$

The derivation of (2.5) requires the use of (2.1), (2.3) and (2.4).

We now regard W as a function of $\boldsymbol{\alpha}^*$ and J *independently, but subject to the constraint* (2.2). We write

$$W \equiv W(\boldsymbol{\alpha}^*, J). \tag{2.6}$$

From (1.2), with the help of (2.1), (2.3), (2.4) and (2.5), we now obtain

$$\mathbf{s} = J^{-1/2}\left\{\frac{\partial W}{\partial \boldsymbol{\alpha}^*} - \left(\frac{1}{2}\boldsymbol{\alpha}^* \cdot \frac{\partial W}{\partial \boldsymbol{\alpha}^*} - JW_J\right)\boldsymbol{\beta}^{*T}\right\}, \tag{2.7}$$

where W_J denotes $\partial W/\partial J$ (at fixed $\boldsymbol{\alpha}^*$).

The term in curly brackets in (2.7) should be compared with the right-hand side of (1.6).

From (1.4) and (2.7) we now obtain

$$J\boldsymbol{\sigma} = \boldsymbol{\alpha}\mathbf{s} = \boldsymbol{\alpha}^* \frac{\partial W}{\partial \boldsymbol{\alpha}^*} + \left(JW_J - \frac{1}{2}\boldsymbol{\alpha}^* \cdot \frac{\partial W}{\partial \boldsymbol{\alpha}^*}\right)\boldsymbol{\delta}. \tag{2.8}$$

It follows immediately from (2.8) that

$$\tfrac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}) = W_J, \tag{2.9}$$

and this should be compared with (1.8).

3. Nearly isochoric deformations. Nearly incompressible materials are characterized by the property that, under given boundary conditions, the deformation differs from that in the corresponding incompressible material by a "small" quantity. Expressed mathematically we have

$$\mathbf{x} = \mathbf{x}_0 + \eta\mathbf{u}, \tag{3.1}$$

where η ($\ll 1$) is essentially the ratio of shear to bulk modulus and $\mathbf{u}(\mathbf{X})$ is typically of the same order as $\mathbf{x}_0$. Full details are given in [1]. We note here that (3.1) is valid provided the order of magnitude of the applied tractions does not exceed that of the shear modulus. Note that the dilatation $\epsilon \equiv J - 1$ is of order η.

It follows that

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \eta\partial\mathbf{u}/\partial\mathbf{X} \tag{3.2}$$

and hence, from (1.1) and (1.5), that

$$J = 1 + \eta(\partial\mathbf{u}/\partial\mathbf{X})\cdot\boldsymbol{\beta}_0^T \equiv 1 + \eta\operatorname{tr}(\partial\mathbf{u}/\partial\mathbf{x}_0) \tag{3.3}$$

to the first order in η. From (2.1) we obtain

$$\boldsymbol{\alpha}^* = \boldsymbol{\alpha}_0 + \eta\{\partial\mathbf{u}/\partial\mathbf{X} - \tfrac{1}{2}\operatorname{tr}(\partial\mathbf{u}/\partial\mathbf{x}_0)\boldsymbol{\alpha}_0\}. \tag{3.4}$$

As in [1] we expand $W(\boldsymbol{\alpha}^*, J)$ as a power series in ϵ. Thus,

$$W(\boldsymbol{\alpha}^*, J) = W(\boldsymbol{\alpha}^*, 1) + \epsilon W_J(\boldsymbol{\alpha}^*, 1) + \tfrac{1}{2}\epsilon^2 W_{JJ}(\boldsymbol{\alpha}^*, 1), \tag{3.5}$$

to the second *power* in ϵ. For our purposes higher-order terms are not needed.

For compatibility with the linear theory we must have

$$W(\boldsymbol{\delta}, 1) = W_J(\boldsymbol{\delta}, 1) = 0, \quad W_{JJ}(\boldsymbol{\delta}, 1) = \kappa \tag{3.6}$$

where κ is the bulk modulus in the undeformed configuration. Thus the term $W_{JJ}(\boldsymbol{\alpha}^*, 1)$ is typically of order κ.

As discussed in [1], the terms $W(\boldsymbol{\alpha}^*, 1)$ and $W_J(\boldsymbol{\alpha}^*, 1)$ are typically of the order of the shear modulus μ in the undeformed configuration. With $\mu/\kappa = \eta$ and, by (3.3), the fact that ϵ is of order η it follows that the second and third terms in (3.5) are each of order $\mu\eta$.

