BERGMAN'S INTEGRAL OPERATOR METHOD IN INHOMOGENEOUS ELASTICITY*

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1. Introduction. Plevako [1] introduced a representation for the displacement and stress in an isotropic elastic solid in which the elastic parameters vary with the Cartesian coordinate z. Here an integral operator method introduced by Bergman [2] in connection with the hodograph equations of subsonic and supersonic gasdynamics is employed in combination with the Plevako equations to obtain representations for the displacement and stress in terms of harmonic functions. If the elastic parameters of the material are taken to be arbitrary functions of z, then the displacement and stress are shown to be expressible in terms of two infinite series involving two harmonic functions and their derivatives. If, however, the elastic parameters take on certain specific forms then these series are shown to consist of only a finite number of terms. In particular, for an incompressible material with shear modulus given by $\mu = \mu_0(1 + c |z|)$ (where μ_0 and c are positive constants) the stress and displacement are written down in terms of a single series which contains only two terms. The representation thus obtained is used to consider the problem of a pressurized crack. The crack problem is reduced to a Fredholm integral equation. The first two terms in the iterative solution of the Fredholm equation are derived and used to obtain an expression for the crack tip stress intensity factor.

2. The basic formulation. The linearized equilibrium equations for an isotropic Hookean elastic solid are

$$\tau_{ij,j} = 0, \qquad i = 1, 2, 3 \ (\tau_{ij} = \tau_{ji}), \qquad i \equiv \partial / \partial x_i,$$
 (2.1)

where

$$\tau_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij}, \qquad i, j = 1, 2, 3.$$
(2.2)

Here, τ_{ij} , E_{ij} , represent Cartesian components of the stress and strain tensors respectively, while λ and μ are the Lamé elastic parameters. The strain displacement relations are

$$E_{ij} = 1/2[\mu_{i,j} + \mu_{j,i}]. \tag{2.3}$$

If we write $(x_1, x_2, x_3) = (x, y, z)$ and if λ , μ and Poisson's ratio v are assumed to vary in the z-direction alone, then the Plevako [1] representation for the displacement and stress is

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as follows:

$$\mathbf{\mathfrak{y}} = \left\{ \frac{-1}{2\mu} \left(\nu \nabla^2 - \frac{\partial^2}{\partial z^2} \right) \frac{\partial L}{\partial x} + \frac{\partial N}{\partial y}, \frac{-1}{2\mu} \left(\nu \nabla^2 - \frac{\partial^2}{\partial z^2} \right) \frac{\partial L}{\partial y} - \frac{\partial N}{\partial x}, \\ \cdot \frac{-1}{\mu} \left(\nabla^2 - \frac{\partial^2}{\partial z^2} \right) \frac{\partial L}{\partial z} + \frac{\partial}{\partial z} \left[\frac{1}{2\mu} \left(\nu \nabla^2 - \frac{\partial^2}{\partial z^2} \right) L \right] \right\},$$
(2.4)

$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = \left\{ \left[\nu \; \frac{\partial^2}{\partial y^2} \nabla^2 + \frac{\partial^4}{\partial x^2 \partial z^2} \right] L + 2\mu \; \frac{\partial^2 N}{\partial x \; \partial y}, \\ \cdot \left[\nu \; \frac{\partial^2}{\partial x^2} \nabla^2 + \frac{\partial^4}{\partial y^2 \partial x^2} \right] L - 2\mu \; \frac{\partial^2 N}{\partial x \; \partial y}, \left[\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right]^2 L \right\}, \qquad (2.5)$$
$$\boldsymbol{\tau} = (\tau_{xy}, \tau_{yz}, \tau_{zx}) = \left\{ - \left[\nu \nabla^2 - \frac{\partial^2}{\partial z^2} \right] \frac{\partial^2 L}{\partial x \; \partial y} - \mu \left[\; \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right] N, \right\}$$

$$-\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\frac{\partial^2 L}{\partial y \partial z} - \mu \frac{\partial^2 N}{\partial x \partial z}, -\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\frac{\partial^2 L}{\partial x \partial z} + \frac{\partial^2 N}{\partial y \partial z}\right], \quad (2.6)$$

where primes denote derivatives and L and N satisfy

$$\nabla^{2}\left\{\frac{1-\nu}{\mu}\nabla^{2}L\right\} - \left\{\left(\nabla^{2}-\frac{\partial^{2}}{\partial z^{2}}\right)L\right\} \left\{\frac{1}{\mu}\right\}^{\prime\prime} = 0, \qquad (2.7)$$

$$\nabla^2 N + \frac{\mu'}{\mu} \frac{\partial N}{\partial z} = 0.$$
 (2.8)