Substitution of (3.5) into (2.7) and use of (3.4) leads to

$$\mathbf{s} = \frac{\partial W}{\partial \boldsymbol{\alpha}_0}(\boldsymbol{\alpha}_0, 1) - \left\{\frac{1}{2}\boldsymbol{\alpha}_0 \cdot \frac{\partial W}{\partial \boldsymbol{\alpha}_0}(\boldsymbol{\alpha}_0, 1) - W_J(\boldsymbol{\alpha}_0, 1) - \epsilon W_{JJ}(\boldsymbol{\alpha}_0, 1)\right\}\boldsymbol{\beta}_0^T + \mu O(\eta) \cdot \tag{3.7}$$

If $W(\boldsymbol{\alpha}_0, 1)$ is identified with the incompressible strain-energy function $W_0(\boldsymbol{\alpha}_0)$ then (3.7) can be rewritten as

$$\mathbf{s} = \mathbf{s}_0\{1 + O(\eta)\}, \tag{3.8}$$

where $\mathbf{s}_0$ is given by (1.6), provided p_0 is identified with the term in curly brackets in (3.7). Thus

$$p_0 = \tfrac{1}{2}\boldsymbol{\alpha}_0 \cdot \frac{\partial W_0}{\partial \boldsymbol{\alpha}_0} - W_J(\boldsymbol{\alpha}_0, 1) - \epsilon W_{JJ}(\boldsymbol{\alpha}_0, 1) \cdot \tag{3.9}$$

The equations of incompressible elasticity are equivalent to the zero-order equations of compressible elasticity when these are expanded as a power series in η. In view of (3.3), $\mathrm{tr}(\partial \mathbf{u}/\partial \mathbf{x}_0)$ is determined from (3.9). The displacement function $\mathbf{u}$ is obtained from the first-order equations, which we do not require here, and this must be compatible with (3.9).

With the help of (1.8) Eq. (3.9) may be rewritten as an equation for $\epsilon \equiv \eta\,\mathrm{tr}(\partial \mathbf{u}/\partial \mathbf{x}_0)$. Thus

$$\epsilon W_{JJ}(\boldsymbol{\alpha}_0, 1) = \tfrac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}_0) - W_J(\boldsymbol{\alpha}_0, 1). \tag{3.10}$$

We note, in particular, that (3.10) provides an expression for the dilatation in terms of the deformation $\boldsymbol{\alpha}_0$ appropriate to the incompressible material with strain-energy function $W_0(\boldsymbol{\alpha}_0)$. It also requires a knowledge of the functions $W_J(\boldsymbol{\alpha}^*, 1)$ and $W_{JJ}(\boldsymbol{\alpha}^*, 1)$ associated with the compressible material.

The formula (3.10) is valid for all elastic materials independently of any material symmetry.

4. Isotropic elastic solids. For an isotropic elastic solid W depends on $\boldsymbol{\alpha}$ through the (positive) principal stretches λ_1, λ_2 and is indifferent to their interchange. We denote the pair (λ_1, λ_2) by $\boldsymbol{\lambda}$ and introduce the pair of 'modified' stretches (see [3] for a description of this terminology in the three-dimensional context) defined by $\boldsymbol{\lambda}^* = J^{-1/2}\boldsymbol{\lambda}$ by analogy with (2.1). Note that $J = \lambda_1\lambda_2$ and $\lambda_1^*\lambda_2^* = 1$. The notation $\boldsymbol{\lambda}_0 = (\lambda_{01}, \lambda_{02})$ is reserved for the stretches associated with the incompressible material.

For convenience we write

$$W(\boldsymbol{\alpha}_0, 1) \equiv W_0(\boldsymbol{\alpha}_0) = \phi(\boldsymbol{\lambda}_0), \qquad W_J(\boldsymbol{\alpha}_0, 1) \doteq \chi(\boldsymbol{\lambda}_0), \qquad W_{JJ}(\boldsymbol{\alpha}_0, 1) = \psi(\boldsymbol{\lambda}_0),$$

in the isotropic context, where ϕ, χ and ψ are symmetric in λ_{01} and λ_{02}.

From (3.6) we then have

$$\phi(1) = \chi(1) = 0, \qquad \psi(1) = \kappa, \tag{4.1}$$

where **1** denotes the pair (1, 1).

The strain-energy function (3.5) can now be written

$$W = \phi(\boldsymbol{\lambda}^*) + \epsilon\chi(\boldsymbol{\lambda}^*) + \tfrac{1}{2}\epsilon^2\psi(\boldsymbol{\lambda}^*), \tag{4.2}$$

and (3.10) becomes

$$\epsilon\psi(\boldsymbol{\lambda}_0) = \tfrac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}_0) - \chi(\boldsymbol{\lambda}_0) \equiv \tfrac{1}{2}\boldsymbol{\lambda}_0\cdot\frac{\partial\phi}{\partial\boldsymbol{\lambda}_0}(\boldsymbol{\lambda}_0) - p_0 - \chi(\boldsymbol{\lambda}_0)\,\cdot \tag{4.3}$$

Noting the approximation $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*(1 + \tfrac{1}{2}\epsilon)$, we have

$$\phi(\boldsymbol{\lambda}) = \phi(\boldsymbol{\lambda}^*) + \tfrac{1}{2}\epsilon\boldsymbol{\lambda}^*\cdot\frac{\partial\phi}{\partial\boldsymbol{\lambda}^*}(\boldsymbol{\lambda}^*) \tag{4.4}$$

to the first order in η, and (4.2) can therefore be rewritten as

$$W = \phi(\boldsymbol{\lambda}) + \epsilon\left\{\chi(\boldsymbol{\lambda}^*) - \tfrac{1}{2}\boldsymbol{\lambda}^*\cdot\frac{\partial\phi}{\partial\boldsymbol{\lambda}^*}(\boldsymbol{\lambda}^*)\right\} + \tfrac{1}{2}\epsilon^2\psi(\boldsymbol{\lambda}^*). \tag{4.5}$$

The difference between the forms of (4.2) and (4.5) is significant and will be explained shortly in relation to rubberlike solids.

In view of (4.3) we remark that the dilatation depends on $\chi(\boldsymbol{\lambda}_0)$ and $\psi(\boldsymbol{\lambda}_0)$ in general as well as on the hydrostatic part of the (plane) stress.

In general ϕ and χ (and also ψ) are independent functions.

5. Applications to rubberlike solids. As discussed fully in [1] and [3], experimental evidence shows that $\psi(\boldsymbol{\lambda}_0)$ is independent of $\boldsymbol{\lambda}_0$ in simple tension for values of the stretches up to about 2. No reliable data are available for larger values of the stretches so we assume, tentatively, that $\psi(\boldsymbol{\lambda}_0) = \kappa$, noting (4.1). Our calculations will in fact be confined to values of the λ_{0i}'s for which this has been shown to be valid.

Eq. (4.3) now becomes

$$\kappa\epsilon = \tfrac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}_0) - \chi(\boldsymbol{\lambda}_0) \equiv \tfrac{1}{2}\boldsymbol{\lambda}_0\cdot\frac{\partial\phi}{\partial\boldsymbol{\lambda}_0}(\boldsymbol{\lambda}_0) - p_0 - \chi(\boldsymbol{\lambda}_0) \tag{5.1}$$

and the strain-energy function can be expressed as

$$W = \phi(\boldsymbol{\lambda}^*) + \epsilon\chi(\boldsymbol{\lambda}^*) + \tfrac{1}{2}\kappa\epsilon^2. \tag{5.2}$$

We recall that attention is restricted to circumstances in which $\epsilon = O(\eta)$.

For rubberlike solids, under the incompressibility approximation, the strain-energy function is derivable as a 'network' response function, and we may write

$$W_0 = \Phi(\boldsymbol{\lambda}_0). \tag{5.3}$$

The function $\Phi(\boldsymbol{\lambda}_0)$ reflects the network structure of the long chain molecules.

When the compressibility is taken into account the network response function is supplemented by a 'liquid-like' contribution to the strain-energy function which depends only on the volume ratio J (Flory [2]). In the present context this is the term $\tfrac{1}{2}\kappa\epsilon^2$. However, there are essentially two distinct ways in which the incompressible network response function $\Phi(\lambda_0)$ can be set in the compressible context. We can interpret this function as either $\Phi(\lambda^*)$ or $\Phi(\lambda)$.

In the former case

$$W = \Phi(\lambda^*) + \tfrac{1}{2}\kappa\epsilon^2, \tag{5.4}$$

while in the latter

$$W = \Phi(\lambda) + \tfrac{1}{2}\kappa\epsilon^2. \tag{5.5}$$

The strain-energy functions (5.4) and (5.5) are equally valid.