3. The Bergman series approach. In the spirit of the work of Bergman [4] which was developed in the context of the hodograph equations of gas dynamics, solutions to (2.7) and (2.8) are sought in the form

$$L = \sum_{n=0}^{\infty} F_n f_n(z), N = \sum_{n=0}^{\infty} G_n g_n(z), \qquad (3.1, 2)$$

where F_n , G_n are harmonic functions of x, y, z obeying the recurrence relations

$$\partial F_n/\partial z = F_{n-1}, \qquad \partial G_n/\partial z = G_{n-1}, \qquad n = 1, 2, \cdots.$$
 (3.3, 4)

This approach is analogous to the formal wavefront expansion method of geometric optics which has been widely used in recent years (see Karal and Keller [3]).

Combination of (3.1) and (2.7) provides

$$\sum_{n=0}^{\infty} \left[a_n'' F_n + \{ 2a_n' + b_n'' \} (\partial F_n / \partial z) + \left\{ 2b_n' + f_n \left\{ \frac{1}{\mu} \right\}'' \right\} (\partial^2 F_n / \partial z^2) \right] = 0, \quad (3.5)$$

where

$$a_n = (1 - \nu) f_n'' / \mu, \qquad b_n = 2(1 - \nu) f_n' / \mu.$$
 (3.6, 7)

Hence, if we set

$$a_{n}'' + \{2a_{n+1}' + b_{n+1}''\} + \left\{2b_{n+2}' + f_{n+2}\left\{\frac{1}{\mu}\right\}''\right\} = 0, n = 0, 1, 2, \cdots,$$
(3.8)

$$2a_0' + b_0'' + 2b_1' + f_1 \left\{ \frac{1}{\mu} \right\}'' = 0, \qquad (3.9)$$

$$2b_{0}' + \left\{\frac{1}{\mu}\right\}'' f_{0} = 0 \tag{3.10}$$

then (3.5) is automatically satisfied.

Insertion of (3.2) into (2.8) provides

$$\sum_{n=0}^{\infty} \left[\left\{ g_n'' + \frac{\mu'}{\mu} g_n' \right\} G_n + \left\{ 2g_n' + \frac{\mu'}{\mu} g_n \right\} \frac{\partial G_n}{\partial z} \right] = 0 \quad (3.11)$$

and this is automatically satisfied if we set

$$g_{n}'' + \frac{\mu'}{\mu}g_{n}' + 2g_{n+1}' + \frac{\mu'}{\mu}g_{n+1} = 0, \qquad n = 0, 1, 2, \cdots,$$
 (3.12)

$$2g_0' + \frac{\mu'}{\mu}g_0 = 0. \tag{3.13}$$

Thus, the relations (3.5)-(3.10) iteratively define the f_n in (3.1) while (3.12)-(3.13) iteratively define the g_n in (3.2). Further, in view of the recurrence relations (3.3) and (3.4), we have

$$L(x, y, z) = \sum_{n=0}^{\infty} f_n(z) \int_{z_0}^{z} F_0(t) [(z-t)^{n-1}/(n-1)!] dt$$

= $\int_{\overline{z}_0}^{z} \left\{ \sum_{n=0}^{\infty} f_n(z) (z-t)^{n-1}/(n-1)! \right\} F_0(t) dt$, (3.14)

$$N(x, y, z) = \sum_{n=0}^{\infty} g_n(z) \int_{\bar{z}_0}^{z} \left\{ \sum_{n=0}^{\infty} G_0(t) [(2-t)^{n-1}/(n-1)!] dt \right\}$$
$$= \int_{\bar{z}_0}^{z} \left\{ \sum_{n=0}^{\infty} g_n(z) (z-t)^{n-1}/(n-1)! \right\} G_0(t) dt, \qquad (3.15)$$

where the interchange of the order of summation and integration is valid in the region of uniform convergence of the kernels

$$\sum_{n=0}^{\infty} f_n(z)(z-t)^{n-1}/(n-1)!, \qquad \sum_{n=0}^{\infty} g_n(z)(z-t)^{n-1}/(n-1)!. \qquad (3.16,17)$$

In (3.14) and (3.15), z_0 and \overline{z}_0 are reference constants. The convergence properties of (3.16 and (3.17) depend on the form of the f_n and g_n which in turn depends on the nature of the inhomogeneity of the elastic parameters.