A comparison of (5.4) with (5.2) shows that

$$\Phi(\lambda^*) = \phi(\lambda^*),\ \chi(\lambda^*) = 0 \tag{5.6}$$

and hence, from (5.1),

$$\kappa\epsilon = \tfrac{1}{2}\operatorname{tr}(\sigma_0) \equiv \tfrac{1}{2}\lambda_0\cdot\frac{\partial\phi}{\partial\lambda_0}(\lambda_0) - p_0\,. \tag{5.7}$$

The Taylor expansion of $\Phi(\lambda)$ in (5.5) shows that

$$\Phi(\lambda^*) = \phi(\lambda^*), \qquad \chi(\lambda^*) = \tfrac{1}{2}\lambda^*\cdot\frac{\partial\phi}{\partial\lambda^*}(\lambda^*) - \tfrac{1}{2}\mathbf{1}\cdot\frac{\partial\phi}{\partial\lambda^*}(\mathbf{1}) \tag{5.8}$$

on comparison with (5.2), and (5.1) in this case becomes

$$\kappa\epsilon = \tfrac{1}{2}\mathbf{1}\cdot\frac{\partial\phi}{\partial\lambda_0}(\mathbf{1}) - p_0. \tag{5.9}$$

The constant term has been introduced in (5.8) to ensure that $\chi(\mathbf{1}) = 0$.

If full generality is retained then $\chi(\lambda^*)$, as given by (5.8), can be treated as independent of $\phi(\lambda^*)$. However, if one starts with a specific form of the network response function this is no longer the case, but one can retain the option of adding an arbitrary term $\epsilon\chi(\lambda^*)$ to either (5.4) or (5.5). The form of $\chi(\lambda^*)$ given by (5.8) in respect of (5.5) can then be absorbed into this new term. A discussion concerning the physical implications and origin of such a term is given in [4].

The distinction between (5.4) and (5.5), and hence between (5.7) and (5.9), is important since (5.7) and (5.9) may predict entirely different behaviour for the dilatation ϵ. This is now illustrated for a particular boundary-value problem.

6. Shear of an annulus: the incompressible solution. Incompressible elastic material is contained within a circular annulus of radii A and B $(> A)$. In the undeformed configuration we take polar coordinates R and Θ with $A \le R \le B$, $0 \le \Theta \le 2\pi$ and, correspondingly, r and θ in the deformed configuration.

Subject to the boundary conditions $r = R$ on $R = A, B$ and $\theta = \Theta$ on $R = A$, $\theta = \Theta + \gamma$ on $R = B$, we may write the deformation in the form

$$r = R, \qquad \theta = \Theta + \omega(R) \tag{6.1}$$

where $\omega(R)$ is a function to be determined from the equilibrium equations. Then

$$\omega(A) = 0, \qquad \omega(B) = \gamma. \tag{6.2}$$

The principal stretches λ_{01} and λ_{02} are such that $\lambda_{01}\lambda_{02} = 1$ and we write

$$\lambda_{01} = \lambda, \qquad \lambda_{02} = \lambda^{-1},$$

assuming, without loss of generality, that $\lambda \geq 1$.

We introduce the notation

$$q = \lambda - \lambda^{-1} \tag{6.3}$$

and it is easily shown that

$$q = R\omega'(R), \tag{6.4}$$

where the prime denotes d/dR. Details may be found in [5].

The material has strain-energy function $W_0 = \Phi(\lambda_0)$. This can be regarded as a function of λ or, equivalently, of q. Following the analysis in [5], we adopt q as the most appropriate variable and write

$$W_0 = \hat{\Phi}(q). \tag{6.5}$$

From [5], in which different notation was used, we have

$$\hat{\Phi}_q(q) = C/R^2, \tag{6.6}$$

where $\hat{\Phi}_q$ denotes $d\hat{\Phi}/dq$ and C is a constant related to the shear stress required to maintain the deformation. In fact

$$C = SB/2\pi, \tag{6.7}$$

where S is the shear stress on $R = B$.

When the form of the strain-energy function is specified (6.6) determines q. The argument presented in [5] indicates that q is determined *uniquely* as a function of R from (6.6). Hence $\omega(R)$ is obtained from (6.4).

The arbitrary hydrostatic pressure p_0 is then obtained from the equilibrium equations which give

$$Q - p_0 = C \int R^{-2}\omega'(R)\, dR + D \tag{6.8}$$

[5], where $Q = \hat{\Phi}_q(q)/q$ and D is a constant.

Detailed results for a number of different strain-energy functions have been given in [5]. For present purposes, however, it is sufficient to confine attention to the neo-Hookean form of strain-energy function which, in the present (plane-strain) context, has the form

$$\Phi(\lambda_0) = \tfrac{1}{2}\mu(\lambda_{01}^2 + \lambda_{02}^2 - 2), \qquad \hat{\Phi}(q) = \tfrac{1}{2}\mu q^2, \tag{6.9}$$

where $\mu(> 0)$ is the shear modulus in the undeformed configuration.

This gives $Q = \mu$, while (6.6) gives $\mu q = C/R^2$. Eq. (6.8) then simplifies to

$$p_0 = \mu - D + \tfrac{1}{4} C^2/\mu R^4. \tag{6.10}$$

In view of $(5.6)_1$ it follows from (6.9), with the help of (6.3), that

$$\lambda_0 \cdot \frac{\partial \phi}{\partial \lambda_0}(\lambda_0) = \mu(q^2 + 2), \qquad 1 \cdot \frac{\partial \phi}{\partial \lambda_0}(1) = 2\mu. \tag{6.11}$$

7. Shear of an annulus: the local dilatation. In respect of the strain-energy tunction (5.4), Eq. (5.7) with (6.10) and (6.11) gives

$$\kappa\epsilon = D + \tfrac{1}{4} C^2/\mu R^4, \tag{7.1}$$

while, for the strain-energy function (5.5), Eq. (5.9) with (6.10) and (6.11) gives

$$\kappa\epsilon = D - \tfrac{1}{4} C^2/\mu R^4. \tag{7.2}$$

In view of the boundary conditions given in Sec. 6 the overall volume of the material is unchanged in the (nearly incompressible) deformation. Thus

$$\int_A^B \epsilon R dR = 0,$$

and (7.1) and (7.2) therefore become

$$\kappa\epsilon = \tfrac{1}{4} C^2 \left(\frac{1}{R^4} - \frac{1}{A^2 B^2} \right) / \mu \tag{7.3}$$

$$\kappa\epsilon = \tfrac{1}{4} C^2 \left(\frac{1}{A^2 B^2} - \frac{1}{R^4} \right) / \mu \tag{7.4}$$

respectively.

In respect of the neo-Hookean strain-energy function the forms (5.4) and (5.5) of nearly incompressible strain-energy function predict opposite values for the dilatation, namely (7.3) and (7.4).

In particular, (7.3) corresponds to (5.4) and therefore (5.2) with $\chi = 0$. With $\chi = 0$, Eq. (5.1) shows that ϵ depends only on the hydrostatic part of the stress. This is also true for more general (three-dimensional) deformations [1, 3]). In [1] and [3] it has been shown that a χ-term is necessary in order that the predictions of the theory agree with experimental data. In the present circumstances, therefore, (5.4) is unrealistic and consequently so also is (7.3).

As for (5.5): this leads to

$$\chi(\lambda^*) = \tfrac{1}{2}\mu(\lambda_1^* - \lambda_2^*)^2 \tag{7.5}$$

which is positive, and from (7.4) we see that

$$\begin{aligned} \epsilon &< 0 \quad \text{for} \quad A \le R < (AB)^{1/2}, \\ \epsilon &> 0 \quad \text{for} \quad (AB)^{1/2} < R \le B, \\ \epsilon &= 0 \quad \text{for} \quad R = (AB)^{1/2}. \end{aligned} \tag{7.6}$$

The experimental evidence discussed in [1] and [3] indicates that χ *should* be positive. There is no experimental data available for the local dilatation accompanying the present deformation but intuitively the results (7.6) are consistent with what one would expect. This view is reinforced if we note that for the neo-Hookean solid it may be shown that

$$\kappa\epsilon = \sigma_{rr},$$

where σ_{rr} is the radial traction on a circle of radius r. This means that local volume decrease (increase) is associated with compressive (tensile) normal stress. For other forms of strain-energy function, however, the situation is less simple. In fact it may be shown that, in general,

$$\kappa\epsilon = \sigma_{rr} + \tfrac{1}{2} 1 \cdot \frac{\partial\phi}{\partial\lambda_0}(1) - Q,$$

where Q is defined in Sec. 6.

Since no volume change data are available for the specific deformation considered here, it is not possible at this stage to determine whether or not (7.5) needs to be supplemented by a term independent of $\phi(\lambda^*)$. Such a term is required in simple tension [1].

The circular shear problem considered here has also been examined from an entirely different standpoint in respect of a *highly* compressible elastic material [6]. It is noteworthy that the results (7.6) have also been found to hold for this material.

In conclusion we remark that the present analysis provides an insight into the properties required of the constitutive law of a compressible elastic material. The theory is in accord with what one might expect on intuitive grounds and is consistent with the limited experimental data which are available (see [1] and [3] for a discussion of this). In particular, we are able to reject (5.4) in favor of (5.5). This is consistent with Penn's deduction [7] which was discussed in [1].

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