4. A particular class of solutions. Attention is now restricted to the case

$$\nu = \text{constant}, \quad \mu(z) = \mu_0 (1 + cz)^b \quad (c > 0).$$
 (4.1, 2)

It is readily seen that the system (3.6) - (3.10) admits solutions of the form

$$f_n = \overline{a}_n (1 + cz)^{\beta - n} \tag{4.3}$$

where, from (3.10),

$$\beta = \frac{1}{2} \left[b + 1 \pm ((b+1)[1 - \nu b/(1 - \nu])^{1/2}] \right] = \frac{1}{2} \left[b + 1 \pm \eta \right]$$
(4.4)

while (3.8), (3.9) provide solutions to (2.7) in the form

$$L = (1 + cz)^{\beta_1} \sum_{n=0}^{\infty} \overline{a}_n^{(1)} (1 + cz)^{-n} F_n^{(1)} + (1 + cz)^{\beta_2} \sum_{n=0}^{\infty} \overline{a}_n^{(2)} (1 + cz)^{-n} F_n^{(2)}, \quad (4.5)$$

where

$$\overline{a}_{n}^{(1)} = \left[\frac{(\eta - 2n + 1)^2 - (b + 2)^2}{8n}\right] c \,\overline{a}_{n-1}^{(1)}, \quad n = 1, 2, \cdots,$$
(4.6)

$$\overline{a}_{n}^{(2)} = \left[\frac{(\eta + 2n - 1)^{2} - (b + 2)^{2}}{8n}\right] c \overline{a}_{n-1}^{(2)}, \quad n = 1, 2, \cdots,$$
(4.7)

$$\beta_1 = \frac{1}{2} [b+1+\eta], \quad \beta_2 = \frac{1}{2} [b+1-\eta]$$
(4.8, 9)

and $\overline{a}_0^{(i)} i = 1, 2$ are arbitrary non-zero constants. Similarly, (2.8) admits solutions of the form (3.2) with

$$g_n = \bar{d}_n (1 + cz)^{\gamma - n}$$
 (4.10)

where (3.13) shows that

$$\gamma = -b/2 \tag{4.11}$$

while (3.12) provides

$$\overline{d}_n = \left[\frac{(2n-1)^2 - (b-1)^2}{8n}\right] c \,\overline{d}_{n-1}, n = 1, 2, \cdots$$
(4.12)

where \overline{d}_0 is an arbitrary non-zero constant. Thus,

$$N = (1 + cz)^{-b/2} \sum_{n=0}^{\infty} \bar{d}_n (1 + cz)^{-n} G_n.$$
(4.13)

It is noted that the recurrence relations (4.6) and (4.7) and (4.13) may be written respectively as

$$\overline{A}_{n+1}^{(1)} = \frac{(1+n)}{(\alpha+n)(\beta+n)} \,\overline{A}_n^{(1)}, n = 0, 1, 2, \cdots,$$

$$\overline{A}_{n+1}^{(2)} = \frac{(1+n)}{(\gamma+n)(\delta+n)} \,\overline{A}_n^{(2)}, n = 0, 1, 2, \cdots,$$

$$\overline{D}_{n+1} = \frac{(1+n)}{(\epsilon+n)(\zeta+n)} \,\overline{D}_n, n = 0, 1, 2, \cdots,$$
(4.14)

where

$$\bar{A}_{n}^{(1)} = \frac{1}{\bar{a}_{n}^{(1)}} \left(\frac{c}{2}\right)^{n}, \ \bar{A}_{n}^{(2)} = \frac{1}{\bar{a}_{n}^{(2)}} \left(\frac{c}{2}\right)^{n}, \ \bar{D}_{n} = \frac{1}{\bar{d}_{n}} \left(\frac{c}{2}\right)^{n},$$
(4.15)

 $\bar{a}_n^{(i)} \neq 0, i = 1, 2, \bar{d}_n \neq 0$, together with

$$\alpha = [b - \eta + 3]/2, \quad \beta = [-b - \eta - 1]/2,$$

$$\gamma = [\eta - 1 - b]/2, \quad \delta = [\eta + b + 3]/2,$$

$$\epsilon = b/2, \quad \zeta = (b - 2)/2.$$
(4.16)

Hence, if at least one of the parameters in each of the pairs $(\alpha, \beta), (\gamma, \delta), (\epsilon, \zeta)$ is unity, the

relation (4.14) generates the coefficients of confluent hypergeometric functions of the type $_{1}F_{1}(1; \lambda; x)$.

For particular inhomogeneities, the constituent series in (4.5) and the series (4.13) terminate. Thus, if

$$\overline{a}_{r+1}^{(1)} = \overline{a}_{s+1}^{(2)} = d_{t+1} = 0 \tag{4.17}$$

so that

$$(\eta - 2r - 1)^2 - (b + 2)^2 = 0,$$
 $(\eta + 2s + 1)^2 - (b + 2)^2 = 0,$
 $(2t + 1)^2 - (b - 1)^2 = 0,$ (4.18)

then solutions are generated in the form

$$L = \sum_{n=0}^{n=r} \overline{a}_n^{(1)} (1+cz)^{r-n} \frac{\partial^{r-n} \phi}{\partial z^{r-n}} + \sum_{n=0}^{n=s} \overline{a}_n^{(2)} (1+cz)^{s-n} \frac{\partial^{s-n} \psi}{\partial a^{s-n}}, \qquad (4.19)$$

$$N = (1 + cz)^{-b/2} \sum_{n=0}^{n=t} \bar{d}_n (1 + cz)^{-n} \frac{\partial^{t-n} \psi}{\partial z^{t-n}}, \qquad (4.20)$$

where ϕ , ψ and χ are arbitrary harmonic functions of x, y, and z.

In plane strain, a particularly simple case is that in which $\nu = 1/2$ (so that the material is incompressible) and b = 1, whence we obtain

$$L = -c\Phi + (1 + cz) \frac{\partial \Phi}{\partial z}, N = 0, \qquad (4.21)$$

where Φ is an arbitrary harmonic function of x and z. The displacements and stresses are given by

$$\mathbf{u} = \left\{ \frac{1}{4\mu} \left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial z^2} \right] \frac{\partial L}{\partial x}, 0, \frac{-1}{\mu} \frac{\partial^3 L}{\partial x^2 \partial z} + \frac{\partial}{\partial z} \left[\frac{1}{4\mu} \left(\frac{\partial^2 L}{\partial x^2} - \frac{\partial^2 L}{\partial z^2} \right) \right] \right\},$$
(4.22)

$$\sigma_x = \frac{\partial^4 L}{\partial x^2 \partial z^2}, \quad \sigma_z = \frac{\partial^4 L}{\partial x^4}, \quad \tau_{xz} = \frac{-\partial^4 L}{\partial x^3 \partial z}. \quad (4.23)$$

5. A crack problem. The stress field distribution in the neighborhood of a crack in an incompressible inhomogeneous material with shear modulus $\mu = \mu_0 (1 + c |z|)$ where $\mu_0 > 0$ and c > 0 are constants is now investigated. The crack is assumed to lie in the plane z = 0 and to occupy the region $-a < x < a, -\infty < y < \infty$. By symmetry, it is necessary to consider only the stress and displacement in the half-space z > 0 where the boundary z = 0 of the half-space is subjected to the boundary conditions

$$u_z = 0 \quad \text{for} \quad |x| > a, \tag{5.1}$$

$$\sigma_z = -p^*(x) \quad \text{for} \quad |x| < a, \tag{5.2}$$

where the normal traction $p^*(x)$ will be restricted to be an even function of x.

An appropriate form for the harmonic function occurring in (4.21) is

$$\Phi = \Re \int_0^\infty A(p) \exp(ip(x+iz)) \, dp \tag{5.3}$$

where A(p) is a function to be determined and \mathfrak{R} denotes the real part of a complex

number. Use of (5.3), (4.22) and (4.23) now provides the following expressions for u_z and σ_z :

$$u_{z} = \mu_{0}^{-1} \Re \int_{0}^{\infty} A(p) p^{4} \exp(ip(x+iz)) dp, \qquad (5.4)$$

$$\sigma_z = -\Re \int_0^\infty \{c + (1 + cz)p\} A(p) p^4 \exp(ip(x + iz)) \, dp.$$
 (5.5)

Hence, from $(5.1) - (\overline{5.5})$ it follows that A(p) must be chosen such that

$$\Re \int_0^\infty A(p) p^4 \exp(ipx) \, dp = 0 \quad \text{for} \quad |x| > a, \tag{5.6}$$

$$\Re \int_0^\infty (p+c) A(p) p^4 \exp(ipx) dp = p^*(x) \quad \text{for} \quad |x| < a.$$
(5.7)

Since p(x) is restricted to be a real function of x, it is sufficient to take A(p) in the form

$$A(p) = p^{-4} \int_0^a r(t) J_0(pt) dt, \qquad (5.8)$$

where r(t) is a real function to be determined and J_0 is the Bessel function of order zero. With this choice of A(p), the condition (5.6) is automatically satisfied while (5.6) yields

$$\int_{0}^{\infty} \cos(px)pdp \int_{0}^{\infty} r(t)J_{0}(pt) dt + c \int_{0}^{\infty} \cos(px) dp \int_{0}^{a} r(t)J_{0}(pt) dt$$

= $p^{*}(x)$ for $0 < x < a$. (5.9)

On interchange of the order of integration and use of standard results for integrals involving Bessel functions, it is seen that

$$\frac{d}{dx} \int_0^x \frac{r(t)}{(x^2 - t^2)^{1/2}} dt + c \int_x^a \frac{r(t) dt}{(t^2 - x^2)^{1/2}} = p^*(x) \quad \text{for} \quad 0 < x < a.$$
(5.10)

Use of the inversion formula for Abel's integral equation and interchange of the order of integration in the resulting double integral now provides

$$r(t) + \frac{2c}{\pi} \int_0^a K(s, t) r(s) \, ds = \frac{2t}{\pi} \int_0^t \frac{p^*(u) \, du}{(t^2 - u^2)^{1/2}} \quad \text{for} \quad 0 < t < a, \tag{5.11}$$

where

$$K(s, t) = t \int_{0}^{\min(s, t)} \frac{du}{[(t^{2} - u^{2})(s^{2} - u^{2})]^{1/2}}$$
$$= \frac{t}{4} \int_{-\pi}^{\pi} \frac{d\phi}{[s^{2} + t^{2} - 2st \cos\phi]^{1/2}} .$$
(5.12)

The stress σ_z near the crack tip or z = 0 for x > a may be now generated in the form

$$\sigma_z(x,0) = -\frac{d}{dx} \int_0^a \frac{r(t)}{(x^2 - t^2)^{1/2}} dt$$
 (5.13)

whence

$$\lim_{x \to a^+} (x - a)^{1/2} \sigma_z(x, 0) = r(a)/(2a)^{1/2}.$$
 (5.14)

Thus, once the Fredholm equation (5.11) has been solved, the stress intensity factor may be determined from (5.14).

To obtain some idea of the behavior of r(a) for small c, the first two terms in the iterative solution of (5.11) will now be obtained for the case of constant applied pressure $p^*(x) = p_0$. From (5.11), the first term is

$$r(t) = p_0 t,$$

while, again from (5.11), the first iteration yields

$$r(t) = p_0 t - \frac{2p_0 c}{\pi} \int_0^a K(s, t) s ds$$

= $p_0 t - \frac{2p_0 c}{\pi} t \int_0^t \frac{du}{(t^2 - u^2)^{1/2}} \int_u^a \frac{s ds}{(s^2 - u^2)^{1/2}}$
= $p_0 t - \frac{2p_0 c t}{\pi} \int_0^{\pi/2} (a^2 - t^2 \sin^2 \theta)^{1/2} d\theta.$

Hence, for small c,

$$r(a) \simeq p_0 a \left[1 - \frac{2c}{\pi} \right]$$
 (5.15)

and so, from (5.14),

$$\lim_{x \to a^+} (x - a)^{1/2} \sigma_z(x, 0) \simeq p_0 a \left[1 - \frac{2c}{\pi} \right]$$
 (5.16)

This indicates that the pressurized crack in an incompressible elastic material with shear modulus $\mu = \mu_0(1 + c |z|)$ is more stable than the corresponding crack in a homogeneous elastic medium with shear modulus μ_0 .

Finally, it is noted that a number of simpler boundary-value problems such as the problem of determining the effect of a distributed load on an inhomogeneous half-space may readily be solved by straightforward modifications of the procedure used in this section.

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